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ON KÄHLERIAN COHERENT STATES*

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Abstract. A reformulation of Rawnsley's Kählerian coherent states (in the framework of geometric quantization) is used in order to investigate the interplay between their local and global properties (projective embeddings) and the relationship with Klauder quantization (via path integrals and the introduction of a metric on the classical phase space). A Klauder type formula is established for the projection operator onto the quantum Hilbert space (the kernel of a Bochner Laplacian) in terms of a phase space path integral. As a further application, a Riemann surface diastatic identity is derived, yielding, via Green function theory, a short proof of the Abel-Jacobi theorem (and conversely), together with some coherent state induced theta function identities.

1. Introduction

Coherent states, originally introduced by Schrödinger [20], and generalized in various directions (see e. g. the recent surveys [2] for a thorough introduction to the subject), provide an extremely useful tool for dealing with many aspects of quantum mechanics and play a relevant role in most quantization prescriptions. In this note we further elaborate the notion of coherent state in the Kählerian setting in geometric quantization, due to J. Rawnsley (see e. g. [19, 7, 22]), in view of establishing a relationship with Klauder's quantization. Specifically, Klauder's approach to quantization (via heat equation regularized path integrals over phase space, after introducing a metric thereon, [10, 11] yields the quantum Hilbert space as the (degenerate, in general) Landau ground state space of a "quantum" Hamiltonian (a generalized Laplacian) (cf. also [9, 13] and references therein, see [23] as well) and this is, in turn, close in spirit to the

*This paper is dedicated to the memory of my friend and collaborator Giorgio Valli (1960–1999).

index theoretic approach of [26] in the complex and almost complex case (see [21, 4] and, as a general reference, [3]).

Our results can be summarized as follows: we give several equivalent local conditions ensuring projective embeddability or, what is the same, the constancy of Rawnsley's function (see [19] and [7]) providing corresponding physical interpretations, slightly improving [7]. In particular, we show that positive powers of a regular line bundle are again regular. Furthermore, we establish a Klauder type formula for the projection operator onto the quantum Hilbert space (the kernel of a Bochner type Laplace operator) in terms of path integrals complementing Odzijewicz's treatment [16], the basic technical inspiration coming from explicit computations in Woodhouse's book [27] (in particular Sect. 9.8). Next we apply the above scheme to the Riemann surface case, which is also physically interesting in view of applications to vortices and anyons [18]. Upon relying on Green function theory, as in [18], we prove a geometrically transparent diastatic identity leading to a reformulation of the Abel–Jacobi theorem. Finally, additional coherent state and diastatic identities are interpreted in terms of theta functions, via the Riemann Factorization Theorem.

2. Coherent States

For the basic notions and notations of geometric quantization we refer to [27] (setting Planck's constant $\hbar = 1$) .

Let (M, ω) be a *compact* prequantizable Kähler manifold. Let $(L, \nabla, (\,,\,))$ be a hermitian holomorphic prequantum bundle (unique up to equivalence if M is simply connected), with ∇ the unique connection compatible with the hermitian metric $(\,,\,)$, with curvature $-2\pi\mathrm{i}\omega$, and the holomorphic structure, namely $\nabla^{0,1} = \overline{\partial}$. Denote simply by L^2 the space of L^2-sections of $L \to M$ (with respect to $(\,,\,)$, the integration being carried out with respect to the Liouville measure $\mathrm{d}m$). The *Quantum Hilbert Space*, will be $H = H^0(L)$, i. e. the (finite dimensional) subspace of L^2 consisting of all holomorphic sections of $L \to M$. If we set $\Delta := \nabla^{0,1*}\nabla^{0,1}$, (with $\nabla^{0,1*}$ denoting the formal adjoint of $\nabla^{0,1}$), then it is clear that $H = \ker \Delta$.

