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SPECIAL KÄHLER GEOMETRY

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ABSTRACT. The geometry that is defined by the scalars in couplings of Einstein–Maxwell theories in $N = 2$ supergravity in 4 dimensions is denoted as special Kähler geometry. There are several equivalent definitions, the most elegant ones involve the symplectic duality group. The original construction used conformal symmetry, which immediately clarifies the symplectic structure and provides a way to make connections to quaternionic geometry and Sasakian manifolds.

1. INTRODUCTION

In the previous workshop in this series on quaternionic geometry, B. de Wit and me gave talks [1] on the classification of quaternionic homogeneous spaces [2]. Results in special geometry had lead to new homogeneous quaternionic spaces. We have discussed on this topic further with D. Alekseevsky and V. Cortés[1]. and realised that it would be useful to have a definition of special Kähler geometry that does not refer to the constructions of supersymmetric actions. The text of the proceedings was a first step in that direction. Meanwhile, in 1994, the second superstring revolution took place. The main issue was that theories which were previously thought as different, are recognized as perturbations around 'vacua' of a master theory. Essential for that are the duality relations which make the connections between the different descriptions. The first example was provided by Seiberg and Witten [4]. They used a model with $N = 2$ supersymmetry in 4 dimensions with vector multiplets, being multiplets involving Maxwell fields. Special Kähler geometry [5] is defined by the couplings of the scalars in the locally supersymmetric theory, i.e. in the coupled Einstein–Maxwell theory. The model used by Seiberg–Witten thus involves a similar geometry, which has been called rigid special Kähler geometry [6], as it appears in rigid supersymmetry. The structure of that geometry was important for the obtained results. In particular the analyticity properties of fields in these theories allowed them to find exact solutions.

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[1]V. Cortés made our results more accessible to the mathematical audience [3]

The so-called vector multiplets in $d = 4$, $N = 2$ supersymmetry are multiplets with spins $(0, 0, \frac{1}{2}, \frac{1}{2}, 1)$, the latter being the vector providing the Maxwell theory. The scalars are moduli, whose values parametrize the different vacua. The two (real) scalars in a multiplet can be combined to a complex one, and the supersymmetry will indeed provide a complex structure. As will become clear below, the structure of special Kähler geometry implies holomorphicity of the resulting field equations. Then the result of Seiberg–Witten is based on the fact that singularities and the asymptotic behaviour determine exact answers. The singularities, see figure 1, are points around

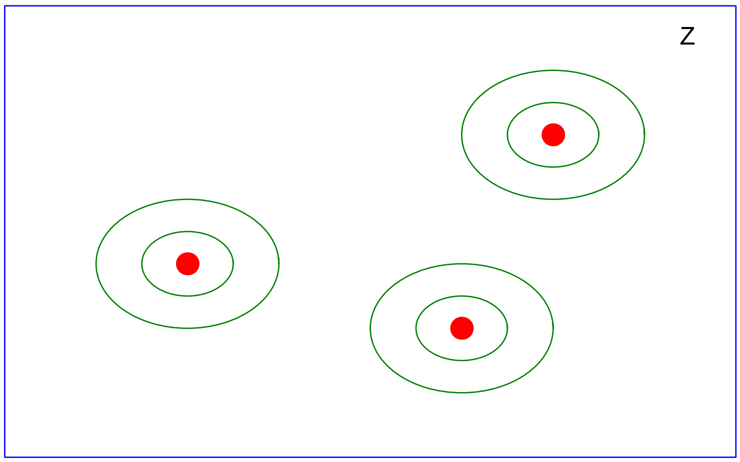

FIGURE 1. *The moduli space with 3 singularities.*

which a classical limit can be considered. The theory allows perturbation expansions around these points. Each one leads classically to a different theory, but there is only one full quantum theory. The singular points form a family of inequivalent vacua.

These developments motivated us to look for a definition of special geometry independent of supersymmetry. A first step in that direction had meanwhile be taken by Strominger [7]. He had in mind the moduli spaces of Calabi–Yau spaces. His definition is already based on the symplectic structure, which we also have emphasized. However, being already in the context of Calabi–Yau moduli spaces, his definition of special Kähler geometry omitted some ingredients that are automatically present in any Calabi–Yau moduli space, but have to be included as necessary ingredients in a generic definition. Another important step was obtained in [8]. Before, special geometry was connected to the existence of a holomorphic prepotential function $F(z)$. The special Kähler manifolds were recognized as those for which the Kähler potential can be determined by this prepotential, in a way to be described below. However, in [8] it was found that one can have $N = 2$ supergravity models coupled to Maxwell multiplets such that there is no such prepotential. These models were constructed by applying a symplectic transformation to a model with prepotential. This fact raised new questions: are all the models without prepotential symplectic dual to models with a prepotential? Can one still define special Kähler geometry starting from the models with a prepotential? Is there a more convenient definition which does not involve this prepotential? These questions have been answered in [9], and are reviewed here.

