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USING NETS IN DEDEKIND, MONOTONE, OR SCOTT INCOMPLETE ORDERED FIELDS AND DEFINABILITY ISSUES

MOJTABA MONIRI AND JAFAR S. EIVAZLOO

Abstract. Given a Dedekind incomplete ordered field, a pair of convergent nets of gaps which are respectively increasing or decreasing to the same point is used to obtain a further equivalent criterion for Dedekind completeness of ordered fields: Every continuous one-to-one function defined on a closed bounded interval maps interior of that interval to the interior of the image. Next, it is shown that over all closed bounded intervals in any monotone incomplete ordered field, there are continuous not uniformly continuous unbounded functions whose ranges are not closed, and continuous 1-1 functions which map every interior point to an interior point (of the image) but are not open. These are achieved using appropriate nets cofinal in gaps or coinitial in their complements. In our third main theorem, an ordered field is constructed which has parametrically definable regular gaps but no $\emptyset$-definable divergent Cauchy functions (while we show that, in either of the two cases where parameters are or are not allowed, any definable divergent Cauchy function gives rise to a definable regular gap). Our proof for the mentioned independence result uses existence of infinite primes in the subring of the ordered field of generalized power series with rational exponents and real coefficients consisting of series with no infinitesimal terms, as recently established by D. Pitteloud.

1. A Dedekind Incompleteness Feature via Convergent Nets of Gaps

A cut of an ordered field F is a subset which is downward closed in F. By a nontrivial cut, we mean a nonempty proper cut. A nontrivial cut is a gap if it does not have a least upper bound in the field. An ordered field is Archimedean (has no infinitesimals) just in case it can be embedded in $\mathbb{R}$.

The following fact presents some of the well known characterizations of the ordered field of real numbers. A more delicate equivalent condition is presented in Theorem 1.2.

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Fact 1.1. *The real ordered field $\mathbb{R}$ is, up to isomorphism, the unique ordered field which satisfies either of the following equivalent conditions:*

(i) *Dedekind Completeness, i.e. not having any gaps,*
(ii) *connectedness,*
(iii) *not being totally disconnected,*
(iv) *every (nonempty) convex subset being an interval (of one of the usual kinds),*
(v) *all convex subsets being connected,*
(vi) *all intervals being connected,*
(vii) *all continuous functions on the field mapping any convex subset onto a convex subset,*
(viii) *all continuous functions on the field satisfying the intermediate value property.*

These are fairly well known, let us give the argument for (iii). Pick a Dedekind incomplete ordered field F. In the Archimedean case, F is a proper subfield of $\mathbb{R}$ which therefore misses some points in any real interval. In the non-Archimedean case, and given any two points $a < b$ of a subset A of F, we have the nontrivial clopen subset of those points x in A such that $\frac{x-a}{b-a}$ is a $\mathbb{Q}$-infinitesimal. Alternatively, one can use linear increasing functions between intervals to map a given gap (somewhere in the field) into a given interval.

Notice that if F is an ordered field and $P(x) \in F[x]$, then $P(x)$ is continuous on F. To see this, it suffices to show that if $a \in F$ and $P(a) < 0$, then P is negative on a neighborhood of a in F. As the same inequality also holds in $RC(F)$, where polynomials have factorizations into linear and irreducible quadratics, there are $b, c \in RC(F)$ with $b < a < c$ such that $\forall x \in (b, c)_{RC(F)}$, $P(x) < 0$. Using cofinality of F in $RC(F)$, one can now see that there are $d, e \in F$ such that $b < d < a < e < c$ (observe that there exists $g \in F$ such that $0 < g < a - b$ and put $d = a - g$, similarly for e). A very mild use of this takes place in the claim within the next theorem.

Theorem 1.2. *An ordered field is Dedekind complete if and only if all continuous 1-1 functions defined on some (equivalently all) non-degenerated closed bounded interval(s) of it map interior points [of the interval(s)] to interior points [of their range(s)].*

Proof. We only need to prove the *if* part.