H might well be $\{0\}$ a priori, but this will not be the case if $L \to M$ is sufficiently positive; moreover, be the Riemann–Roch theorem, sufficient positivity entails that $\dim H := h^0(L)$ is indeed a topological invariant, and the Kodaira Embedding theorem allows one to see M as an analytic (indeed algebraic, by Chow's theorem) subvariety of the projective space $P(H)$. In view of ensuring a proper semiclassical interpretation, we shall normalize the volume of M in such a way that

$$\mathrm{vol}(M) := \int_M \mathrm{d}m = h^0(L)\,. \tag{2.1}$$

This will yield the value $\lambda = 1$ for the Rawnsley constant (see below). The above equality tells us that the volume of the classical phase space equals the dimension of the quantum Hilbert space and this is one of the forms of Bohr's Correspondence Principle (BCP).

The above set-up can be easily described via Rawnsley's coherent states, which we now recall.

Denote by π the natural projection onto the base M, and by L_m the fibre $\pi^{-1}(m)$ over m. Continuity of the evaluation map $\mathrm{ev}_m : H \to L_m$ given by $\mathrm{ev}_m(s) = s(m)$ yields, by Riesz's theorem $s(m) = \langle e_q, s\rangle \cdot q$, with $e_q \in H$, and $\pi(q) = m$. Clearly $e_{cq} = \bar{c}^{-1} e_q$, for $c \in \mathbb{C}^*$.

We now introduce an a priori different notion of coherent state, which will be soon shown to agree with the one above (cf. [27]).

Definition 2.1. *The* coherent state vectors $s_m \in H$, $m \in M$, *are defined so that they maximize* $(s, s)(m)$ *over all vectors* $s \in H$ *such that* $\langle s, s\rangle = 1$. *They are of course determined up to a phase factor.*

A coherent state vector s_m is thus an eigenvector of the positive operator $A_m : H \to H$ defined by $(s_1, s_2)(m) = \langle s_1, A_m s_2\rangle$. Set $\Lambda(m) = \lambda_m := (s_m, s_m)(m)$, the largest eigenvalue of A_m.

Lemma 2.1. *The coherent states s_m essentially coincide with Rawnsley's coherent states.*

Proof: Let $\pi(q) = m$. From $\langle e_q, s\rangle \cdot q = s(m)$, and $\|s\|^2 = 1$ we get

$$(s, s)(m) \leq |q|^2 \|e_q\|^2$$

with equality if and only if s is proportional to e_q, namely $s_m = ce_q$. If $\|e_q\|^2 = 1$, then $\Lambda(m) = (s_m, s_m)(m) = |q|^2$.

Let $P(H)$ denote the projective space pertaining to H and let $[\;] : H\backslash\{0\} \to P(H)$ is the canonical map. We wish to define a *coherent state map* $\epsilon : M \to P(H)$ by

$$\epsilon(m) = [s_m]\,. \tag{2.2}$$

In case it is well defined, the point $[s_m] \in P(H)$ is called a *coherent state*. Henceforth we shall often drop the distinction between coherent state vectors and coherent states, in case no confusion arises. We now require fulfillment of the following conditions:

A0: Kodaira vanishing theorem, implying, via Riemann–Roch theorem, $\dim H = h^0(L) = \int_M \Lambda(m)\,\mathrm{d}m$ (see [7]).

A1: ϵ is well defined (absence of base points).

A2: ϵ is injective.

An important case in which the **A1** is fulfilled whereas **A2** is not is given by the canonical bundle K on a hyperelliptic Riemann surface, i. e. a 2-1 branched covering of the Riemann sphere, see e. g. [8].

Conditions **A1** and **A2** entail that the eigenspace pertaining to λ_m is one-dimensional and that the real valued function $\Lambda\colon m \mapsto \Lambda(m)$ is strictly positive. We introduce a further condition:

A3: The function Λ is constant ($\Lambda(m) = \lambda = 1$).

The last equality comes from **A0** and BCP. This is indeed equivalent to the *regularity* condition of the line bundle $L \to M$ stated in [7]. One of our main tasks will be the formulation of **A3** in several different guises, leading to a slight improvement of some results in [7]. Kodaira's embedding theorem states that if $L \to M$ is positive, then a suitable power thereof indeed fulfills conditions **A0-3**, since they will be equivalent to projective embeddability (this is referred to as *ampleness* of $L \to M$). Observe that **A3** is a sort of "weak homogeneity" condition which is of course satisfied by coadjoint orbits and more generally by G-hamiltonian symplectic manifolds).

