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SOME EQUIVALENCES FOR MARTIN'S AXIOM IN ASYMMETRIC TOPOLOGY

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Abstract. We find some statements in the language of asymmetric topology and continuous partial orders which are equivalent to the statements $\kappa < \mathfrak{m}$ or $\kappa < \mathfrak{p}$.

We think of asymmetric topology as those parts of topology in which the specialization order, $x \leq y$ if and only if $x \in c\{y\}$, need not be symmetric. (See [5] for some of the motivations.)

Martin's Axiom has many equivalent statements, consequences, and variations in the literature which can be stated in topological terms. Most of the treatments we have seen so far from set-theoretic topologists assume that spaces are Hausdorff. In view of recent interest in asymmetric topology, in which even T_1 spaces are a highly symmetric special case, we give some equivalences for Martin's Axiom which utilize the terms of this field.

Our reference for properties related to Martin's Axiom is [2], and for properties related to continuous lattices we referred to [6].

Definition 1. A partially ordered set $(\mathcal{P}, \leq)$ is *upwards-ccc* if any uncountable subset of $\mathcal{P}$ must have two distinct members which have a common upper bound in $\mathcal{P}$. The cardinal $\mathfrak{m}$ is the least cardinal such that there exists a non-empty upwards-ccc partially ordered set $(\mathcal{P}, \leq)$ and a collection $\{D_\alpha | \ \alpha < \mathfrak{m}\}$ of cofinal subsets of $\mathcal{P}$ such that no upwards-directed subset of $\mathcal{P}$ meets each D_α.

It can be shown that $\omega_1 \leq \mathfrak{m} \leq \mathfrak{c}$. The Martin's Axiom of the title is the statement $\mathfrak{m} = \mathfrak{c}$.

Definition 2. A topological space is *ccc* if any uncountable collection of open sets has two distinct members which are not disjoint. A space is *locally compact* if every open set contains a compact neighborhood of each of its points. For two sets U and V, we say U *is compact in* V, denoted $U \ll V$, if for every open cover of V there is a finite subcollection which covers U. A

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space is *core compact* if every open set O is the union of open sets O' with $O' \ll O$.

Definition 3. A subset C of a space is *irreducible* if it is closed and the non-empty open sets in the subspace topology on C form a filterbase. A space is *sober* if it is T_0 and every irreducible subset is the closure of a point. A space is *supersober* if it is T_0 and every ultrafilter on the space has either the empty set or the closure of a point as its set of limits. A filter $\mathcal{F}$ of open sets on a space is *Scott open* if whenever $\mathcal{D}$ is a collection of open sets which is directed upward under inclusion, $\bigcup \mathcal{D} \in \mathcal{F}$ implies that some element of $\mathcal{D}$ is a member of $\mathcal{F}$.

We will make use below of the Hofmann-Mislove Theorem, that on a sober space the intersection of a Scott open filter of open sets is non-empty. (See [3] for the original reference, and see [4] for a new proof of the form of the theorem that we have just stated.)

Definition 4. In a partially ordered set $(\mathcal{P}, \leq)$, for two elements x and y we say *x is way below y*, denoted $x \ll y$, if for any directed subset D of $\mathcal{P}$ with supremum $\sup D$, if $y \leq \sup D$ then there is some $d \in D$ with $x \leq d$. $(\mathcal{P}, \leq)$ is *upwards-continuous* if for all $y \in \mathcal{P}$ the set $\{x \in \mathcal{P} |\ x \ll y\}$ is a directed set with supremum y.

Definition 5. A lattice is *distributive* if it satisfies the distributive laws,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A complete lattice is *downwards-ccc* if in any uncountable subset there are two distinct members x and y with $x \wedge y \neq 0$, where 0 is the least element. An element x in a complete lattice is a *non-zero divisor* if $x \wedge y \neq 0$ for any $y \neq 0$. An element x is *irreducible* if whenever $x = a \wedge b$ then $x = a$ or $x = b$.