Claim 1.2.1. *Given two convex subsets A and B of an ordered field F, with A bounded and B not a singleton, there exists a nonconstant linear (and so 1-1 continuous) function from A into B.*

Proof of Claim. Let $A \subseteq [a_1, a_2]$, $[b_1, b_2] \subseteq B$, with $a_1 \neq a_2$ and $b_1 \neq b_2$. Consider the function

$$f(x) = b_1 + \frac{b_2 - b_1}{a_2 - a_1}(x - a_1),$$

whose restriction to A does the required job. $\square$

Let F be a Dedekind incomplete ordered field of cofinality λ. Consider an interval $[a,b]$ in F with a point $c \in (a,b)$. Pick a strictly increasing net $(a_\alpha)_{\alpha<\lambda}$ and a strictly decreasing one $(b_\alpha)_{\alpha<\lambda}$ in $[a,b]$ with $a_0 = a$ and $b_0 = b$ which converge in F to c. For each α less than λ, let U_α be a gap in $(a_\alpha, a_{\alpha+1})$ and V_α a gap in $(b_{\alpha+1}, b_\alpha)$. Put

$$S_0 = U_0 \cap [a,b],\ T_0 = [a,b] \setminus V_0,$$

$$S_{\alpha+1} = U_{\alpha+1} \setminus U_\alpha,\ T_{\alpha+1} = V_\alpha \setminus V_{\alpha+1}$$

for $\alpha < \lambda$, and for limit $\beta < \lambda$,

$$S_\beta = U_\beta \setminus \cup_{\gamma<\beta} U_\gamma,\ T_\beta = (\cap_{\gamma<\beta} V_\gamma) \setminus V_\beta.$$

Note that $[a,c) = \cup_{\alpha<\lambda} S_\alpha$, and $(c,b] = \cup_{\alpha<\lambda} T_\alpha$, and that both unions are disjoint. Using the claim, consider a function f on $[a,b]$ which linearly injects, for $\alpha < \lambda$, the set S_α into $T_{\alpha\cdot 2}$, T_α into $T_{(\alpha\cdot 2)+1}$, and finally c to c. Then f maps the interior point c of $[a,b]$ to the boundary point c of $f([a,b])$. $\square$

Corollary 1.3. *An ordered field is Dedekind complete if and only if all continuous 1-1 functions on some (equivalently all) non-degenerated closed bounded interval(s) are open.*

Proof. For the non-trivial *if* part, notice that all open maps on any ordered field map interior to interior. $\square$

2. Some Consequences of Monotone Incompleteness via Nets Cofinal in Gaps

An ordered field F is Scott complete, see [6], if it satisfies either of the following three equivalent conditions:

(i) It does not have any proper extensions to an ordered field in which it is dense,

(ii) It does not have any regular gaps, where a gap $C \subset F$ is regular if $\forall \epsilon \in F^{>0}$, $C + \epsilon \not\subseteq C$,

(iii) All functions $f : F \to F$ which are Cauchy at ∞_F, are convergent there.

We abbreviate the (equivalent non first order) axiomatizations obtained from the theory OF of ordered fields by adding condition (ii) or (iii) as $S_{rc}COF$ and $S_{cf}COF$ respectively.

Observe that an ordered field of cofinality λ is Scott complete if and only if it has no divergent Cauchy nets of length λ.

If an ordered field F is not Scott complete, then for all $a < b \in F$, there is a regular gap V of F with $a \in V$ and $b \in F \setminus V$ (and so a divergent Cauchy net in (a,b) of length equal to cofinality of the field). In fact, any regular gap of F has an additive translation to a regular gap in (a,b). To see this, let U be a regular gap of F. We can pick $c \in U$ and $d \in F \setminus U$ such that

$d - c < b - a$. Let $V = U + (a - c)$. It is straightforward to see that V is a regular gap and $a < V < b$.

It was proved in [6] (Theorem 1), that any ordered field F has a (unique, up to an isomorphism of ordered fields which is identity on F) Scott completion. It is characterized by being Scott complete and having F dense in it. Furthermore F is dense in $RC(F)$ if and only if its Scott completion is real closed, see [6] (Theorem 2).

As shown in [2] (Lemma 2.3), any uncountable ordered field has a dense transcendence basis over the rationals. Therefore all Scott complete ordered fields necessarily have proper dense subfields and so can be obtained by Scott completion of a proper subfield (the field extension, inside the original ordered field, of rationals by the just mentioned basis minus a point does the job).

We will state and prove a number of results on Scott completeness and its two first order versions in section 3. In this section, we are aiming at a result dealing with the stronger notion of monotone completeness. Monotone complete ordered fields were introduced in [3]. They are ordered fields with no bounded strictly increasing divergent functions. Scott complete ordered fields that are κ-Archimedean for some regular cardinal κ are Monotone Complete (by κ-Archimedean, one means that there are subsets of cardinality κ whose distinct elements are at least a unit apart and all such sets are unbounded), see [1].