Moreover, the above mentioned BCP in conjunction with Kodaira vanishing theorem lends further motivation for the Λ-constancy requirement. Notice, as a consistency check, that **A0-3** and Riemann–Roch theorem entail

$$1 = \langle s_m, s_m \rangle < \mathrm{vol}(M) = h^0(L)$$

as it should be, in order to have a non trivial projective space.

Now, let s_0 be a local unitary frame ($(s_0, s_0) = 1$), defined in an open neighbourhood U of a fixed point $m \in M$ and set for (the restriction of) a section in H, $s = \tilde{s} s_0$, with $\tilde{s}$ a (local) smooth function. Also set $\nabla s_0 = \vartheta^0 s_0$.

We also observe that, given **A0-3**, in *local* calculations one may adjust the phase in s_m so as to have $\tilde{s}_m(m) = 1$. We shall tacitly assume this in the sequel.

We gather various properties fulfilled by coherent states in the following theorem, resorting to the previously established notation.

Theorem 2.1. *Under assumptions **A0-3** the coherent states possess the following properties.*

i) *The following (strict) inequality holds*

$$(s_{m'}, s_{m'})(m) < (s_m, s_m)(m)\,. \tag{2.3}$$

ii) *The function* $m' \mapsto (s_m, s_m)(m')$ *has a (strict) maximum at* m*:*

$$(s_m, s_m)(m') < (s_m, s_m)(m) = 1\,. \tag{2.4}$$

iii) *(reproducing property)*

$$\langle s_m, s\rangle = (s_m, s)(m). \tag{2.5}$$

iv) *(generalized resolution of the identity)*

$$\langle s_1, s_2\rangle = \int_M \langle s_1, s_m\rangle\langle s_m, s_2\rangle\,\mathrm{d}m\,. \tag{2.6}$$

v) *(overcompleteness)* $\langle s_m, s\rangle = 0$ *for all* m *implies* $s = 0$.

vi) *In a local unitary frame, formula (2.5) reads*

$$\tilde{s}(m) = \tilde{s}_m(m)\langle s_m, s\rangle = \langle s_m, s\rangle\,. \tag{2.7}$$

vii) *The following formula holds, for all* m, m' *in* M*:*

$$(s_{m'}, s_{m'})(m) = (s_m, s_m)(m') = |\langle s_m, s_{m'}\rangle|^2. \tag{2.8}$$

viii) *For fixed* $m \in M$*, and* $\forall m' \in U \setminus \{m\}$ *one has:*

$$|\tilde{s}_m(m')| < |\tilde{s}_m(m)|\,. \tag{2.9}$$

ix) *Let* $\nu \neq \lambda_m$ *be an eigenvalue of* A_m*. Then* $\nu = 0$.

Proof:

i) The very definition of coherent state yields $\leq$ in (2.3). Uniqueness gives $<$.

ii) follows from i) after exchanging the roles of m and m', and by the homogeneity assumption **A3**.

iii) One has

$$\langle s_m, s\rangle = \langle A_m s_m, s\rangle = (s_m, s)(m)\,.$$

iv) The r.h.s. of (2.6) equals $\int_M (s_1, s_m)(m)(s_m, s_2)(m)\,\mathrm{d}m$. But the integrand equals $(s_1, s_2)(m)$ (this is easily verified by using a local unitary frame), so we get, after integration, the l.h.s..

v) follows directly from iv), setting $s = s_1 = s_2$.

vi) and viii) are immediate.

vii) follows at once from vi). This formula gives the transition probability between two coherent states in terms of local data (i. e. the shape of the coherent state wave function).

ix) Let $s \in H$ be an eigenvector pertaining to ν. Then $\langle s, s_m\rangle = 0$ so, by (2.7), $\tilde{s}(m) = 0$ which, in turn, implies $(s, s)(m) = 0 = \langle A_m s, s\rangle$, whence $\nu = 0$.

The following lemma is crucial.