Section 2 introduces some ingredients. I give some elements of the algebraic context of $N = 2$ supersymmetry, and how the geometric quantities are encoded in the action. Then I show the emergence of symplectic transformations in the actions with vector fields coupled to scalars. Rigid $N = 2$ supersymmetry and the associated rigid special Kähler geometry is discussed in section 3. Section 4 will then discuss the supergravity case. For that, it is useful to look first at the conformal group, as a formulation from that perspective will show more structure, in particular it clarifies the role of the symplectic transformations, and gives the connection with Sasakian manifolds. This is the central section where the definitions, their equivalence and some examples are discussed. The special Kähler manifolds appear in moduli spaces of Riemann surfaces for the rigid version and in those of Calabi–Yau manifolds for the local version. That is illustrated in section 5. A summary is given in section 6. We briefly discuss there also the usage of the same construction methods for quaternionic geometry as recently applied in [10].

2. Ingredients

For supersymmetry in 4-dimensional spacetime, the fermionic charges should belong to a spinor representation of $SO(3,1)$. Therefore, in the minimal supersymmetric case, the supercharges have 4 real components. This minimal situation is called $N = 1$. Field theory allows realizations up to $N = 8$ supersymmetry, i.e. with 32 real supercharges. Special Kähler geometry appears in the context of $N = 2$ supersymmetry. The 8 real spinor supercharges are denoted as Q^i_α, where $\alpha = 1, \ldots, 4$ and $i = 1, 2$. They satisfy the anticommutation rule

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \gamma^\mu_{\alpha\beta} P_\mu \delta^{ij}\,, \tag{2.1}$$

thus involving the translation operator P_μ in 4-dimensional spacetime. There are representations with spins

$$\begin{array}{rll} (0,0,0,0,\frac{1}{2},\frac{1}{2}): & \text{hypermultiplet} & \text{quaternionic scalars} \\ (0,0,\frac{1}{2},\frac{1}{2},1): & \text{vector multiplet} & \text{complex scalars} \\ (1,\frac{3}{2},\frac{3}{2},2): & \text{supergravity} & , \end{array} \tag{2.2}$$

where I have indicated their names and the types of scalars. The quaternionic and complex structures are guaranteed by the supersymmetry.

The ingredients of the geometry are found in the action. In general, having scalars $\phi^i(x)$, vectors with field strength $\mathcal{F}^I_{\mu\nu}(x)$, and possibly a non-trivial spacetime metric

$g_{\mu\nu}(x)$, the bosonic kinetic part of the action has the general form

$$\begin{aligned} S &= \int d^4x \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j G_{ij}(\phi) \\ &\quad + \tfrac{1}{4} \sqrt{g} g^{\mu\rho} g^{\nu\sigma} (\Im \mathcal{N}_{IJ})(\phi) \mathcal{F}^I_{\mu\nu} \mathcal{F}^J_{\rho\sigma} - i8 (\Re \mathcal{N}_{IJ})(\phi) \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}^I_{\mu\nu} \mathcal{F}^J_{\rho\sigma} \\ &\quad + \ldots . \end{aligned} \tag{2.3}$$

$G_{ij}(\phi)$ is identified as the metric of the manifold of scalars. The complex symmetric matrix $\mathcal{N}_{IJ}$ determines the kinetic terms of the vectors, and its meaning will be clarified below.

Supersymmetry relates bosons and fermions, e.g. for the scalars

$$\delta \phi^i(x) = \bar{\epsilon}\; \chi^i(x) \,, \tag{2.4}$$

where ϵ are the supersymmetry parameters and $\chi^i(x)$ are the fermions. In the context of local supersymmetry the parameters depend on spacetime, and we thus have

$$\delta \phi^i(x) = \overline{\epsilon(x)}\, \chi^i(x) \,. \tag{2.5}$$

In order to have an action invariant under these local symmetries, one needs connection fields, which are the gravitini for the supersymmetry. Due to the algebra (2.1) this should be related to local translations, i.e. general coordinate transformations, whose connection field is the (spin 2) graviton.