As we have already mentioned, all open maps on any interval in an arbitrary ordered field map every interior point to an interior point of the image. However, the converse property, that of mapping interior to interior implying being open even when restricted to continuous 1-1 functions, is strong enough to imply monotone completeness as the next theorem shows.

Theorem 2.1. *Over all closed bounded intervals in arbitrary monotone incomplete ordered fields, there are*

(i) *continuous non uniformly continuous unbounded functions whose ranges are not closed,*
(ii) *continuous 1-1 functions mapping all interior points to interior of the image which, nevertheless, are not open.*

Proof. Let F be a monotone incomplete ordered field of cofinality λ. We first observe the following.

Claim 2.1.1. *Any nondegenerated interval of F contains a strictly increasing divergent net of length λ [compare with* [3] *(Proposition 1(a))].*

Proof of Claim. Given a divergent bounded strictly increasing net of length λ, consider an interval containing its range. Then for any other interval, one may apply the linear strictly increasing function mapping the former onto the latter interval, thereby producing a strictly increasing divergent net λ in the latter. This concludes the claim. □

(i). Given a closed bounded interval $[a,b]$ of F, using the claim, we can pick divergent λ-nets $(a_\alpha)_{\alpha<\lambda}$ and $(b_\alpha)_{\alpha<\lambda}$ with $a_0 = a$, $b_0 = b$, $a_\alpha < b_\beta$, for all $\alpha, \beta < \lambda$, which are respectively strictly increasing or strictly decreasing. Also, let $(u_\alpha)_{\alpha<\lambda}$ and $(d_\alpha)_{\alpha<\lambda}$ be both strictly increasing, the former being unbounded and having consecutive terms which are at least a unit apart, the latter converging in F to 0. Now, for each $\alpha < \lambda$, let f map $[a_\alpha, a_{\alpha+1})$ linearly and increasingly onto $[u_\alpha, u_{\alpha+1})$ and $(b_{\alpha+1}, b_\alpha]$ linearly and in a decreasing manner onto $[d_\alpha, d_{\alpha+1})$ and be equal to 1 on the rest of $[a,b]$. We may assume without loss of generality that for any $\delta > 0$, there exists $\alpha_0 < \lambda$ such that $a_{\alpha_0+1} - a_{\alpha_0} < \delta$. To see that such a reduction indeed causes no loss of generality, take a net $(c_\alpha)_{\alpha<\lambda}$ converging to 0 such that $0 < c_\alpha < a_{\alpha+1} - a_\alpha$ and consider the net obtained from $(a_\alpha)_{\alpha<\lambda}$ by right-shifting those of its terms which have limit ordinal indices by the corresponding same-indexed terms in $(c_\alpha)_{\alpha<\lambda}$. The function f is a continuous non uniformly continuous function on $[a,b]$ whose values are F-unbounded on the downward closure of $(a_\alpha)_{\alpha<\lambda}$ in $[a,b]$ and asymptotic to zero on the upward closure of $(b_\alpha)_{\alpha<\lambda}$ in $[a,b]$ traversed backwards. To see that f is not uniformly continuous, let $\epsilon = \frac{1}{2}$. Then for any $\delta > 0$, by the assumption we made above, there exists $\alpha_0 < \lambda$ such that $a_{\alpha_0+1} - a_{\alpha_0} < \delta$. Now by $u_{\alpha+1} - u_\alpha \geq 1$ and the way f is constructed, we have

$$f(a_{\alpha_0+1}) - f(a_{\alpha_0}) \geq 1 > \epsilon.$$

(ii). Consider the interval $[a,b]$ with a point $c \in (a,b)$. We now follow a construction similar to the one in Theorem 1.2. Using the same notation as there, the change we make is the following. We linearly inject T_α, for $1 \leq \alpha < \lambda$, into $T_{(\alpha\cdot 2)+1}$ but T_0 onto $[a,c)$. The latter can be done as in part (i) in a piece-wise manner. Here we may assume that T_0 is the upward closure of a strictly decreasing divergent λ-net in $(b_1, b_0]$. Then c will be a boundary point of the image of the open interval (a_1, b_1) under f, so that image is not open. On the other hand, by construction, f sends interior to interior. □

3. P-Definable Regular Gaps Not Traversed by ∅-Definable Cauchy Functions

We now consider two notions of being definably (with or without parameters) Scott complete for ordered fields, those corresponding to $S_{rc}COF$ and $S_{cf}COF$. They are ordered fields with no definable regular gaps, respectively no definable functions which are Cauchy at infinity but divergent there. We denote the corresponding theories by $D_pS_{rc}COF$, $D_pS_{cf}COF$, $D_\emptyset S_{rc}COF$, and $D_\emptyset S_{cf}COF$.

