Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 315–317

INEQUALITIES AMONG THE NUMBER OF THE GENERATORS AND RELATIONS OF A KÄHLER GROUP

AZNIV KASPARIAN

Department of Mathematics and Informatics, Kliment Ohridski University
5 James Bourchier Blvd, 1126 Sofia, Bulgaria

Abstract. The present note announces some inequalities on the number of the generators and relations of a Kähler group $\pi_1(X)$, involving the irregularity $q(X)$, the Albanese dimension $a(X)$ and the Albanese genera $g_k(X)$, $1 \leq k \leq a(X)$, of the corresponding compact Kähler manifold X. The principal ideas for their derivation are outlined and the proofs are postponed to be published elsewhere.

Let X be an irregular compact Kähler manifold, i. e., with an irregularity $q = q(X) := \dim_{\mathbb{C}} H^{1,0}(X) > 0$. The **Albanese variety** $\mathrm{Alb}(X) = H^{1,0}(X)^*/H_1(X,\mathbb{Z})_{\text{free}}$ admits a holomorphic Albanese map $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$, given by integration $\mathrm{alb}_X(x)(\omega) := \int_{x_0}^{x} \omega$ of holomorphic $(1,0)$-forms $\omega \in H^{1,0}(X)$ from a base point $x_0 \in X$ to $x \in X$. The complex rank of the Albanese map alb_X is called an **Albanese dimension** $a = a(X)$ of X.
A compact Kähler manifold Y is said to be Albanese general if $\dim_{\mathbb{C}} Y = a(Y) < q(Y)$. The surjective holomorphic maps $f_k\colon X \to Y_k$ of a compact Kähler manifold X onto Albanese general Y_k are referred to as Albanese general k-fibrations of X. The maximum irregularity $q(Y_k)$ of a base Y_k of an Albanese general k-fibration $f_k\colon X \to Y_k$ is called k-th Albanese genus of X and denoted by $g_k = g_k(X)$. The present note states lower bounds on the Betti numbers $b_i(\pi_1(X)) := rk_{\mathbb{Z}} H^i(\pi_1(X),\mathbb{Z})$ of the fundamental group $\pi_1(X)$, in terms of the irregularity $q(X)$, the Albanese dimension $a(X)$ and the Albanese genera $g_k(X)$, $1 \leq k \leq a(X)$.
On the other hand, $b_i(\pi_1(X))$ are estimated above by the number of the generators s and the number of the relations r of $\pi_1(X)$ and, eventually, by the irregularity $q(X)$, exploiting to this end few abstract results on the group cohomologies.

Proposition 1. *Let X be a compact Kähler manifold with Albanese dimension $a \geq 2$, irregularity $q \geq a$ and Albanese genera g_k, $1 \leq k \leq a$. Put*

$$\mu^{2,0} := \max\left\{ \binom{\max\{a, g_k\,;\, g_k > 0, 2 \leq k \leq a\}}{2}\,,\ \delta^0_{g_1}(2q-3) \right\},$$

$$\mu^{1,1} := \max\left\{ \binom{a}{2}\,,\ 2a-1, g_k - 1,\ \delta^0_{g_1}(2q-3)\,;\ g_k > 0,\ 2 \leq k \leq a \right\}$$

where $\delta^0_{g_1}$ stands for Kronecker's delta. Denote by $b_2(\pi_1(X)) := rk_{\mathbb{Z}} H^2(\pi_1(X), \mathbb{Z})$ the second Betti number of the fundamental group of X and suppose that $\pi_1(X)$ admits a finite presentation with s generators and r relations. Then

$$r - s + 2q \geq b_2(\pi_1(X)) \geq 2\mu^{0,2} + \mu^{1,1}\,.$$

If F is a free group and R is a normal subgroup of F then Hopf's Theorem is equivalent to the presence of the exact sequence

$$0 \to H_2(F/R, \mathbb{Z}) \to H_1(R, \mathbb{Z})_F \to H_1(F, \mathbb{Z}) \to H_1(F/R, \mathbb{Z}) \to 0$$

where $H_1(R, \mathbb{Z})_F$ stands for the F-coinvariants of $H_1(R, \mathbb{Z})$ (cf. [4] or [1]). In particular, for a Kähler group $\pi_1(X)$ with s generators and r relations there follows $r - s + 2q \geq b_2(\pi_1(X))$.