Lemma 2.2.

i) *With the above notations, we have*

$$(\nabla s_m)(m) = 0 \qquad \forall m \in M\,. \tag{2.10}$$

ii) *In a local unitary frame the previous formula reads*

$$\mathrm{d}\tilde{s}_m(m) = -\vartheta^0(m)\tilde{s}_m(m) = -\vartheta^0(m)\,. \tag{2.11}$$

Proof:

i) In view of the hermiticity of ∇, one has, for any $s \in H$,

$$\mathrm{d}(s,s) = (\nabla s, s) + (s, \nabla s).$$

Applying the above formula to s_m and evaluating at m we have

$$0 = (\nabla s_m, s_m)(m) + (s_m, \nabla s_m)(m),$$

i. e. upon using s_m as a local holomorphic frame, setting $\nabla s_m := \vartheta^{(m)} s_m$, with $\vartheta^{(m)}$ a type $(1,0)$ form (since $\nabla^{0,1} s_m = 0$):

$$0 = \vartheta^{(m)}(m) + \overline{\vartheta^{(m)}}(m),$$

whence $\vartheta^{(m)}(m) = 0$.

ii) One has, from i), $0 = (\nabla s_m)(m) = (d\tilde{s}_m(m) + \vartheta^0(m)\tilde{s}_m(m))s_0$, whence (2.11). □

Let $\mathcal{O}(1)$ denote the hyperplane section bundle over $P(H)$, dual to the tautological line bundle (we regard it as a *smooth* line bundle and consider its canonical Fubini-Study connection ∇_{can} with connection form α given by

$$\alpha_v = -\frac{\langle v, \mathrm{d}v\rangle}{\|v\|^2}\,, \tag{2.12}$$

where $H \ni v \neq 0$. The curvature of ∇_{can} equals $-2\pi \mathrm{i}\Omega$.

Proposition 2.1. *(cf. [19]). Let conditions* ***A0-3*** *be fulfilled. Then:*

i) *The mapping $\epsilon : M \to P(H)$ given by*

$$m \mapsto \epsilon(m) := [s_m] \tag{2.13}$$

yields a symplectic embedding, i. e. ϵ and ϵ_* are injective and

$$\epsilon^*(\Omega) = \omega\,. \tag{2.14}$$

ii) *One has $\epsilon^*(\mathcal{O}(1), \nabla_{can}) = (L, \nabla)$.*

Proof: We already know that the coherent state map ϵ is injective. If we prove (2.14), then ϵ_* will be automatically injective (since ω is non degenerate).
We shall now prove that, locally around m,

$$\epsilon^*(\alpha) = \vartheta^0, \tag{2.15}$$

which, in turn, implies (2.14).
Choose a point $m \in M$. Consider an orthonormal basis $(s_1 = s_m, s_2, \ldots, s_N)$ of A_m $(N = \dim H)$. Express any coherent state $s_{m'}, m' \neq m$ as

$$s_{m'} = \sum_{i=1}^{N} \langle s_i, s_{m'} \rangle s_i \,.$$

Then, working in a unitary frame as above, we find

$$\epsilon^*(\alpha)(m') = -\langle s_{m'},\, \mathrm{d}s_{m'} \rangle = -\sum_{i=1}^{N} \overline{\tilde{s}_i(m')}\, \mathrm{d}\tilde{s}_i(m')\,.$$

Now, if we let $m' \to m$, we have, by observing that $\tilde{s}_i(m) = 0$ for $i \neq 1$ (by Theorem 2.1.).

$$\epsilon^*(\alpha)(m) = -(\tilde{s}_m(m))^{-1}(\,\mathrm{d}\tilde{s}_m)(m) = \vartheta^0(m)$$

by Lemma 2.2. □
Let $D_m(m') := D(m, m')$ denote the *Calabi diastasis function* (see e. g. [6], [7], [24]). Recall that it can be manifactured from any Kähler potential f, via the (local) formula

$$D(m, m') = f(m, m) + f(m', m') - f(m', m) - f(m, m')$$

(a sesquiholomorphic extension of f being understood), and that, fixing, say, m, it gives a canonical Kähler potential D_m as a function of m', which we henceforth call *diastatic*. The following property of the diastasis function is important.