Proposition 1. *Let κ be a cardinal. The following are equivalent.*

(1) $\kappa < \mathfrak{m}$
(2) *In a ccc locally compact sober space the intersection of κ or fewer open dense sets is non-empty.*
(3) *In a ccc core compact space if $\mathcal{D}$ is a collection of κ or fewer open dense sets then there is an irreducible set meeting every member of $\mathcal{D}$.*
(4) *Let $\mathcal{L}$ be a downwards-ccc upwards-continuous distributive complete lattice. If D is a set of κ or fewer non-zero divisors of $\mathcal{L}$ then there is an irreducible $p \in \mathcal{L}$ such that no element of D is below p.*

Proof. (1) implies (2). Given a ccc locally compact sober space $(X, \mathcal{T})$ let $\mathcal{P} = \mathcal{T} - \{\phi\}$ and define a partial order on $\mathcal{P}$ by $O_1 \leq O_2$ if either $O_1 = O_2$ or O_2 is compact in O_1. By local compactness, two open sets O_1 and O_2 have a common upper bound in $(\mathcal{P}, \leq)$ if and only if $O_1 \cap O_2 \neq \phi$. So $(\mathcal{P}, \leq)$ is upwards-ccc. Given a collection $\{O_\alpha |\ \alpha < \kappa\}$ of open dense subsets of

X, for each α define $D_\alpha = \{O \in \mathcal{P} | \ O \subseteq O_\alpha\}$. Then each D_α is cofinal in $(\mathcal{P}, \leq)$, again by local compactness. Since $\kappa < \mathfrak{m}$ there is a directed set $S \subseteq \mathcal{P}$ that meets each D_α. We may assume that S is closed upwards under $\subseteq$, i.e., S is a filter of open sets on $(X, \mathcal{T})$.

If $\bigcap S \neq \phi$ we are done since an element of $\bigcap S$ must be contained in some member of each D_α and so is in $\bigcap_{\alpha<\kappa} O_\alpha$. If S has a maximal element under $\leq$ then it is $\bigcap S$ and we are done. Otherwise S is Scott open, since if $\mathcal{D}$ is a directed collection of open sets with $\bigcup \mathcal{D} \in S$ then some element $O \in S$ is strictly greater than $\bigcup \mathcal{D}$, which means it is compact in $\bigcup \mathcal{D}$, and so it is a subset of some element of $\mathcal{D}$. By the Hofmann-Mislove Theorem $\bigcap S \neq \phi$.

(2) implies (4). Given a downwards-ccc upwards-continuous distributive complete lattice $\mathcal{L}$, by Theorem 1.2 in [6] there exists a locally compact sober space $(X, \mathcal{T})$ such that $\mathcal{L}$ is order isomorphic to $\mathcal{T}$ with inclusion. We note that this $(X, \mathcal{T})$ must be a ccc space. Given $\{d_\alpha | \ \alpha < \kappa\}$, a set of non-zero divisors, the image of each d_α under the isomorphism must be an open dense subset O_α of X. So there is a point $x \in \bigcap_{\alpha<\kappa} O_\alpha$. Let $O = X - c\{x\}$. Then O corresponds in $\mathcal{L}$ to an irreducible p, and since x is in each O_α we can't have $d_\alpha \leq p$ for any α.

(4) implies (3). Given a ccc core compact space $(X, \mathcal{T})$, by Theorem 1.1 in [6] $(\mathcal{T}, \subseteq)$ is an upwards-continuous lattice. It is clear that it is also downwards-ccc, distributive, and complete. Given a collection $\{O_\alpha | \ \alpha < \kappa\}$ of open dense subsets of X, we see as above that each O_α is a non-zero divisor. So there is an irreducible O in the lattice which does not contain any O_α. $C = X - O$ is an irreducible set for $(X, \mathcal{T})$, and C meets every O_α.

(3) implies (1). Given an upwards-ccc partially ordered set $(\mathcal{P}, \leq)$ consider the partial order topology $\mathcal{T}$ generated by all sets

$$\uparrow x = \{y \in \mathcal{P} | \ x \leq y\}$$

for $x \in \mathcal{P}$. $(\mathcal{P}, \mathcal{T})$ is a ccc space. It is locally compact, hence core compact. Given a collection $\{C_\alpha | \ \alpha < \kappa\}$ of cofinal subsets of $\mathcal{P}$, each C_α is dense, so there is an irreducible set C meeting each C_α. But in the partial order topology a set is irreducible only if it is directed, and so we are done. □

We note that each of properties (2) or (3) in Proposition 1 also implies the Baire category formulation of $\kappa < \mathfrak{m}$, to wit, no ccc compact Hausdorff space is the union of κ or fewer nowhere dense sets. We gave in the proof an argument that (3) implies (1) so that we may now suggest that (3) is a simultaneous generalization of the partial order and Baire category forms of $\kappa < \mathfrak{m}$.

We also wish to point out that the equivalence of properties (2) and (3) may be established directly using the sobrification of the space $(X, \mathcal{T})$. The sobrification of $(X, \mathcal{T})$ is the collection of irreducible subsets of X, topologized by the lower Vietoris topology, and $(X, \mathcal{T})$ can be mapped continuously to its sobrification by sending each x to $c\{x\}$ (and thus the image of $(X, \mathcal{T})$

is the T_0-identification of $(X, \mathcal{T})$). A space is core compact if and only if its sobrification is locally compact [6].