It is easy to see that, as long as there are no definability concerns, all regular gaps can be traversed by suitable (divergent Cauchy) functions and vice versa, every divergent Cauchy function induces a regular gap. The latter converse is shown in Theorem 3.4(i) to be true in either of the p-definable or ∅-definable cases. For the former direction, however, we are only able in

Theorem 3.4(ii) to prove an independence result in a mixed $\emptyset$-definable / p-definable case.

Lemma 3.1. *If the ordered field F is a proper dense sub-field of its real closure, then $F \nvDash D_pS_{rc}COF$.*

Proof. Pick out an element $r \in RC(F) \setminus F$ and consider the set

$$C = \{x \in F : \ RC(F) \vDash x < r\}.$$

It is obviously a gap of F. As F is dense in $RC(F)$, C is regular. Finally, C is p-definable in F: consider the minimal polynomial of r over F and the number of roots of that polynomial in $RC(F)$ which are less than r. □

For any ordered field F and ordered abelian group G, the set $[[F^G]]$ of all functions $G \to F$ whose supports are well ordered in G equipped with pointwise sum and Cauchy product

$$(f_1f_2)(g) = \sum_{i+j=g} f_1(i)f_2(j)$$

(a finite sum by the condition on the supports) forms a field. It can be ordered by comparison of values at the minimum of support of the difference. Elements of $[[F^G]]$ can also be thought of as those formal power series $\sum_{g\in G} f(g)t^g$ which have well ordered supports. The indeterminate t is taken to be a positive F-infinitesimal. The field $[[F^G]]$ is real closed if and only if F is so and G is divisible, see [5] (6.10).

In the proposition below, χ_S stands for the characteristic function of (a set) S.

Proposition 3.2. *For any ordered abelian group G and ordered field F, the generalized power series field $[[F^G]]$ is Scott complete if F is so.*

Proof. Let λ be the cofinality of G and consider a strictly decreasing λ-net $(a_\theta)_{\theta<\lambda}$ in $G^{<0}$ which is unbounded below there. Assume $\mathcal{F} : [[F^G]] \to [[F^G]]$ is Cauchy. For each $g \in G$, define $f_g : \lambda \to F$ by

$$f_g(\theta) = (\mathcal{F}(\chi_{\{a_\theta\}}))(g), \quad \text{for } \theta < \lambda.$$

Claim 3.2.1. *For all $g \in G$, the net $(f_g(\theta))_{\theta<\lambda}$ is convergent in F.*

Proof of Claim 3.2.1. Fix $g \in G$ and $\epsilon \in F^{>0}$. By the Cauchy condition for $\mathcal{F}$, there exists $\theta_0 < \lambda$ such that $\forall \alpha, \beta \in [[F^G]]$ with $\alpha, \beta \geq \chi_{\{a_{\theta_0}\}}$, we have $|\mathcal{F}(\alpha) - \mathcal{F}(\beta)| < \epsilon\chi_{\{g\}}$. This shows that $|\mathcal{F}(\alpha)(g) - \mathcal{F}(\beta)(g)| < \epsilon$. Therefore $\forall \theta_1, \theta_2 \geq \theta_0$,

$$|f_g(\theta_1) - f_g(\theta_2)| = |(\mathcal{F}(\chi_{\{a_{\theta_1}\}}))(g) - (\mathcal{F}(\chi_{\{a_{\theta_2}\}}))(g)| < \epsilon,$$

since $\chi_{\{a_{\theta_1}\}}, \chi_{\{a_{\theta_2}\}} \geq \chi_{\{a_{\theta_0}\}}$. Hence the net $(f_g(\theta))_{\theta<\lambda}$ is Cauchy in F and so convergent there, since F is Scott complete. □

Let $\gamma : G \to F$ be defined by $\gamma(g) = \lim(f_g(\theta))_{\theta<\lambda}$.

Claim 3.2.2. $(\forall \eta < \lambda)(\exists \theta_0 < \lambda)(\forall \theta \geq \theta_0)(\forall g \in G^{<-a_\eta})\mathcal{F}(\chi_{\{a_\theta\}})(g) = \gamma(g)$.