Lemma 2.3. (Calabi, Bochner, see [6]) *Locally around m, there exists a holomorphic coordinate system centred at m such that the diastasis takes the form (obvious notation):*

$$D(m, m') = \sum_{i=1}^{n} |z_i(m')|^2 + \text{ higher order terms in } \; z, \overline{z}$$

We are in a position to state one of the main results of this paper:

Theorem 2.2. *Under conditions **A0-2**, the following are equivalent:*

i) ***A3** holds, i. e. the function Λ is constant $(= \lambda = 1)$.*

ii) *The coherent state map is an embedding.*
iii)

$$d(s_m, s_m)|_{m'=m} = 0\,, \quad \forall m \in M\,. \tag{2.16}$$

iv)

$$\log(s_m, s_m)(m') = -D_m(m') \tag{2.17}$$

for all m' in a suitable m-neighbourhood $\mathcal{U} \subset M$.
v) *A* Reciprocity Law *holds in the form:*

$$(s_m, s_m)(m') = (s_{m'}, s_{m'})(m) \tag{2.18}$$

for all m' in a suitable m-neighbourhood $\mathcal{U} \subset M$.

Proof: Our goal will be accomplished once we prove the following:
1. i) ⇒ ii)
2. i) ⇔ iii)
3. ii) ⇒ iv).
4. iv) ⇒ v) ⇒ i).

1. The implication ⇒ follows from Proposition 2.1.
2. One direction comes from Theorem 2.1, the other follows from the very definition of coherent states and elementary differential calculus.
3. Straightforward computation in projective space.
4. Symmetry of the diastasis function gives the first assertion, whereas the second one is immediate. □

Remark: The reciprocity law has a natural physical meaning: it states the the probability density of finding the system described by the coherent state s_m at any point m' is equal the the one obtained by exchanging the roles of m and m'. In particular, we observe that two coherent states s_m and $s_{m'}$ are orthogonal if and only if, say, $(s_m, s_m)(m') = 0$. Also, observe that the coherent state wave function s_m has a Gaussian-like shape (modelled by the diastasis function, in view of the Bochner–Calabi lemma) peaked around m, in accordance with physical intuition:

$$(s_m, s_m)(m') = \mathrm{e}^{-D(m,m')} = (s_{m'}, s_{m'})(m)\,. \tag{2.19}$$

Notice that the implication iv) ⇒ iii) is also a direct consequence of Lemma 2.3. We draw some consequences of the above theorem.

Corollary 2.1. *The function* $\exp(-D(m, m'))$ *is globally defined and* $D(m, m') = 0$ *if and only if* $m \equiv m'$*. (Propositions 2 and 3 in [7])*

Corollary 2.2. *If* $L \to M$ *is a regular line bundle (i. e.* Λ *is constant), then* $L^k \to M$ *is also regular,* $k \geq 1$.

Proof: One has $L = \epsilon^* \mathcal{O}(1)$, so $L^k = \epsilon^* \mathcal{O}(k)$. From this it easily follows that the coherent state $s_m^{(k)}$ pertaining to L^k equals s_m^k, whence

$$\mathrm{d}(s_m^{(k)}, s_m^{(k)})|_{m'=m} = \mathrm{d}(s_m, s_m)^k|_{m'=m} = k(s_m, s_m)^{k-1}\,\mathrm{d}(s_m, s_m)|_{m'=m} = 0$$

for any $m \in M$. So Theorem 2.2 yields the desired result. □

Let us also notice the following consequence of the above formalism in the framework of the Borel–Weil theory in its reformulation given in [17]). We consider a highest weight (projective) unitary representation $g \to U(g)$ of a compact simple Lie group G.

