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THE CANTOR SET OF LINEAR ORDERS ON $\mathbb{N}$ IS THE UNIVERSAL MINIMAL S_∞-SYSTEM

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Abstract. Each topological group G admits a unique universal minimal dynamical system $(M(G), G)$. When G is a non-compact locally compact group the phase space $M(G)$ of this universal system is non-metrizable. There are however topological groups for which $M(G)$ is the trivial one point system (extremely amenable groups), as well as topological groups G for which $M(G)$ is a metrizable space and for which there is an explicit description of the dynamical system $(M(G), G)$. One such group is the topological group S_∞ of all permutations of the integers $\mathbb{Z}$, with the topology of pointwise convergence. We show that $(M(S_\infty), S_\infty)$ is a symbolic dynamical system (hence in particular $M(S_\infty)$ is a Cantor set), and give a full description of all its symbolic factors. Among other facts we show that $(M(G), G)$ (and hence also every minimal S_∞) has the structure of a two-to-one group extension of proximal system and that it is uniquely ergodic.

This is a summary of a talk given at the Prague Topological Symposium of 2001 in which I described results obtained in a joint paper with B. Weiss. The paper is going to appear soon in GAFA [3].

Given a topological group G and a compact Hausdorff space X, a dynamical system (X, G) is a jointly continuous action of G on X. If (Y, G) is a second dynamical system then a continuous onto map $\pi : (X, G) \to (Y, G)$ which intertwines the G actions is called *a homomorphism*. The dynamical system (X, G) is *point transitive* if there exists a point $x_0 \in X$ whose *orbit* Gx_0 is dense in X. (X, G) is *minimal* if every orbit is dense. It can be easily shown that there exists a unique (up to isomorphism of dynamical G-systems) universal point transitive G-system $(\mathbf{L}, G)$. One way of presenting this universal object is via the Gelfand space of the C^*-algebra $\mathcal{L}_l(G)$ of left uniformly $\mathbb{C}$-valued continuous functions on G. From the existence of $(\mathbf{L}, G)$ one easily deduces the existence of a universal minimal dynamical system; i.e. a system $(M(G), G)$ such that for every minimal system (X, G) there exists a homomorphism $\pi : (M(G), G) \to (X, G)$. Ellis' theory shows

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that up to isomorphism this universal minimal dynamical system is unique, see e.g. [1].

The existence of uncountably many characters of the discrete group $\mathbb{Z}$ already shows that the phase space $M(\mathbb{Z})$ is non-metrizable. In fact one can show that $M(G)$ is non-metrizable whenever G is non-compact locally compact group.

A topological group G has the *fixed point on compacta property* (f.p.c.) (or is *extremely amenable*) if whenever it acts continuously on a compact space, it has a fixed point. Thus the group G has the f.p.c. property iff its universal minimal dynamical system is the trivial one point system.

A triple (X, d, μ), where (X, d) is a metric space and μ a probability measure on X, is called an *mm-space*. For $A \subseteq X$, $\mu(A) \geq 1/2$, and $\epsilon > 0$ let A_ϵ be the set of all points whose distance from A is at most ϵ.

A family of *mm* spaces (X_n, d_n, μ_n) is called a *Lévy family* if for every ϵ, $\alpha_n(\epsilon) \to 0$, where $\alpha(\epsilon) = 1 - \inf\{\mu(A_\epsilon) : A \subseteq X,\ \mu(A) \geq 1/2\}$. When a Polish group (G, d) contains an increasing sequence of compact subgroups $\{G_n : n \in \mathbb{N}\}$ whose union is dense in G and such that with respect to the corresponding sequence of Haar measures μ_n, the family (G_n, d, μ_n) forms a Lévy family, then G is called a *Lévy group*.

In [5] Gromov and Milman prove that every Lévy group G has the f.p.c. property. Many of the examples presently known of extremely amenable groups are obtained via this theorem. There are however other methods of obtaining such groups. Here is a partial list:

(1) The unitary group $U(\infty) = \cup_{n=1}^{\infty} U(n)$ with the uniform operator topology (Gromov-Milman, [5]).
(2) The monothetic Polish group $L_m(I, S^1)$, consisting of all (classes) of measurable maps from the unit interval I into the circle group S^1 with the topology of convergence in measure induced by, say, Lebesgue measure on I (Glasner, [4]; Furstenberg-Weiss). More generally, $L_m(I, G)$, where G is any locally compact amenable group (Pestov, [8]).
(3) The group of measurable automorphisms $\mathrm{Aut}(X, \mu)$ of a standard sigma-finite measure space (X, μ), with respect to the weak topology (Giordano-Pestov [2]).
(4) Using Ramsey's theorem, Pestov has shown that the group $\mathrm{Aut}(\mathbb{Q}, <)$, of order automorphism of the rational numbers with pointwise convergence topology, is extremely amenable, [6].

Thus, as we have seen, the universal minimal system $(M(G), G)$ corresponding to a non-compact G is usually non-metrizable but can be, in some cases, trivial. Are there non-compact topological groups for which $M(G)$ is metrizable but non-trivial? The first such example was pointed out by Pestov [6] who used claim 4 above to show that the universal minimal dynamical system of the group G of orientation-preserving homeomorphisms of the circle coincides with the natural action of G on S^1.

In [9] V. Uspenskij shows that the action of a topological group G on its universal minimal system $M(G)$ is never 3-transitive. As a direct corollary he shows that for manifolds X of dimension > 1 (as well as for $X = Q$, the Hilbert cube) the corresponding group G of orientation preserving homeomorphisms, $(M(G), G)$ does not coincide with the natural action of G on X.

