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AN INTRINSIC CHARACTERIZATION OF p-SYMMETRIC HEEGAARD SPLITTINGS

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ABSTRACT. We show that every p-fold strictly-cyclic branched covering of a b-bridge link in $\mathbf{S}^3$ admits a p-symmetric Heegaard splitting of genus $g = (b-1)(p-1)$. This gives a complete converse to a result of Birman and Hilden, and gives an intrinsic characterization of p-symmetric Heegaard splittings as p-fold strictly-cyclic branched coverings of links.

1. Introduction

The concept of p-symmetric Heegard splittings has been introduced by Birman and Hilden (see [2]) in an extrinsic way, depending on a particular embedding of the handlebodies of the splitting in the ambient space $\mathbf{E}^3$. The definition of such particular splittings was motivated by the aim to prove that every closed, orientable 3-manifold of Heegaard genus $g \leq 2$ is a 2-fold covering of $\mathbf{S}^3$ branched over a link of bridge number $g+1$ and that, conversely, the 2-fold covering of $\mathbf{S}^3$ branched over a link of bridge number $b \leq 3$ is a closed, orientable 3-manifold of Heegaard genus $b-1$ (compare also [7]).

A genus g Heegaard splitting $M = Y_g \cup_\phi Y'_g$ is called *p-symmetric*, with $p > 1$, if there exist a disjoint embedding of Y_g and Y'_g into $\mathbf{E}^3$ such that $Y'_g = \tau(Y_g)$, for a translation τ of $\mathbf{E}^3$, and an orientation-preserving homeomorphism $\rho : \mathbf{E}^3 \to \mathbf{E}^3$ of period p, such that $\rho(Y_g) = Y_g$ and, if $\mathcal{G}$ denotes the cyclic group of order p generated by ρ and $\Phi : \partial Y_g \to \partial Y_g$ is the orientation-preserving homeomorphism $\Phi = \tau^{-1}_{|\partial Y'_g}\phi$, the following conditions are fulfilled:

i) $Y_g/\mathcal{G}$ is homeomorphic to a 3-ball;

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ii) $\mathrm{Fix}(\rho^h_{|Y_g}) = \mathrm{Fix}(\rho_{|Y_g})$, for each $1 \leq h \leq p-1$;

iii) $\mathrm{Fix}(\rho_{|Y_g})/\mathcal{G}$ is an unknotted set of arcs[1] in the ball $Y_g/\mathcal{G}$;

iv) there exists an integer p_0 such that $\Phi\rho_{|\partial Y_g}\Phi^{-1} = (\rho_{|\partial Y_g})^{p_0}$.

Remark 1. *By the positive solution of the Smith Conjecture* [4] *it is easy to see that necessarily* $p_0 \equiv \pm 1$ *mod* p.

The map $\rho' = \tau\rho\tau^{-1}$ is obviously an orientation-preserving homeomorphism of period p of $\mathbf{E}^3$ with the same properties as ρ, with respect to Y'_g, and the relation $\phi\rho_{|\partial Y_g}\phi^{-1} = (\rho'_{|\partial Y'_g})^{p_0}$ easily holds.

The *p-symmetric Heegaard genus* $g_p(M)$ of a 3-manifold M is the smallest integer g such that M admits a p-symmetric Heegaard splitting of genus g.

The following results have been established in [2]:

(1) Every closed, orientable 3-manifold of p-symmetric Heegaard genus g admits a representation as a p-fold cyclic covering of $\mathbf{S}^3$, branched over a link which admits a b-bridge presentation, where $g = (b-1)(p-1)$.

(2) The p-fold cyclic covering of $\mathbf{S}^3$ branched over a knot of braid number b is a closed, orientable 3-manifold M which admits a p-symmetric Heegaard splitting of genus $g = (b-1)(p-1)$.

Note that statement 2 is not a complete converse of 1, since it only concerns knots and, moreover, b denotes the braid number, which is greater than or equal to (often greater than) the bridge number. In this paper we fill this gap, giving a complete converse to statement 1. Since the coverings involved in 1 are strictly-cyclic (see next section for details on strictly-cyclic branched coverings of links), our statement will concern this kind of coverings. More precisely, we shall prove in Theorem 4 that a p-fold strictly-cyclic covering of $\mathbf{S}^3$, branched over a link of bridge number b, is a closed, orientable 3-manifold M which admits a p-symmetric Heegaard splitting of genus $g = (b-1)(p-1)$, and therefore has p-symmetric Heegaard genus $g_p(M) \leq (b-1)(p-1)$. This result gives an intrinsic interpretation of p-symmetric Heegaard splittings as p-fold strictly-cyclic branched coverings of links.