The isomorphism $H^1(\pi_1(X), \mathbb{C}) \simeq H^1(X, \mathbb{C})$ of the first cohomologies of $\pi_1(X)$ and X allows to introduce Hodge decomposition $H^1(\pi_1(X), \mathbb{C}) = H^{1,0}(\pi_1(X)) \oplus H^{0,1}(\pi_1(X))$ on the group cohomologies. After constructing an Eilenberg–MacLane space $K(\pi_1(X), 1)$ by glueing to X cells of real dimension ≥ 3, one observes that the complex rank of the cup product of group cohomologies

$$\zeta^{i,j}_{\pi_1(X)} :\ \wedge^i H^{1,0}(\pi_1(X)) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(\pi_1(X)) \to H^{i+j}(\pi_1(X), \mathbb{C})$$

dominates the complex rank of the cup product of de Rham cohomologies

$$\zeta^{i,j}_X :\ \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) \to H^{i+j}(X, \mathbb{C})\,.$$

The quantities $\mu^{i,j}$ are lower bounds on $rk_{\mathbb{C}} \zeta^{i,j}_X$. They are derived by the means of the cohomological descriptions of a and g_k, due to Catanese (cf. [2]). On one hand, $rk_{\mathbb{C}} \zeta^{i,j}_X \geq \binom{a}{i+j}$ for all non-negative integers i, j with $2 \leq i + j \leq a$. On the other hand, $rk_{\mathbb{C}} \zeta^{i,j}_X \geq \binom{g_k - i}{j}$ for $0 \leq i \leq j$ and $2 \leq i + j \leq k \leq a$. Further, $rk_{\mathbb{C}} \zeta^{i,j}_X \geq (i+j)(q-i-j)+1$ provided $g_1 = \cdots = g_{i+j-1} = 0$. Finally, $rk^{1,1}_X \geq 2a - 1$.

Proposition 2. *Let X be a compact Kähler manifold whose fundamental group $\pi_1(X)$ admits a finite presentation with s generators and r relations. Then the complex rank of the cup products $\zeta^{i,j}_{\pi_1(X)}$ and $\zeta^{i,j}_X$ are bounded below by*

$$\mu^{i,j} := \max\left\{\binom{a}{i+j}, \binom{g_k - i}{j}, \delta^0_{g_1} \cdots \delta^0_{g_{i+j-1}}(i+j)(q-i-j)+1\right\}$$

where a stands for the Albanese dimension of X, g_k, $1 \leq k \leq a$, are the Albanese genera, $q > 0$ is the irregularity, $\delta^0_{g_s}$ denote Kronecker's deltas and the maximum is taken over the positive Albanese genera g_k, labeled by $i+j \leq k \leq a$. The Betti numbers $b_i(\pi_1(X)) := rk_{\mathbb{Z}} H^i(\pi_1(X), \mathbb{Z})$ are subject to the inequalities

$$sr^k \geq b_{2k+1}(\pi_1(X)) \geq 2\sum_{i=0}^{k} \mu^{i,2k+1-i} \quad \text{for } 1 \leq k \leq \frac{a-1}{2}$$

and

$$r^k \geq b_{2k}(\pi_1(X)) \geq 2\sum_{i=0}^{k-1} \mu^{i,2k-i} + \mu^{k,k} \quad \text{for } 2 \leq k \leq \frac{a}{2}\,.$$

The upper bounds on the higher Betti numbers of $\pi_1(X)$ are derived by Gruenberg's free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\pi_1(X)]$-module (cf. [3]).

Acknowledgements

The author is extremely grateful to Tony Pantev for the useful advices and the encouragement.

References

[1] Brown K., *Cohomology of Groups*, Springer, New York, Heidelberg, Berlin 1982.

[2] Catanese F., *Moduli and Classification of Irregular Kähler Manifolds (and Algebraic Varieties) with Albanese General Type Fibrations*, Inv. Math. **104** (1991) 263–289.

[3] Gruenberg K., *Resolutions by Relations*, J. London Math. Soc. **35** (1960) 481–494.

[4] Hopf H., *Fundamentalgruppe und Zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1942) 257–309.