A prerequisite to understand the following development, is the understanding of the meaning of the symplectic transformations. These are the duality symmetries of 4 dimensions, the generalizations of the Maxwell dualities. They were first discussed in [11]. Consider the kinetic terms of the vector fields as in (2.3) with $I = 1, ..., m$. $\mathcal{N}_{IJ}$ are coupling constants or functions of scalars. One defines (anti)selfdual combinations as

$$\mathcal{F}^{\pm}_{\mu\nu} = \tfrac{1}{2} \left(\mathcal{F}_{\mu\nu} \pm \tfrac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma} \right) \,. \tag{2.6}$$

The conventions[2] are such that the complex conjugate of $\mathcal{F}^+$ is $\mathcal{F}^-$. Defining

$$G^{\mu\nu}_{+I} \equiv 2i \frac{\partial \mathcal{L}}{\partial \mathcal{F}^{+I}_{\mu\nu}} = \mathcal{N}_{IJ} \mathcal{F}^{+J\,\mu\nu} \,, \tag{2.7}$$

the Bianchi identities and field equations can be written as

$$\begin{aligned} \partial^\mu \Im \mathcal{F}^{+I}_{\mu\nu} &= 0 \qquad \text{Bianchi identities} \\ \partial_\mu \Im G^{\mu\nu}_{+I} &= 0 \qquad \text{Equations of motion.} \end{aligned} \tag{2.8}$$

This set of equations is invariant under $GL(2m, \mathbb{R})$:

$$\begin{pmatrix} \widetilde{\mathcal{F}}^+ \\ \widetilde{G}_+ \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} \,. \tag{2.9}$$

[2] The Levi–Civita symbol has $\varepsilon_{0123} = i$.

In order that this transformation be consistent with (2.7), we should have

$$\widetilde{G}^+ = (C + D\mathcal{N})\mathcal{F}^+ = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\widetilde{\mathcal{F}}^+$$

$$\to \boxed{\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}} \quad . \tag{2.10}$$

However, this matrix should remain symmetric, $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}^T$, which implies that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m, \mathbb{R}), \tag{2.11}$$

as the explicit condition is

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \qquad \text{where} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{2.12}$$

Thus the remaining transformations are real symplectic ones in dimension $2m$, where m is the number of vector fields.

In the following we will denote by symplectic vectors, those vectors V such that its symplectic transformed is $\widetilde{V} = \mathcal{S}V$. The prime example is thus $V = \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix}$. An invariant inner product of symplectic vectors is defined by

$$\langle V, W \rangle \equiv V^T \Omega W . \tag{2.13}$$

The important properties for the matrix $\mathcal{N}$ is that it should be symmetric and $\Im\mathcal{N} < 0$ in order to have positive kinetic terms. These properties are preserved under symplectic transformations defined by (2.10).

3. Rigid special Kähler geometry

As mentioned in the introduction, the 'rigid' special Kähler geometry is the geometric structure encountered in rigid $N = 2$ supersymmetry in 4 dimensions. This supersymmetry has as field representations multiplets with spins $(0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$, the hypermultiplet, and multiplets with spin $(0, 0, \frac{1}{2}, \frac{1}{2}, 1)$, the vector multiplet. For the former, the scalar field geometry is based on quaternions, and is a hyper-Kähler structure. Here, we will consider the vector multiplets, for which the scalars combine to complex fields, whose geometry is Kähler ian. A natural description for such multiplets uses $N = 2$ superspace, that is an extension of usual spacetime (with points labelled by x) by fermionic coordinates θ, such that the superspace is a representation of the superalgebra. The vector multiplets are then described by superfields $\Phi^A(x, \theta)$ that satisfy some constraints, restricting the way in which they depend on the θ. The result is some superfield

$$\Phi^A(x, \theta) = X^A(x) + \bar{\theta}\chi^A_\alpha(x) + \bar{\theta}\gamma^{\mu\nu}\theta\mathcal{F}_{\mu\nu}(x) + \dots , \tag{3.1}$$

where the lowest components X^A are complex fields. $A = 1, \dots, n$ labels different vector multiplets. To build an action, one integrates a general holomorphic function F over one half of the θ variables (the chiral superspace). The above mentioned constraints have, between other restrictions, restricted the superfields to depend only on this chiral superspace. With

$$S = \int d^4x \int d^4\theta\; F(\Phi)\; + c.c.\,, \tag{3.2}$$

one obtains that the scalars have a metric of Kähler ian type:

$$\begin{aligned} G_{A\bar B}(X,\bar X) &= \partial_A\partial_{\bar B}K(X,\bar X)\\ K(X,\bar X) &= i(\bar F_A(\bar X)X^A - F_A(X)\bar X^A)\\ \mathcal{N}_{AB} &= F_{AB}\,, \end{aligned} \tag{3.3}$$

where the latter defines the kinetic term of the vectors as in (2.3). Further, $F_A(X) = \frac{\partial}{\partial X^A}F(X)$ or $\bar F_A(\bar X) = \frac{\partial}{\partial \bar X^A}\bar F(\bar X)$, $F_{AB}(X) = \frac{\partial}{\partial X^A}\frac{\partial}{\partial X^B}F(X)$.