In view of the Baire category form of $\kappa < \mathfrak{m}$ mentioned above, the core compactness in property (3) may be replaced by any stronger property that is weaker than compact Hausdorff. Likewise, locally compact sober in property (2) may be replaced by anything stronger which is implied by compact Hausdorff, including locally compact supersober. Sober may also be weakened to *quasisober*, which is sober without the T_0 assumption, since that is sufficient for the Hofmann-Mislove Theorem.

Corollary 1. *The following are equivalent.*

(1) *Martin's Axiom.*
(2) *In a ccc locally compact sober space the intersection of fewer than open dense sets is non-empty.*
(3) *In a ccc core compact space if $\mathcal{D}$ is a collection of fewer than $\mathfrak{c}$ open dense sets then there is an irreducible set meeting every member of $\mathcal{D}$.*
(4) *Let $\mathcal{L}$ be a downwards-ccc upwards-continuous distributive complete lattice. If D is a set of fewer than $\mathfrak{c}$ non-zero divisors of $\mathcal{L}$ then there is an irreducible $p \in \mathcal{L}$ such that no element of D is below p.*

There is another cardinal, $\mathfrak{p}$, for which we can get some similar results.

Definition 6. In a partially ordered set $(\mathcal{P}, \leq)$ a set R is *upwards-centered* if every finite subset of R has an upper bound in $\mathcal{P}$. $(\mathcal{P}, \leq)$ is *upwards-σ-centered* if $\mathcal{P}$ is the union of countably many centered subsets. The cardinal $\mathfrak{p}$ is the least cardinal such that there exists a non-empty upwards-σ-centered partially ordered set $(\mathcal{P}, \leq)$ and a collection $\{D_\alpha | \alpha < \mathfrak{p}\}$ of cofinal subsets of $\mathcal{P}$ such that no upwards-directed subset of $\mathcal{P}$ meets each D_α.[1] A space $(X, \mathcal{T})$ is *σ-centered* if $\mathcal{T} - \{\phi\}$ with reverse inclusion is upwards-σ-centered. A complete lattice $\mathcal{L}$ is *downwards-σ-centered* if $\mathcal{L} - \{0\}$ with order reversed is upwards-σ-centered.

The Baire category formulation of $\kappa < \mathfrak{p}$ is that no separable compact Hausdorff space is the union of fewer than $\mathfrak{p}$ nowhere dense sets. It can be shown that $\mathfrak{m} \leq \mathfrak{p} \leq \mathfrak{c}$.

Proposition 2. *Let κ be a cardinal. The following are equivalent.*

(1) $\kappa < \mathfrak{p}$
(2) *In a separable locally compact sober space the intersection of κ or fewer open dense sets is non-empty.*

[1]This is not the original definition. if $\mathcal{A}$ is a collection of infinite sets we say that an infinite set B is a *pseudo-intersection* of $\mathcal{A}$ if $B - A$ is finite for every $A \in \mathcal{A}$. Then $\mathfrak{p}$ is the least cardinal such that there exists a collection $\mathcal{A}$ of subsets of ω, with the cardinaltiy of $\mathcal{A}$ equal to $\mathfrak{p}$, and although every finite subset of $\mathcal{A}$ has an infinite intersection, $\mathcal{A}$ has no infinite pseudo-intersection. It was Murray Bell [1] who proved that this definition is equivalent to the σ-centered partial order definition given above, and Fremlin [2] refers on page 25 to this result as "Bell's Theorem."

(3) *In a separable core compact space if $\mathcal{D}$ is a collection of κ or fewer open dense sets then there is an irreducible set meeting every member of $\mathcal{D}$.*
(4) *In a σ-centered locally compact sober space the intersection of κ or fewer open dense sets is non-empty.*
(5) *In a σ-centered core compact space if $\mathcal{D}$ is a collection of κ or fewer open dense sets then there is an irreducible set meeting every member of $\mathcal{D}$.*
(6) *Let $\mathcal{L}$ be a downwards-σ-centered upwards-continuous distributive complete lattice. If D is a set of κ or fewer non-zero divisors of $\mathcal{L}$ then there is an irreducible $p \in \mathcal{L}$ such that no element of D is below p.*

Proof. The equivalence of properties (1), (4), (5), and (6) may be established as in the proof of Proposition 1. Properties (2) and (3) are implied by (4) and (5), respectively, and each of them implies the Baire category form of property (1). □

We do not know if σ-centered is equivalent to separable for locally compact sober spaces, but this is true for locally compact supersober spaces.

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