Proof of Claim 3.2.2. For any $\epsilon = \chi_{\{-a_\eta\}}$, with $\eta < \lambda$, there exists $\theta_0 < \lambda$ such that $\forall \theta_1, \theta_2 < \lambda$ with $\theta_1, \theta_2 \geq \theta_0$, we have

$$|\mathcal{F}(\chi_{\{a_{\theta_1}\}}) - \mathcal{F}(\chi_{\{a_{\theta_2}\}})| < \chi_{\{-a_\eta\}}.$$

This shows that for all $\theta_1, \theta_2 \geq \theta_0$ and $g < -a_\eta$, we have

$$(\mathcal{F}(\chi_{\{a_{\theta_1}\}}))(g) = (\mathcal{F}(\chi_{\{a_{\theta_2}\}}))(g) = \gamma(g).$$

Therefore, for all $\theta \geq \theta_0$ and $g \in G^{<-a_\eta}$, we have $\mathcal{F}(\chi_{\{a_\theta\}})(g) = \gamma(g)$. □

Claim 3.2.3. $\gamma \in [[F^G]]$.

Proof of Claim 3.2.3. It suffices to show that the support of γ is well ordered. For all $g \in G$, there exists $\eta < \lambda$ such that $g < -a_\eta$. Let θ_0 be as in Claim 3.2.2. As g can not be the initial term of any infinite strictly decreasing sequence in the support of $\mathcal{F}(\chi_{\{a_{\theta_0}\}})$, Claim 3.2.2 shows that the same holds for the support of γ. □

Claim 3.2.4. *The function $\mathcal{F}$ on $[[F^G]]$ tends to γ at infinity.*

Proof of Claim 3.2.4. It is enough, by the Cauchy criterion for $\mathcal{F}$, to apply $\mathcal{F}$ on those f's in $[[F^G]]$ that are of the form $\chi_{\{a_\theta\}}$, for $\theta < \lambda$ and let θ tend to λ. The result is then immediate from Claim 3.2.2. □

The above claims give the result. □

Lemma 3.3. *Suppose that F is a Scott complete ordered field, G is a 2-divisible ordered abelian group, and K is a dense ordered sub-field of $[[F^G]]$ which contains F. Assume furthermore that K is closed under the automorphism on $[[F^G]]$ sending $\chi_{\{g\}}$ to $\chi_{\{g+g\}}$ and its inverse. Then K satisfies $D_\emptyset S_{cf} COF$.*

Proof. If $K = [[F^G]]$, then the conclusion will trivially hold since $[[F^G]]$ is Scott complete. Suppose $K \subsetneq [[F^G]]$. Assume for the purpose of a contradiction that $\mathcal{F}$ is a $\emptyset$-definable divergent Cauchy function on K. Fix a net $(k_\alpha)_{\alpha<\lambda}$ of elements of K, where λ is cofinality of $[[F^G]]$, which is cofinal in the latter. For any $f \in [[F^G]]$, let $\lambda(f)$ be the least ordinal less than λ such that $f < k_{\lambda(f)}$. Consider the function $\tilde{\mathcal{F}}$ on $[[F^G]]$ with $\tilde{\mathcal{F}}(f) = \mathcal{F}(k_{\lambda(f)})$. As a Cauchy function on the Scott complete ordered field $[[F^G]]$, it will converge to some $f_0 \in [[F^G]]$. Clearly $f_0 \notin K$ and therefore $f_0 \notin F$. Let $\mathcal{A}$ be the ordered field automorphism on $[[F^G]]$ described in the statement of the lemma. The only fixed points of $[[F^G]]$ under $\mathcal{A}$ are elements of F (since if there is a leading F-infinitely large or a highest F-infinitely small term, then the result of applying $\mathcal{A}$ to such elements will have a different leading F-infinitely large, respectively highest F-infinitely small term and so must be different). Therefore, $\mathcal{A}(f_0) \neq f_0$. Let

$$C = \{f \in K : f \text{ is cofinally many times dominated in } K \text{ by values of } \mathcal{F}\}.$$

Now since in $[[F^G]]$, $\mathcal{F}$ tends to f_0, while $\mathcal{A}(\mathcal{F})$ approaches $\mathcal{A}(f_0)$ there (as $\mathcal{A}$ is continuous) and also K is dense in $[[F^G]]$, we get $C \neq \mathcal{A}(C)$.

This contradicts $\emptyset$-definability of $\mathcal{F}$, since $\mathcal{A}$ restricted to K is an (onto) automorphism. $\square$

Theorem 3.4.