The equations of motion turn out to be those equations that determine that $F_A(\Phi)$ satisfy the same superfield constraints as Φ^A. Comparing with (2.8), the superfield constraints on Φ^A contain the first equations (Bianchi identities) while the same superfield equations on F_A contain the second line.

It is therefore appropriate to combine the superfield in a 'symplectic vector'

$$\tilde\Phi = \begin{pmatrix}\Phi^A\\ F_A(\Phi)\end{pmatrix} \qquad \text{chiral superfields which satisfy extra constraints.} \tag{3.4}$$

The scalars, i.e. the $\theta = 0$ part of this vector form also a symplectic vector

$$V \equiv \begin{pmatrix}X^A\\ F_A(X)\end{pmatrix} \qquad \text{is a symplectic vector.} \tag{3.5}$$

A further improvement is to allow general coordinates. So far, we parametrize the scalars as X^A, which are special coordinates (occurring in the superfields). We can, however, allow arbitrary coordinates [12] z^α with $\alpha = 1,\ldots,n$. Then the special coordinates are holomorphic functions of the z^α, i.e. $X^A(z^\alpha)$, such that $e^A_\alpha \equiv \partial_\alpha X^A(z)$ is invertible.

Now we have all the ingredients to give definitions [9].

Definition 1 of rigid special Kähler geometry.

A rigid special Kähler manifold is an n-dimensional Kähler manifold with on any chart n holomorphic functions $X^A(z)$ and a holomorphic function $F(X)$ such that

$$K(z,\bar z) = i\left(X^A\frac{\partial}{\partial \bar X^A}\bar F(\bar X) - \bar X^A\frac{\partial}{\partial X^A}F(X)\right). \tag{3.6}$$

On overlap of charts these functions should be related by (inhomogeneous) symplectic transformations $ISp(2n,\mathbb{R})$:

$$\begin{pmatrix}X\\ \partial F\end{pmatrix}_{(i)} = e^{ic_{ij}}M_{ij}\begin{pmatrix}X\\ \partial F\end{pmatrix}_{(j)} + b_{ij}\,, \tag{3.7}$$

with

$$c_{ij} \in \mathbb{R} \; ; \qquad M_{ij} \in Sp(2n,\mathbb{R}) \; ; \qquad b_{ij} \in \mathbb{C}^{2n} \, , \tag{3.8}$$

satisfying the cocycle condition on overlaps of 3 charts.

There is, however, a second definition of rigid special Kähler manifolds, which is based on the symplectic structure, rather than on the prepotential.

Definition 2 of rigid special Kähler geometry.

A Kähler manifold is the base manifold of a $U(1) \times ISp(2n,\mathbb{R})$ bundle. A holomorphic section $V(z)$ defines the Kähler potential by

$$K(z,\bar{z}) = i\langle V, \bar{V}\rangle \, , \tag{3.9}$$

and it should satisfy the constraint

$$\langle \partial_\alpha V, \partial_\beta V\rangle = 0 \, . \tag{3.10}$$

One can show that the prepotential exists locally, but it is thus not essential for the definition. In rigid special geometry the choice of definition is rather a question of esthetics. However, in the local case, it will be important to have the analogue of the second definition available. The kinetic matrix for the vectors is

$$\mathcal{N}_{AB} = (\partial_\alpha F_A(z))\, e_B^\alpha \, . \tag{3.11}$$

The condition (3.10) guarantees that this matrix is symmetric. Finally, let us remark that the symplectic metric Ω should in general not assume the canonical form (2.12), but can be an arbitrary non-degenerate real antisymmetric matrix. However, in order to distinguish X^A and F_A components, and thus to write a prepotential, one should bring it to this canonical form.

4. $N = 2$ SUPERGRAVITY AND SPECIAL KÄHLER GEOMETRY

In this section we introduce the 'local' special Kähler geometry, which is the one generally denoted as special Kähler geometry. It is this one which was found in [5], and has most interesting applications. It was introduced in the context of supergravity. To explain its structure, it is useful to consider again its origin.