Corollary 2.3. *The Onofri's* [17] *potential* $-2 \log |\langle 0|U(g)0\rangle|$ *(abuse of notation) is diastatic.*

We also recall, for the sake of completeness and illustration of the above formalism, that under assumptions **A0-3**, the semiclassical evolution property and the minimal uncertainty properties of Rawnsley's coherent states can be established (see e. g. [21]). We recover the former for the sake of completeness and as an easy application of the above formalism. Given an observable (Hamiltonian) f with (complete) vector field $f^\sharp$ and consider its prequantum operator F given by

$$Fs = -\mathrm{i}\nabla_{f^\sharp} s + fs\,. \tag{2.20}$$

The *quantum* operator associated to f is, by definition $Q(f) := PFP \colon H \to H$. The observable f is said to be quantizable if $Q(f) = FP$. Then we have

Proposition 2.2. *Let* f *be quantizable. The following semiclassical evolution property holds*

$$\langle s_m, Q(f)s_m\rangle = f(m)\,. \tag{2.21}$$

Proof: We have: $\langle s_m, Q(f)s_m\rangle = \langle s_m, Fs_m\rangle$ (this is always true) $= (s_m, Fs_m)(m)$ (since $Fs_m \in H$, being f quantizable) $= f(m)$ (in view of (2.20) and Lemma 2.2). □

3. A Path Integral Formula

In this section we wish to give a Klauder type formula for the projection operator $P : L^2 \to H$ in terms of a path integral over M, or rather, for the matrix elements of P over coherent states (reproducing kernel) (cf. [10] and [11]). The computation below mimicks the one in [27].

Starting from the resolution of the identity (2.6), with $s_i = s_{m_i}$, $i = 1, 2$ one easily finds, upon iteration, for $n \geq 1$

$$\langle s_1, s_2 \rangle = \int_{M^n} \langle s_1, s_{\mu_1} \rangle \langle s_{\mu_1}, s_{\mu_2} \rangle \ldots \langle s_{\mu_n}, s_2 \rangle \, \mathrm{d}\mu_1 \ldots \mathrm{d}\mu_n \, . \tag{3.1}$$

Working in a unitary frame, we have, in view of (2.5),

$$\langle s_{\mu_i}, s_{\mu_{i+1}} \rangle = \bar{\bar{s}}_{\mu_i}(\mu_{i+1}) \, . \tag{3.2}$$

Let $\mathcal{P}(m_1, m_2)$ denote the space of (continuous) paths γ in M connecting m_1 with m_2. From now on (up to the end of this section) we work heuristically, though our reasoning could be made rigorous, by resorting to the Feynman-Kac formula for the heat kernel of the Laplace operator Δ. If we let $n \to \infty$ taking formula (2.12) into due account, and introducing the path integral "measure" $\mathcal{D}(\gamma)$, we are led to the following result:

Theorem 3.1. *Under the above assumptions, the orthoprojector $P : L^2 \to H$ has a "reproducing kernel" $\Pi(m_1, m_2)$ defined via the phase space path integral*

$$\Pi(m_1, m_2) := \langle s_{m_1}, s_{m_2} \rangle = \int_{\mathcal{P}(m_1, m_2)} T_\nabla(\gamma) \mathcal{D}(\gamma) \tag{3.3}$$

where $T_\nabla(\gamma)$ is the parallel transport (holonomy) operator attached to ∇: in a local unitary frame it reads

$$T_\nabla(\gamma) = \mathrm{e}^{\int_\gamma \vartheta^0} \, . \tag{3.4}$$

Remarks: (i) This construction also matches with the coherent state approach developed by Odzijewicz [16], and further confirms that it is mandatory to act in the projective embedding case (as was postulated from the beginning in [16]). It also yields an example of *topological dynamics* (the Hamiltonian being set equal to zero), with the integrand appearing as the holonomy of the prequantum connection (cf. [1]).

(ii) We also notice that in the path integrand only the *topological* information (encoded in the connection, via the Weil-Kostant theorem) survives, as it should be, in the Kähler case, under Kodaira vanishing assumptions.

4. Applications to Riemann Surfaces

In this section we wish to establish a link with the theory of Green functions on a Riemann surface (Σ, h) of genus g, where h is a Kähler metric with (see e. g. [12] and [18] as well) Kähler form $\kappa = \frac{\mathrm{i}}{2} h_{\overline{z}z}\, \mathrm{d}z \wedge \mathrm{d}\overline{z}$. In this section, the diastasis will be denoted by $\mathcal{D}$. The following proposition is easily established,

Proposition 4.1. *On the Riemann surface Σ the Green function G_P associated to a divisor P (a point in Σ) and to a normalized volume form, e. g. the Kähler structure κ, reads:*

$$G_P(Q) = \log |f(Q)|^2 - \mathcal{D}_P(Q) \tag{4.1}$$

for all Q in a suitable open dense neighbourhood $\mathcal{U} \subset \Sigma$ (with $f(Q)$ being a local meromorphic function having a (simple) zero at P (it plays the role of a Bochner coordinate around P).