2. Main results

Let

$$\beta = \{(p_k(t), t) \mid 1 \leq k \leq 2n\,,\ t \in [0,1]\} \subset \mathbf{E}^2 \times [0,1]$$

be a geometric $2n$-string braid of $\mathbf{E}^3$ [1], where $p_1, \ldots, p_{2n} : [0,1] \to \mathbf{E}^2$ are continuous maps such that $p_k(t) \neq p_{k'}(t)$, for every $k \neq k'$ and $t \in [0,1]$,

[1] A set of mutually disjoint arcs $\{t_1, \ldots, t_n\}$ properly embedded in a handlebody Y is *unknotted* if there is a set of mutually disjoint discs $D = \{D_1, \ldots, D_n\}$ properly embedded in Y such that $t_i \cap D_i = t_i \cap \partial D_i = t_i$, $t_i \cap D_j = \emptyset$ and $\partial D_i - t_i \subset \partial Y$ for $1 \leq i, j \leq n$ and $i \neq j$.

and such that $\{p_1(0), \ldots, p_{2n}(0)\} = \{p_1(1), \ldots, p_{2n}(1)\}$. We set $P_k = p_k(0)$, for each $k = 1, \ldots, 2n$, and

$$A_i = (P_{2i-1}, 0), B_i = (P_{2i}, 0), A'_i = (P_{2i-1}, 1), B'_i = (P_{2i}, 1),$$

for each $i = 1, \ldots, n$ (see Figure 1). Moreover, we set $\mathcal{F} = \{P_1, \ldots, P_{2n}\}$, $\mathcal{F}_1 = \{P_1, P_3 \ldots, P_{2n-1}\}$ and $\mathcal{F}_2 = \{P_2, P_4, \ldots, P_{2n}\}$.

The braid β is realized through an ambient isotopy

$$\widehat{\beta} : \mathbf{E}^2 \times [0,1] \to \mathbf{E}^2 \times [0,1], \ \widehat{\beta}(x,t) = (\beta_t(x), t),$$

where β_t is an homeomorphism of $\mathbf{E}^2$ such that $\beta_0 = \mathrm{Id}_{\mathbf{E}^2}$ and $\beta_t(P_i) = p_i(t)$, for every $t \in [0,1]$. Therefore, the braid β naturally defines an orientation-preserving homeomorphism $\widetilde{\beta} = \beta_1 : \mathbf{E}^2 \to \mathbf{E}^2$, which fixes the set $\mathcal{F}$. Note that β uniquely defines $\widetilde{\beta}$, up to isotopy of $\mathbf{E}^2$ mod $\mathcal{F}$.

Connecting the point A_i with B_i by a circular arc α_i (called *top arc*) and the point A'_i with B'_i by a circular arc α'_i (called *bottom arc*), as in Figure 1, for each $i = 1, \ldots, n$, we obtain a $2n$-plat presentation of a link L in $\mathbf{E}^3$, or equivalently in $\mathbf{S}^3$. As is well known, every link admits plat presentations and, moreover, a $2n$-plat presentation corresponds to an n-bridge presentation of the link. So, the bridge number $b(L)$ of a link L is the smallest positive integer n such that L admits a representation by a $2n$-plat. For further details on braid, plat and bridge presentations of links we refer to [1].

FIGURE 1. A $2n$-plat presentation of a link.

Remark 2. *A $2n$-plat presentation of a link $L \subset \mathbf{E}^3 \subset \mathbf{S}^3 = \mathbf{E}^3 \cup \{\infty\}$ furnishes a $(0,n)$-decomposition* [5] $(\mathbf{S}^3, L) = (D, A_n) \cup_{\phi'} (D', A'_n)$ *of the link, where D and D' are the 3-balls*

$$D = (\mathbf{E}^2 \times]-\infty, 0]) \cup \{\infty\} \text{ and } D' = (\mathbf{E}^2 \times [1, +\infty[) \cup \{\infty\},$$

$$A_n = \alpha_1 \cup \cdots \cup \alpha_n, \ A'_n = \alpha'_1 \cup \cdots \cup \alpha'_n$$

and $\phi' : \partial D \to \partial D'$ is defined by $\phi'(\infty) = \infty$ and $\phi'(x,0) = (\widetilde{\beta}(x), 1)$, for each $x \in \mathbf{E}^2$.

If a $2n$-plat presentation of a μ-component link $L = \bigcup_{j=1}^{\mu} L_j$ is given, each component L_j of L contains n_j top arcs and n_j bottom arcs. Obviously, $\sum_{j=1}^{\mu} n_j = n$. A $2n$-plat presentation of a link L will be called *special* if:

(1) the top arcs and the bottom arcs belonging to L_1 are $\alpha_1, \ldots, \alpha_{n_1}$ and $\alpha'_1, \ldots, \alpha'_{n_1}$ respectively, the top arcs and the bottom arcs belonging to L_2 are $\alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2}$ and $\alpha'_{n_1+1}, \ldots, \alpha'_{n_1+n_2}$ respectively, ..., the top arcs and the bottom arcs belonging on L_μ are

$$\alpha_{n_1+\cdots+n_{\mu-1}+1}, \ldots, \alpha_{n_1+\cdots+n_\mu} = \alpha_n$$

and

$$\alpha'_{n_1+\cdots+n_{\mu-1}+1}, \ldots, \alpha'_{n_1+\cdots+n_\mu} = \alpha'_n$$

respectively;

(2) $p_{2i-1}(1) \in \mathcal{F}_1$ and $p_{2i}(1) \in \mathcal{F}_2$, for each $i = 1, \ldots, n$.