To describe a supergravity theory, there are several methods. One of them is the introduction of a **superspace**. This formalism shows a lot of structure of the theory. It is very transparent for rigid supersymmetry. However, in its local version, necessary for supergravity, there appear a lot of extra superfield symmetries. These symmetries are an artifact of the formalism. They have to be gauge-fixed to obtain the physical theory.

Superconformal tensor calculus is in-between. Also here extra gauge symmetries occur, and these are in fact the symmetries of the superconformal group, the basic ingredient of the formalism. The experience tells us that these symmetries are the relevant ones to display the structure of the theory, but this formalism does not have the many other symmetries present in the superspace approach. It turns out that we just remain with those that are useful to get insight in complicated formulae. Also for the calculation of the action, the superconformal symmetries are just appropriate to simplify the construction. This is particularly interesting in our case. The superconformal tensor calculus gives the proper setup for the symplectic (duality-adapted) formulation.

The idea is to start by constructing an action invariant under superconformal group. Then, one choose gauges for the extra gauge invariances of the superconformal group, such that the remaining theory has just the super-Poincaré symmetries.

The formalism can be used for theories in various dimensions and amount of supersymmetry. Let us review here the structure for 4 dimensions with 8 real supersymmetry generators ($N = 2$). The superconformal group contains first of all the conformal group (translations, Lorentz rotations, dilatations and special conformal transformations). This group is $S0(4,2) = SU(2,2)$. The supersymmetries should sit in a spinor representation of this group. This singles out the supergroup $SU(2,2|2)$, which means essentially that the group can be represented by matrices of the form

$$\begin{pmatrix} SU(2,2) & SUSY \\ SUSY & SU(2)\times U(1) \end{pmatrix} . \tag{4.1}$$

The off-diagonal blocks are the fermionic symmetries. The diagonal blocks are the bosonic ones. They split up in the above-mentioned conformal group and an 'R-symmetry group', $SU(2)\times U(1)$. This extra group plays an important role:

- the gauge connection of $U(1)$ will be the Kähler curvature. It acts on the manifold of scalars in vector multiplets,
- the gauge connection of $SU(2)$ promotes the hyperKähler manifold of hypermultiplets to a quaternionic manifold.

As we neglect here the hypermultiplets, we have to consider the basic supergravity multiplet and the vector multiplets. The physical content that one should have (from representation theory of the super-Poincaré group) can be represented as follows:

$$\begin{array}{ccccccccc} & SUGRA & & & & & vectorm. & & \\ & 2 & & & & & & & \\ \frac{3}{2} & & \frac{3}{2} & & & & & & \\ & 1 & & & & & 1 & & \rightarrow n+1 \\ & & & +n\ * & \frac{1}{2} & & & \frac{1}{2} & \\ & & & & 0 & & & 0 & \end{array} \tag{4.2}$$

The supergravity sector contains the graviton, 2 gravitini and a so-called graviphoton. That spin-1 field gets, by coupling to n vector multiplets, part of a set of $n+1$ vectors, which will be uniformly described by the special Kähler geometry. The scalars appear as n complex ones z^α, with $\alpha = 1, \ldots, n$.

To describe this, we start with $n+1$ superconformal vector multiplets with scalars X^I with $I = 0, \ldots, n$. The action is determined by a holomorphic function $F(X)$. Compared with the rigid case, there is one additional requirement. The conformal invariance requires $F(X)$ to be homogeneous of weight 2, where the X fields carry weight 1. These scalar fields transform also under a local $U(1)$ symmetry.

The obtained metric is a cone [13, 10]. To see this, one splits the $n+1$ complex variables $\{X\}$ in $\{\rho, \theta, z^\alpha\}$

- r is scale which is a gauge degree of freedom for translations
- θ is the $U(1)$ degree of freedom;
- the n complex variables z^α.

The metric now takes the form

$$\begin{aligned} ds^2 \;=\; & dr^2 + \tfrac{1}{18} r^2 \left[A + d\theta + i\left(\partial_\alpha K(z,\bar z)\, dz^\alpha - \partial_{\bar\alpha} K(z,\bar z)\, d\bar z^{\bar\alpha}\right)\right]^2 + \\ & r^2 \partial_\alpha \partial_{\bar\alpha} K(z,\bar z)\, dz^\alpha d\bar z^{\bar\alpha}\,, \end{aligned} \tag{4.3}$$

where A is the one-form gauging the $U(1)$ group, and $K(z, \bar z)$ is a function of the holomorphic prepotential $F(X)$, to be explained below. With $A = 0$, this defines the cone over a Sasakian manifold. However, in supergravity, the field equation of A implies that it is a composite field, given by (minus) the other parts of the second term of (4.3). With fixed ρ (gauge fixing the superfluous dilatations), the remaining manifold is Kähler, with the Kähler potential determined by $F(X)$. That gives the special Kähler metric.