Proof: The proof follows from the very definition of Green and Calabi functions, using the fact that the diastasis is, fixing an argument, a Kähler potential (see [12, 18]).

Warning: We are requiring that the smooth part of the Green function vanishes at P, and moreover a priori, depending on conventions, there is just proportionality between $\mathcal{D}$ and the Green function; the latter can be redefined so as to get the above formula). □

Let us prove the following:

Theorem 4.1. *Let $D = \sum_i k_i P_i$ be a degree zero divisor on Σ. Then D is the divisor of a (global) meromorphic function f if and only if the following diastatic identity holds:*

$$\sum_i k_i \mathcal{D}(P_i, Q) = \mathrm{const} \quad \forall Q \in \Sigma\,. \tag{4.2}$$

Proof: Recall that a Green function exists for any divisor (cf. e. g. the proof by Coleman given in [12], which is independent of Abel–Jacobi, see below). Another way of formulating the above statement is the following: the degree zero divisor D pertains to a meromorphic function if and only if the smooth parts of the Green functions associated with the points in the divisor add up to a *global* harmonic function (hence a constant).

From the above result we get

Theorem 4.2. (Abel–Jacobi) *There exists a meromorphic function associated to D if and only if $A(D) = 0$, with A the Abel–Jacobi map.*

Proof: We essentially employ notations of [8]. In view of the above theorem and of the naturality of Calabi's function one can compute on the Jacobian $J(\Sigma) = \mathbb{C}^g/\Lambda$ where the diastasis is, locally, the one on $\mathbb{C}^g$. Given a basis for the abelian differentials $\omega = (\omega_i)$, $i = 1, \dots g$, the Abel–Jacobi map $A\colon \Sigma \to J(\Sigma)$ reads (after choosing a reference point P_0 on Σ):

$$A(P) = \mathbf{z} = \int_{P_0}^{P} \omega\,. \tag{4.3}$$

Now, if $D = \sum_\lambda P_\lambda - Q_\lambda$ (for simplicity, as in [8]) we also set $\mathbf{z}_\lambda^+ = A(P_\lambda)$, $\mathbf{z}_\lambda^- = A(Q_\lambda)$, $\mathbf{z}_\lambda = A(P_\lambda) - A(Q_\lambda) = \mathbf{z}_\lambda^+ - \mathbf{z}_\lambda^-$. Then the image of D through the Abel–Jacobi map reads:

$$A(D) = \sum_\lambda \mathbf{z}_\lambda^+ - \mathbf{z}_\lambda^- = \sum_\lambda \mathbf{z}_\lambda \tag{4.4}$$

Eventually, we write

$$\sum_\lambda \mathcal{D}(P_\lambda, P) - \sum_\lambda \mathcal{D}(Q_\lambda, P) = \sum_\lambda \|\mathbf{z} - \mathbf{z}_\lambda^+\|^2 - \sum_\lambda \|\mathbf{z} - \mathbf{z}_\lambda^-\|^2$$

which is locally constant if and only if $\sum_\lambda \mathbf{z}_\lambda = 0$. Globalizing yields

$$\sum_\lambda \mathbf{z}_\lambda \equiv 0 \mod (\Lambda)\,. \tag{4.5}$$

which is a form of the Abel–Jacobi theorem (cf. [8]). The constant has clearly the value

$$\sum_i k_i \|\mathbf{z}_i\|^2. \tag{4.6}$$

□

We notice that, conversely, one can directly prove the diastatic identity on the Jacobian through the Abel–Jacobi theorem, then pulling back to Σ via the Abel–Jacobi map yields the diastatic identity thereon.

