

ON THE BIANCHI IDENTITIES IN A GENERALIZED WEYL SPACE*

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Abstract. In this paper, we show that the first Bianchi identity is valid for a generalized Weyl space having a semi-symmetric E -connection and that the second Bianchi identity is satisfied for a recurrent generalized Weyl space provided that the recurrence vector ψ_l and the Vranceanu vector Ω_l are related by $\psi_l = \frac{2}{n-1} \Omega_l$.

1. Introduction

An n -dimensional differentiable manifold W_n^* having an asymmetric connection ∇^* and asymmetric conformal metric tensor g^* preserved by ∇^* is called a **generalized Weyl space** [1]. For a such a space, in local coordinates, we have the compatibility condition

$$\nabla_k^* g_{ij}^* - 2T_k^* g_{ij}^* = 0, \quad (1.1)$$

where T_k^* are the components of a covariant vector field called the complementary vector field of the generalized Weyl space.

The coefficients L_{jk}^i of the connection ∇^* are obtained from the compatibility condition as [2]

$$L_{jk}^i = \Gamma_{jk}^i + \frac{1}{2} \left[\Omega_{kl}^h g_{(jh)}^* + \Omega_{jl}^h g_{(hk)}^* + \Omega_{jk}^h g_{(hl)}^* \right] g^{*(li)} \quad (1.2)$$

or, putting

$$Q_{jk}^i = \frac{1}{2} \left[\Omega_{kl}^h g_{(jh)}^* + \Omega_{jl}^h g_{(hk)}^* + \Omega_{jk}^h g_{(hl)}^* \right] g^{*(li)} \quad (1.3)$$

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we have

$$L_{jk}^i = \Gamma_{jk}^i + Q_{jk}^i \quad (1.4)$$

where Γ_{jk}^i and Ω_{jk}^i are, respectively, the coefficients of a Weyl connection and the torsion tensor of W_n^* given by

$$\Gamma_{jk}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - (\delta_j^i T_k + \delta_k^i T_j - g^{li} g_{jk} T_l), \quad (1.5)$$

and

$$\Omega_{jk}^i = L_{jk}^i - L_{kj}^i = 2L_{[jk]}^i. \quad (1.6)$$

According to Norden [3], if under a renormalization of the fundamental tensor g of the form $\tilde{g} = \lambda^2 g$, an object A admitting a transformation of the form $\tilde{A} = \lambda^p A$ is called a **satellite with weight** $\{p\}$ of the tensor g . The **prolonged covariant derivative** of the satellite A relative to the symmetric connection ∇ , denoted by $\dot{\nabla}A$, is defined by [4]

$$\dot{\nabla}_k A = \nabla_k A - p T_k A. \quad (1.7)$$

The prolonged covariant derivative of the satellite A relative to ∇^* will be denoted by $\dot{\nabla}_k^* A$, is defined by

$$\dot{\nabla}_k^* A = \nabla_k^* A - p T_k^* A. \quad (1.8)$$

2. Bianchi Identities

Let v^i be the contravariant components of vector field v in W_n^* . For the second order covariant derivative of v relative to ∇^* we have

$$\begin{aligned} \nabla_k^* \nabla_l^* v^i &= \partial_k \partial_l v^i + (\partial_k L_{hl}^i) v^h + (\partial_k v^h) L_{hl}^i \\ &+ L_{jk}^i \partial_l v^j + L_{jk}^i L_{hl}^j v^h - L_{lk}^j (\partial_j v^i) - L_{lk}^j L_{hj}^i v^h. \end{aligned} \quad (2.1)$$

Interchanging the indices k and l in (2.1) we obtain

$$\begin{aligned} \nabla_l^* \nabla_k^* v^i &= \partial_l \partial_k v^i + (\partial_l L_{hk}^i) v^h + (\partial_l v^h) L_{hk}^i + L_{jl}^i \partial_k v^j \\ &+ L_{jl}^i L_{hk}^j v^h - L_{kl}^j (\partial_j v^i) - L_{kl}^j L_{hj}^i v^h. \end{aligned} \quad (2.2)$$

Subtracting (2.2) from (2.1) we get

$$\nabla_k^* \nabla_l^* v^i - \nabla_l^* \nabla_k^* v^i = L_{hkl}^i v^h + \Omega_{kl}^j \nabla_j^* v^i, \quad (2.3)$$

where

$$L_{ijk}^l = \partial_j L_{ik}^l - \partial_k L_{ij}^l + L_{ik}^h L_{hj}^l - L_{ij}^h L_{hk}^l. \quad (2.4)$$

This is the curvature tensor corresponding to the connection ∇^* .