Let us explain this now in more detail, using at the same time more of the symplectic formalism. The dilatational gauge fixing (the fixing of r above), is done by the condition

$$X^I \bar F_I(\bar X) - \bar X^I F_I(X) = i\,. \tag{4.4}$$

This condition is chosen in order to decouple kinetic terms of the graviton from those of the scalars. Using again symplectic vectors

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \tag{4.5}$$

this can be written as the condition on the symplectic inner product:

$$< V, \bar V >= i\,. \tag{4.6}$$

To solve this condition, we define

$$V = e^{K(z,\bar z)/2} v(z)\,, \tag{4.7}$$

where $v(z)$ is a holomorphic symplectic vector,

$$v(z) = \begin{pmatrix} Z^I(z) \\ \frac{\partial}{\partial Z^I} F(Z) \end{pmatrix}. \tag{4.8}$$

The upper components here are arbitrary functions (up to conditions for non-degeneracy), reflecting the freedom of choice of coordinates z^α. The Kähler potential is

$$e^{-K(z,\bar z)} = -i \langle v, \bar v \rangle\,. \tag{4.9}$$

The kinetic matrix for the vectors is given by

$$\mathcal{N}_{IJ} = \begin{pmatrix} F_I & \mathcal{D}_{\bar\alpha}\bar F_I(\bar X) \end{pmatrix} \begin{pmatrix} X^J & \mathcal{D}_{\bar\alpha}\bar X^J \end{pmatrix}^{-1}, \tag{4.10}$$

where the matrices are $(n+1)\times(n+1)$ and

$$\mathcal{D}_{\bar\alpha}\bar F_I(\bar X) = \partial_{\bar\alpha}\bar F_I(\bar X) + \tfrac{1}{2}(\partial_{\bar\alpha}K)\bar F_I(\bar X), \qquad \mathcal{D}_{\bar\alpha}\bar X^J = \partial_{\bar\alpha}\bar X^J + \tfrac{1}{2}(\partial_{\bar\alpha}K)\bar X^J. \tag{4.11}$$

Before continuing with general statements, it is time for an example. Consider the prepotential $F = -iX^0X^1$. This is a model with $n = 1$. There is thus just one coordinate z. One has to choose a parametrization to be used in the upper part of (4.8). Let us take a simple choice: $Z^0 = 1$ and $Z^1 = z$. The full symplectic vector is then (as e.g. $F_0(Z) = -iZ^1(z)$)

$$v = \begin{pmatrix} Z^0 \\ Z^1 \\ F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix}. \tag{4.12}$$

The Kähler potential is then directly obtained from (4.9), determining the metric:

$$e^{-K} = 2(z+\bar z)\;; \qquad g_{z\bar z} = \partial_z\partial_{\bar z}K = (z+\bar z)^{-2}. \tag{4.13}$$

The kinetic matrix for the vectors is diagonal. From (4.10) follows

$$\mathcal{N} = \begin{pmatrix} -iz & 0 \\ 0 & -i\frac{1}{z} \end{pmatrix}. \tag{4.14}$$

Therefore the action contains

$$e^{-1}\mathcal{L}_1 = -\tfrac{1}{2}\Re\left[z\left(F^{+0}_{\mu\nu}\right)^2 + z^{-1}\left(F^{+1}_{\mu\nu}\right)^2\right]. \tag{4.15}$$

The domain of positivity for both metrics is $\Re z > 0$.

We formulate again two definitions, the first using the prepotential, and the second one using only the symplectic vectors.

Definition 1 of (local) special Kähler geometry.

A special Kähler manifold is an n-dimensional Hodge-Kähler manifold with on any chart $n+1$ holomorphic functions $Z^I(z)$ and a holomorphic function $F(Z)$, homogeneous of second degree, such that, with (4.8), the Kähler potential is given by

$$e^{-K(z,\bar z)} = -i\langle v, \bar v\rangle, \tag{4.16}$$

and on overlap of charts, the $v(z)$ are connected by symplectic transformations $Sp(2(n+1),\mathbb{R})$ and/or Kähler transformations.

$$v(z) \to e^{f(z)}\mathcal{S}v(z). \tag{4.17}$$

Definition 2 of (local) special Kähler geometry.