It is clear that, because of (2), the homeomorphism $\widetilde{\beta}$, associated to a $2n$-string braid β defining a special plat presentation, keeps fixed both the sets $\mathcal{F}_1$ and $\mathcal{F}_2$. Although a special plat presentation of a link is a very particular case, we shall prove that every link admits such kind of presentation.

Proposition 3. *Every link L admits a special $2n$-plat presentation, for each $n \geq b(L)$.*

Proof. Let L be presented by a $2n$-plat. We show that this presentation is equivalent to a special one, by using a finite sequence of moves on the plat presentation which changes neither the link type nor the number of plats. The moves are of the four types I, I', II and II' depicted in Figure 2. First of all, it is straightforward that condition (1) can be satisfied by applying a suitable sequence of moves of type I and I'. Furthermore, condition (2) is equivalent to the following: (2′) there exists an orientation of L such that, for each $i = 1, \ldots, n$, the top arc α_i is oriented from A_i to B_i and the bottom arc α'_i is oriented from B'_i to A'_i. Therefore, choose any orientation on L and apply moves of type II (resp. moves of type II') to the top arcs (resp. bottom arcs) which are oriented from B_i to A_i (resp. from A'_i to B'_i). □

Figure 2. Moves on plat presentations.

A p-fold branched cyclic covering of an oriented μ-component link $L = \bigcup_{j=1}^{\mu} L_j \subset \mathbf{S}^3$ is completely determined (up to equivalence) by assigning to each component L_j an integer $c_j \in \mathbf{Z}_p - \{0\}$, such that the set $\{c_1, \ldots, c_\mu\}$ generates the group $\mathbf{Z}_p$. The monodromy associated to the covering sends each meridian of L_j, coherently oriented with the chosen orientations of L and $\mathbf{S}^3$, to the permutation $(1\,2\,\cdots\,p)^{c_j} \in \Sigma_p$. Multiplying each c_j by the same invertible element of $\mathbf{Z}_p$, we obtain an equivalent covering.

Following [3] we shall call a branched cyclic covering:

a) *strictly-cyclic* if $c_{j'} = c_{j''}$, for every $j', j'' \in \{1, \ldots, \mu\}$,
b) *almost-strictly-cyclic* if $c_{j'} = \pm c_{j''}$, for every $j', j'' \in \{1, \ldots, \mu\}$,
c) *meridian-cyclic* if $\gcd(b, c_j) = 1$, for every $j \in \{1, \ldots, \mu\}$,
d) *singly-cyclic* if $\gcd(b, c_j) = 1$, for some $j \in \{1, \ldots, \mu\}$,
e) *monodromy-cyclic* if it is cyclic.

The following implications are straightforward:

$$\text{a)} \Rightarrow \text{ b)} \Rightarrow \text{ c)} \Rightarrow \text{ d)} \Rightarrow \text{ e)}.$$

Moreover, the five definitions are equivalent when L is a knot. Similar definitions and properties also hold for a p-fold cyclic covering of a 3-ball, branched over a set of properly embedded (oriented) arcs.

It is easy to see that, by a suitable reorientation of the link, an almost-strictly-cyclic covering becomes a strictly-cyclic one. As a consequence, it follows from Remark 1 that every branched cyclic covering of a link arising from a p-symmetric Heegaard splitting – according to Birman-Hilden construction – is strictly-cyclic.

Now we show that, conversely, every p-fold branched strictly-cyclic covering of a link admits a p-symmetric Heegaard splitting.

Theorem 4. *A p-fold strictly-cyclic covering of $\mathbf{S}^3$ branched over a link L of bridge number b is a closed, orientable 3-manifold M which admits a p-symmetric Heegaard splitting of genus $g = (b-1)(p-1)$. So the p-symmetric Heegaard genus of M is*

$$g_p(M) \leq (b-1)(p-1).$$

Proof. Let L be presented by a special $2b$-plat arising from a braid β, and let $(\mathbf{S}^3, L) = (D, A_b) \cup_{\phi'} (D', A'_b)$ be the $(0, b)$-decomposition described in Remark 2. Now, all arguments of the proofs of Theorem 3 of [2] entirely apply and the condition of Lemma 4 of [2] is satisfied, since the homeomorphism $\widetilde{\beta}$ associated to β fixes both the sets $\mathcal{F}_1$ and $\mathcal{F}_2$. □

As a consequence of Theorem 4 and Birman-Hilden results, there is a natural one-to-one correspondence between p-symmetric Heegaard splittings and p-fold strictly-cyclic branched coverings of links.

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