(i) $D_pS_{rc}COF \vdash D_pS_{cf}COF$, $D_\emptyset S_{rc}COF \vdash D_\emptyset S_{cf}COF$.
(ii) $D_\emptyset S_{cf}COF \nvdash D_pS_{rc}COF$.

Proof. (i). Let $F \vDash D_pS_{rc}COF$ (respectively $F \vDash D_\emptyset S_{rc}COF$) and $f : F \to F$ be a p-definable (respectively $\emptyset$-definable) Cauchy function. Let $\psi(z)$ be the formula expressing, using f as a shorthand, $(\forall t)(\exists x \geq t)(z \leq f(x))$. We claim that the set C defined by ψ in F is a regular cut whose supremum is the limit of f.

To see regularity of C, fix $\epsilon \in F^{>0}$. From the Cauchy criterion for f, there exists $d \in F$ such that $\forall x, y \geq d$, $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let $z = f(d) - \frac{\epsilon}{2}$. Then $F \vDash \psi(z)$ and $F \vDash \neg\psi(z + \epsilon)$.

Now let $\sup(C) = \alpha$. To show $\lim f(x) = \alpha$, as x becomes arbitrarily large in F, let $\epsilon \in F^{>0}$ and d be as before. By the definition of α, there exists $z \in C$ with $z > \alpha - \frac{\epsilon}{2}$. From this, we get $\exists x_0 \geq d$ with $\alpha - \frac{\epsilon}{2} < z \leq f(x_0)$. On the other hand, for any $t \in F$, the element $x = \max\{t, d\}$ satisfies $f(x_0) - \frac{\epsilon}{2} < f(x)$ and so $f(x_0) - \frac{\epsilon}{2} \leq \alpha$. Therefore, $|f(x_0) - \alpha| \leq \frac{\epsilon}{2}$. Hence,

$$\forall x \geq d,\ |f(x) - \alpha| \leq |f(x) - f(x_0)| + |f(x_0) - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(ii). Let R be the ordered ring $[[\mathbb{R}^{\mathbb{Q}^{\leq 0}}]]$ and K its fraction field. We claim that K is a witness for the independence assertion at hand. By Lemmas 3.1 and 3.3 and $[[\mathbb{R}^{\mathbb{Q}}]]$ being real closed, it suffices to show that K is a non real closed dense sub-field of $[[\mathbb{R}^{\mathbb{Q}}]]$ closed under the automorphism mentioned there and its inverse. The latter is immediate from the same property for R.

As to K being proper in its real closure, we use a result which has recently been shown in [4]. It states that in R, all elements $a+1$ where a has a strictly increasing ω-sequence support converging to 0, are prime. The series

$$t^{-1} + t^{-\frac{1}{2}} + t^{-\frac{1}{3}} + \cdots + 1$$

is therefore an infinite prime in R. The square root of any (positive) infinite prime in R (which has cofinal positive-integer powers there) witnesses that K is not real closed.

To see that K is dense in $[[\mathbb{R}^{\mathbb{Q}}]]$, notice that R approximates within 1 any element of $[[\mathbb{R}^{\mathbb{Q}}]]$. $\square$

It is interesting to note that, by Pitteloud's results again, the same infinite prime remains prime in all $[[\mathbb{R}^{(\mathbb{R}^\alpha)^{\leq 0}}]]$ for any ordinal α. The p-definable regular gap corresponding to its square root is therefore never realized in the fraction fields of such ordered extension fields and can not be traversed by $\emptyset$-definable functions over them.

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Note added in proof. The above proof for Proposition 3.2 will be supplemented elsewhere by an appropriate argument to show that $[[F^G]]$ is always Scott complete, no matter whether F is so or not (of course, there G will be a non-zero group). Lemma 3.3 therefore holds without assuming that F is Scott complete. Theorem 3.4(ii) will also be improved to $D_\emptyset S_{rc}COF \nvdash D_p S_{rc}COF$ (with the same witness K as above). Consequently, either $D_\emptyset S_{cf}COF \nvdash^{(?)} D_p S_{cf}COF$ or $D_p S_{cf}COF \nvdash^{(?)} D_p S_{rc}COF$ (or both). The parameter-free version of the latter remains open also.

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Department of Mathematics, Tarbiat Modarres University, Tehran, Iran
E-mail address: moniri_m@modares.ac.ir
E-mail address: eivazl_j@modares.ac.ir