A special Kähler manifold is an n-dimensional Kähler–Hodge manifold, that is the base manifold of a $Sp(2(n+1)) \times U(1)$ bundle. There should exist a holomorphic section $v(z)$ such that the Kähler potential can be written as

$$e^{-K(z,\bar{z})} = -i\langle v, \bar{v}\rangle\,, \tag{4.18}$$

and it should satisfy the condition

$$\langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v\rangle = 0\,. \tag{4.19}$$

Note that the latter condition guarantees the symmetry of $\mathcal{N}_{IJ}$. This condition did not appear in [7], where the author had in mind Calabi–Yau manifolds. As we will see below, in those applications, this condition is automatically fulfilled. For $n > 1$ the condition can be replaced by the equivalent condition

$$\langle \mathcal{D}_\alpha v, v\rangle = 0\,. \tag{4.20}$$

For $n = 1$, the condition (4.19) is empty, while (4.20) is not. In [14] it has been shown that models with $n = 1$ not satisfying (4.20) can be formulated.

The appearance of 'Hodge' manifold in the definitions refers to a global requirement. The $U(1)$ curvature should be of even integer cohomology. This has been considered first in [15], and for an explanation on the normalization, one can consult [9]. Note that in the mathematics literature 'Hodge' refers to integer cohomology. Here, however, the presence of fermions makes the condition stronger by a factor of two: one needs even integers.

Let us come back to the example, on which we will perform a symplectic mapping:

$$\widetilde{v} = \mathcal{S}v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ i \\ -iz \\ z \end{pmatrix}\,. \tag{4.21}$$

After this mapping, z is not any more a good coordinate for $(\widetilde{Z}^0, \widetilde{Z}^1)$, the upper two components of the symplectic vector z. This means that the symplectic vector can not be obtained from a prepotential. We can not obtain the symplectic vector from a form (4.8). No function $\widetilde{F}(\widetilde{Z}^0, \widetilde{Z}^1)$ exists. Therefore, the first definition is not applicable. However, nothing prevents us from using the second definition. The Kähler metric is still the same, (4.13), and one can again compute the vector kinetic matrix, either directly from (4.10), as the denominator is still invertible, or from (2.10):

$$\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} = -iX^1(X^0)^{-1}\mathbb{1} = -iz\mathbb{1}\,. \tag{4.22}$$

In this parametrization, the action is thus

$$e^{-1}\mathcal{L}_1 = -\tfrac{1}{2}\Re\left[z\left(F_{\mu\nu}^{+0}\right)^2 + z\left(F_{\mu\nu}^{+1}\right)^2\right]\,. \tag{4.23}$$

This action is not the same as the one before, but is a 'dual formulation' of the same theory, being obtained from (4.15) by a duality transformation. The straightforward construction in superspace or superconformal tensor calculus does not allow to construct actions without a superpotential. However, in [14] it has been shown that the field equations of these models can also be obtained from the superconformal tensor calculus. One just has to give up the concept of a superconformal invariant action.

It is thus legitimate to ask about the equivalence of the two definitions. Indeed, we saw that in some cases definition 2 is satisfied, but one can not obtain a prepotential F. However, that example, as others in [8], was obtained from performing a symplectic transformation from a formulation where the prepotential does exist. In [9] it was shown that this is true in general. If definition 2 is applicable, then there exists a symplectic transformation to a basis such that $F(Z)$ exists. Note, however, that in the way physical problems are handled, the existence of formulations without prepotentials is important. Going to a dual formulation, one obtains a formulation with different symmetries in perturbation theory. The example that we used here appears in a reduction to $N = 2$ of two versions of $N = 4$ supergravity, known respectively as the '$SO(4)$ formulation' [16] and the '$SU(4)$ formulation' of pure $N = 4$ supergravity [17].

Finally let us note that we still could apply (4.10) because the matrix

$$\begin{pmatrix} X^I & \mathcal{D}_\alpha \bar{X}^I \end{pmatrix} \tag{4.24}$$

is always invertible if the metric $g_{\alpha\bar{\alpha}} = \partial_\alpha \partial_{\bar{\alpha}} K(z,\bar{z})$ is positive definite. Therefore, the inverse exists, and $\mathcal{N}_{IJ}$ can be constructed. However, the matrix

$$\begin{pmatrix} X^I & \mathcal{D}_\alpha X^I \end{pmatrix} \tag{4.25}$$

is not invertible in the formulation (4.21). **If** that matrix is invertible, then a prepotential exists [9].

5. Realizations in moduli spaces of Riemann surfaces and Calabi–Yau manifolds

The realizations of special Kähler geometry that are mostly studied in physics these days, are the moduli spaces of Riemann surfaces for the rigid case, and those of Calabi–Yau 3-folds for the local case.