By cyclic permutation of i, j and k in (2.4)

$$L^l_{jki} = \partial_k L^l_{ji} - \partial_i L^l_{jk} + L^h_{ji} L^l_{hk} - L^h_{jk} L^l_{hi}, \tag{2.5}$$

$$L^l_{kij} = \partial_i L^l_{kj} - \partial_j L^l_{ki} + L^h_{kj} L^l_{hi} - L^h_{ki} L^l_{hj} \tag{2.6}$$

and summing (2.4), (2.5) and (2.6) side by side we obtain

$$\begin{aligned} L^l_{ijk} + L^l_{jki} + L^l_{kij} &= \partial_k \Omega^l_{ji} + \partial_i \Omega^l_{kj} + \partial_j \Omega^l_{ik} \\ &\quad + L^l_{hk} \Omega^h_{ji} + L^l_{hi} \Omega^h_{kj} + L^l_{hj} \Omega^h_{ik} \end{aligned} \tag{2.7}$$

showing that the first Bianchi identity is not satisfied in general.

If the connection ∇^* is semi-symmetric, i. e. if

$$\Omega^i_{jk} = \frac{1}{n-1} (\delta^i_j \Omega_k - \delta^i_k \Omega_j), \tag{2.8}$$

the identity (2.7) reduces to

$$\begin{aligned} L^l_{ijk} + L^l_{jki} + L^l_{kij} &= \frac{1}{n-1} \left[\delta^l_j (\Omega_{i,k} - \Omega_{k,i}) + \delta^l_k (\Omega_{j,i} - \Omega_{i,j}) \right. \\ &\quad \left. + \delta^l_i (\Omega_{k,j} - \Omega_{j,k}) \right], \quad n \neq 1 \end{aligned} \tag{2.9}$$

where $\Omega_j = \Omega^i_{ij}$ is the Vranceanu vector of the connection ∇^* and $\Omega_{i,k} = \frac{\partial \Omega_i}{\partial u^k}$.

From this we obtain the

Theorem 2.1. *If the Vranceanu vector is a gradient then the first Bianchi identity is satisfied.*

Definition. The connection ∇^* is said to be a E -connection if the condition

$$\nabla^*_k \Omega_i - \nabla^*_i \Omega_k = 0 \tag{2.10}$$

holds [5].

Theorem 2.2. *For a generalized Weyl space having a semi-symmetric E -connection the first Bianchi identity is satisfied.*

Proof: The covariant derivative of the Vranceanu vector Ω_i with respect to the coordinates u^k is

$$\nabla^*_k \Omega_i = \frac{\partial \Omega_i}{\partial u^k} - L^h_{ik} \Omega_h. \tag{2.11}$$

Subtracting from (2.11) the equation is obtained by interchanging the indices i and k we find that

$$\nabla_k^* \Omega_i - \nabla_i^* \Omega_k = \frac{\partial \Omega_i}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^i} + 2\Omega_{ki}^h \Omega_h. \quad (2.12)$$

On the other hand, for the generalized Weyl space W_n^* with semi-symmetric E -connection (2.12) is reduced to

$$\frac{\partial \Omega_i}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^i} = 0. \quad (2.13)$$

Using (2.9), from (2.13) we get

$$L_{ijk}^l + L_{jki}^l + L_{kij}^l = 0$$

so the proof is completed. \square

Corollary 2.1. *For a generalized Weyl space having a semi-symmetric connection, the Vranceanu vector and the complementary vector are related by*

$$L_{[ik]} - nT_{[i,k]} = \frac{n-2}{n-1} \Omega_{[i,k]} \quad (n \neq 1)$$

where L_{ik} denotes the Ricci tensor of W_n^* .