Let S_∞ be the group of all permutations of the integers $\mathbb{Z}$. With respect to the topology of pointwise convergence on $\mathbb{Z}$, S_∞ is a Polish topological group. The subgroup $S_0 \subset S_\infty$ consisting of the permutations which fix all but a finite set in $\mathbb{Z}$ is an amenable dense subgroup (being the union of an increasing sequence of finite groups) and therefore S_∞ is amenable as well.

In [5] Gromov and Milman conjectured, in view of the concentration of measure on S_n with respect to Hamming distance, that S_∞ has the f.p.c. property. In [6] and [7] V. Pestov has shown that, on the contrary, S_∞ acts effectively on $M(S_\infty)$ and that, in fact, there is no Hausdorff topology making S_0 a topological group with the f.p.c. property. He as well as A. Kechris (in private communication) asked for explicit examples of S_∞-minimal systems.

The main result of our work [3] is the fact that the universal minimal system $(M(S_\infty), S_\infty)$ is a metrizable system, in fact a system whose phase space is the Cantor set. We also give in this work an explicit description of $(M(S_\infty), S_\infty)$ as a "symbolic" dynamical system and exhibit explicit formulas for all of its symbolic factors. Let me now describe these results in more details.

For every integer $k \geq 2$ let

$$\mathbb{Z}^k_* = \{(i_1, i_2, \dots, i_k) \in \mathbb{Z}^k : i_1, i_2, \dots, i_k \text{ are distinct elements of } \mathbb{Z}\},$$

and set $\Omega^k = \{1, -1\}^{\mathbb{Z}^k_*}$. Consider the dynamical system (Ω^k, S_∞), where for $\alpha \in S_\infty$ and $\omega \in \Omega^k$ we let

$$(\alpha\omega)(i_1, i_2, \dots, i_k) = \omega(\alpha^{-1} i_1, \alpha^{-1} i_2, \dots, \alpha^{-1} i_k).$$

Let $\Omega^k_{alt} \subset \Omega^k$ consist of all the *alternating* configurations, that is those elements $\omega \in \Omega^k$ satisfying

$$\omega(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k)) = \mathrm{sgn}(\sigma)\omega(i_1, i_2, \dots, i_k),$$

for all $\sigma \in S_k$ and $(i_1, i_2, \dots, i_k) \in \mathbb{Z}^k_*$. Clearly $\Omega^k_{alt}$is a closed and S_∞-invariant subset of Ω^k.

A configuration $\omega \in \Omega^2$ *determines a linear order* on $\mathbb{Z}$ if it is alternating, and satisfies the conditions:

$$\omega(m, n) = 1 \ \wedge \ \omega(n, l) = 1 \quad \Rightarrow \quad \omega(m, l) = 1.$$

Let $<_\omega$ be the corresponding linear order on $\mathbb{Z}$, where $m <_\omega n$ iff $\omega(m, n) = 1$. Let $X = \Omega^2_{lo}$ be the subset of Ω^2 consisting of all the configurations which determine a linear order. The correspondence $\omega \longleftrightarrow <_\omega$ is a surjective bijection between Ω^2_{lo} and the collection of linear orders on $\mathbb{Z}$. Clearly X is a closed S_∞-invariant set and using Ramsey's theorem we shall show that (X, S_∞) is a minimal system.

Say that a configuration $\omega \in \Omega^3$ is *determined by a circular order* if there exists a sequence $\{z_m : m \in \mathbb{Z}\} \subset S^1$ with $m \neq n \Rightarrow z_m \neq z_n$ such that: $\omega(l, m, n) = 1$ for $(l, m, n) \in \mathbb{Z}^3_*$ iff the directed arc in S^1 defined by the ordered triple (z_l, z_m, z_n) is oriented in the positive direction. Let $Y = \Omega^3_c \subset \Omega^3_{alt}$ denote the collection of all the configurations in Ω^3 which are determined by a circular order. It follows that the set $Y = \Omega^3_c$ is closed and invariant and using Ramsey's theorem one can show that it is minimal.

If we go now to Ω^4_{alt}, can one find a sequence of points $\{z_n\}$ on the sphere S^2 in general position such that the tetrahedron defined by any four points $z_{n_1}, z_{n_2}, z_{n_3}, z_{n_4}$ has positive orientation when $n_1 < n_2 < n_3 < n_4$?

Starting with any sequence $\{z_n\} \subset S^2$ in general position one can use Ramsey's theorem to find a subsequence with the required property. Another way to see this is to use the 'moment curve'

$$t \mapsto (t, t^2, t^3).$$

Again it turns out that the orbit closure in Ω^4_{alt} which is determined by such a sequence forms a minimal dynamical system.

It now seems as if going up to Ω^k_{alt} with larger and larger k's we encounter more and more complicated minimal systems. However, as we show in [3], this is not the case and the entire story is already encoded in the simplest symbolic dynamical system Ω^2_{lo}.

Theorem 1. *Ω^2_{lo} is the universal minimal S_∞-system.*

The fact that the topology on S_∞ is zero-dimensional, and in fact given by a sequence of clopen subgroups, enables us to reduce this theorem to the following one.

Theorem 2. *Every minimal subsystem Σ of the system (Ω^k, S_∞) is a factor of the minimal system $(\Omega^2_{lo}, S_\infty)$.*

Finally let me mention two more facts concerning the system $(M(S_\infty), S_\infty)$.

Theorem 3. *The universal minimal system $(\Omega^2_{lo}, S_\infty)$ has the structure of a two-to-one group extension of a proximal system.*

Theorem 4. *The universal minimal system $(\Omega^2_{lo}, S_\infty)$ is uniquely ergodic and therefore so is every minimal S_∞-system.*

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