We recall the *Riemann factorization theorem*, which gives an explicit formula for a meromorphic function on a Riemann surface pertaining to a degree zero divisor fulfilling $A(D) = 0$ (see [15] and [18]):

$$f(x) = c\,\mathrm{e}^{\int_{P_0}^{x}\omega} \prod_{k=1}^{n} \frac{\vartheta(A(x) - A(P_k) - \zeta)}{\vartheta(A(x) - A(Q_k) - \zeta)} \tag{4.7}$$

(valid for a generic ζ in the *theta divisor* of $J(\Sigma)$) and with $x \in \Sigma$ and $c \in \mathbb{C}$).

Here $A(x)$, etc. is defined as above via integration from a fixed point P_0 on the surface, or, equivalenty, on takes, e. g. $(A(x) - A(P_k) = A(x - P_k)$, etc.

The role of ζ is the following: the function $x \mapsto \vartheta(A(x) - A(P) - \zeta)$ has a zero in P and in other $g - 1$ points depending only on ζ but not on P. In the final formula these additional points drop out. Finally, ω is a suitable abelian differential (holomorphic 1-form).

Let us now assume that $L \to \Sigma$ is such that $\deg L > 2g$ so that it is very ample, and moreover, that it corresponds to the (effective) divisor D_0. In view of the classical theory, holomorphic sections of $L \to \Sigma$ correspond to meromorphic functions having $(f) = D - D_0$, the correspondence being induced by

$$s = f_s \cdot s_0$$

with s_0 a reference holomorphic section with divisor D_0. Setting $D = \sum_i k_i m_i$, with $\sum_i k_i = 0$ (we take $D = D' - D_0$, with $\deg D' = \deg D_0$)

Denoting by f_m the meromorphic function pertaining to s_m etc., applying (4.2) together with the Riemann factorization theorem we get

$$\prod_i (s_m, s_m)^{k_i}(m_i) = \prod_i (s_{m_i}, s_{m_i})^{k_i}(m) = \text{const} \tag{4.8}$$

entailing (recall that $\sum_i k_i = 0$)

$$\prod_i |f_{m_i}|^{2k_i}(m) = \text{const} \tag{4.9}$$

and, discarding the modulus (squared), by holomorphy

$$\prod_i f_{m_i}^{k_i}(m) = \text{const} \tag{4.10}$$

which is easily seen to be a (non trivial) theta identity by virtue of the Riemann factorization theorem.

We also notice that the above also holds for $L = K$ ($\deg K = 2g - 2 < 2g$) provided Σ is non-hyperelliptic, which is certainly true, generically, for $g \geq 3$.

We give an application of the above formalism in the framework of elliptic function theory on the appropriate line bundle pertaining to the divisor $3p_0$, with p_0 denoting the flex $(0 : 0 : 1)$ of the Weierstrass cubic (see [8] for details).

Theorem 4.3. *The Theta function identity above takes the following form*

$$\prod_{i,j} \vartheta_{11}(z - a^i_j)^{k_i h_j} = \text{const} \tag{4.11}$$

where, for fixed i, (a^i_j) denotes the collection of zeros and poles of f_{m_i}, with multiplicity h_j (negative for poles) and ϑ_{11} is an appropriate theta function with characteristics, (see [14]).

Notice that this entails that *the zeros and poles of the coherent state vector pertaining to any point building up the divisor appear among the zeros and poles (of some) of the others.*

This fact could have interesting physical consequences in vortex and anyon theory (see [9], [13] and [18]). Furthermore, it would be interesting to investigate in depth the possible relationship between "classical" theta identities and these "coherent state" induced ones. This problem will be possibly tackled elsewhere.

We finally notice, in the Riemann surface context, and for regular L, the following

Proposition 4.2. *The number of coherent states orthogonal to a given one, say s_m, is $n = \mathrm{vol}(\Sigma) + g - 1$*

Proof: By virtue of Theorem 2.1 this number is given by the zeros of the holomorphic section s_m, namely $c_1(L)$, which, in turn, can be obtained from the Riemann–Roch formula. □

Remark: In conclusion, we wish to point out that D. Borthwick and A. Uribe recently succeded in extending the Kodaira Embedding Theorem to almost complex manifolds via Rawnsley's coherent states, which retain their meaning in this more general context, (essentially by elliptic regularity, [5], [4], see also [25]).

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