Proof: If, in (2.9) a contraction on l and j is made we have

$$L_{ilk}^l + L_{lki}^l + L_{kil}^l = \frac{n-2}{n-1} (\Omega_{i,k} - \Omega_{k,i}) \quad (2.14)$$

from which we get

$$L_{[ik]} - nT_{[i,k]}^* = \frac{n-2}{n-1} \Omega_{[i,k]}, \quad \left(T_{i,k}^* = \frac{\partial T_i^*}{\partial u^k}, \quad \Omega_{i,k} = \frac{\partial \Omega_i}{\partial u^k} \right)$$

where we have used the facts that

$$L_{ilk}^l = L_{ik}, \quad L_{kil}^l = -L_{ki}$$

so the proof is completed. \square

The prolonged covariant derivative of the curvature tensor L_{ijk}^h , of weight $\{0\}$, is

$$\begin{aligned} \dot{\nabla}_l^* L_{ijk}^h &= \nabla_l^* L_{ijk}^h \\ &= \partial_l L_{ijk}^h + L_{ml}^h L_{ijk}^m - L_{il}^m L_{mjk}^h - L_{jl}^m L_{imk}^h - L_{kl}^m L_{ijm}^h. \end{aligned} \quad (2.15)$$

If the indices j , k and l are changed cyclically in (2.15) the equations

$$\dot{\nabla}_j^* L_{ikl}^h = \nabla_j^* L_{ikl}^h = \partial_j L_{ikl}^h + L_{mj}^h L_{ikl}^m - L_{ij}^m L_{mkl}^h - L_{kj}^m L_{iml}^h - L_{lj}^m L_{ikm}^h \quad (2.16)$$

and

$$\dot{\nabla}_k^* L_{ilj}^h = \nabla_k^* L_{ilj}^h = \partial_k L_{ilj}^h + L_{mk}^h L_{ilj}^m - L_{ik}^m L_{mlj}^h - L_{lk}^m L_{imj}^h - L_{jk}^m L_{ilm}^h \quad (2.17)$$

are obtained respectively.

Summing (2.15), (2.16) and (2.17) we get

$$\dot{\nabla}_l^* L_{ijk}^h + \dot{\nabla}_j^* L_{ikl}^h + \dot{\nabla}_k^* L_{ilj}^h = \Omega_{lj}^m L_{imk}^h + \Omega_{jk}^m L_{iml}^h + \Omega_{kl}^m L_{imj}^h. \quad (2.18)$$

This shows that the second Bianchi identity is not valid in W_n^* .

The generalized Weyl space W_n^* is called recurrent if its curvature tensor L_{ijk}^l satisfies the condition

$$\dot{\nabla}_l^* L_{ijk}^h = \psi_l L_{ijk}^h \quad (2.19)$$

where ψ_l is a 1-form called the recurrence vector of W_n^* .

Let W_n^* be a recurrent generalized Weyl space having a semi-symmetric connection. Then, using (2.8) and (2.19), the identity (2.18) is reduced to

$$a_l L_{ijk}^h + a_j L_{ikl}^h + a_k L_{ilj}^h = 0. \quad (2.20)$$

where

$$a_l = \psi_l - \frac{2}{n-1} \Omega_l.$$

Thus we proved the

Theorem 2.3. *For a generalized recurrent Weyl space having a semi-symmetric connection the second Bianchi identity is satisfied provided that the recurrence vector ψ_l and the Vranceanu vector Ω_l are related by $\psi_l = \frac{2}{n-1} \Omega_l$, $\psi_l \neq \Omega_l$.*

References

- [1] Murgescu V., *Espaces de Weyl generalises*, Rev. Roum. Math. Pures et Appl. **XV** No 2, Bucarest (1970) 293–301.
- [2] Çivi G. and Özdeger A., *Theory of Nets in Generalized Weyl Spaces*, Webs and Quasigroups, Tver State Univ. (in print).
- [3] Norden A., *Affinely Connected Spaces* (in Russian), GRMFML, Moscow 1976.
- [4] Norden A. and Yafarov S., *Theory of Non-geodesic Vector Field in Two-dimensional Affinely Connected Spaces* (in Russian), Izv. Vuzov, Matem., No 12 (1974) 29–34.
- [5] Enghiş P., *E-conexiuni semi-simetrice*, Studia Univ. Babeş-Bolyai Math. **29** (1984) 66–69.