First, consider the Hodge diamond of Riemann surfaces, listing the number of non-trivial (anti)holomorphic (p,q) forms:

$$\begin{array}{ccc} & h^{00}=1 & \\ h^{10}=g & & h^{01}=g \\ & h^{11}=1 & \end{array}$$

Rigid special Kähler geometry is obtained for the moduli spaces of such Riemann surfaces when we consider

- with n complex moduli z^α
- $n \leq g$ holomorphic 1-forms γ_α $(\alpha = 1, \dots n)$
- $2n$ cycles c_Λ that form a complete basis for 1-cycles for which $\int_c \gamma_\alpha \neq 0$.

In this situation

$$\gamma_\alpha(z) = \partial_\alpha \lambda(z) + d\eta_\alpha(z)\,, \tag{5.1}$$

where $\lambda(z)$ is a meromorphic 1-form with zero residues. The symplectic formulation of rigid special Kähler geometry is obtained with as symplectic vector the vector of periods of λ over the chosen cycles:

$$V = \int_{c_\Lambda} \lambda\,. \tag{5.2}$$

The intersection matrix of the cycles plays the role of the symplectic metric. This type of realizations was used in Seiberg–Witten models. The general features have been discussed in [9].

To obtain local special Kähler manifolds, one considers the moduli space of Calabi–Yau 3-folds. In this case the Hodge diamond of the manifold is

$$\begin{array}{ccccccc} & & & h^{00}=1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11}=m & & 0 & \\ h^{30}=1 & & h^{21}=n & & h^{12}=n & & h^{03}=1 \\ & 0 & & h^{22}=m & & 0 & \\ & & 0 & & 0 & & \\ & & & h^{33}=1 & & & \end{array}$$

These manifolds have $h^{21} = n$ complex structure moduli, which play the role of the variables z^α of the previous section. There are $2(n+1)$ 3-cycles c_Λ, with intersection matrix $Q_{\Lambda\Sigma} = c_\Lambda \cap c_\Sigma$. The canonical form is obtained with so-called A and B cycles, and then Q takes the form of Ω in (2.12). Symplectic vectors are identified again as vectors of integrals over the $2(n+1)$ 3-cycles:

$$v = \int_{c_\Lambda} \Omega^{(3,0)}\,, \qquad \mathcal{D}_\alpha v = \int_{c_\Lambda} \Omega^{(2,1)}_\alpha\,. \tag{5.3}$$

$\Omega^{(3,0)}$ is the unique $(3,0)$ form that characterizes the Calabi–Yau manifold. $\Omega^{(2,1)}_\alpha$ is a basis of the $(2,1)$ forms, determined by the choice of basis for z^α. That these moduli spaces give rise to special Kähler geometry became clear in [18]. Details on the relation between the geometric quantities and the fundamentals of special Kähler geometry have been discussed in [19, 9].

The defining equations of special Kähler geometry are automatically satisfied. E.g. one can easily see how the crucial equation (4.19) is realized:

$$\begin{aligned} \langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v\rangle &= \int_{c_\Lambda} \Omega^{(2,1)}_{(\alpha)} \cdot Q^{\Lambda\Sigma} \cdot \int_{c_\Sigma} \Omega^{(2,1)}_{(\beta)} \\ &= \int_{CY} \Omega^{(2,1)}_{(\alpha)} \wedge \Omega^{(2,1)}_{(\beta)} = 0\,. \end{aligned} \tag{5.4}$$

The symplectic transformations correspond now to changes of the basis of the cycles used to construct the symplectic vectors. The statement that a formulation with a prepotential can always be obtained in special Kähler geometry by using a symplectic transformation, can now be translated to the statement that the geometry can be obtained from a prepotential for some choice of cycles.

Finally, it is interesting that singularities of Calabi–Yau manifolds may be used to obtain a 'rigid limit'. Indeed, in [20] it is shown how Calabi–Yau manifolds that are $K3$ fibrations can be reduced near the singularity to fibrations of ALE manifolds. Then the special geometry of the moduli space of the Calabi–Yau manifold reduces to the rigid special geometry with Kähler potential determined by the ALE manifold. This mechanism is considered further in [21]. There it has been shown how the Kähler potential of special geometry approaches the one of rigid special geometry, and how the periods of the local theory behave around the singular points and thus around the rigid limit. In the superstring theory this allows to get the gravity corrections to the rigid theory, which can be used for applications [22].

6. Summary and connection with quaternionic manifolds

Special Kähler geometry is defined by the couplings of $N = 2$ supersymmetric theories ('rigid' special Kähler) or supergravity theories ((local) special Kähler) with vector multiplets. There are several ways to describe the geometry. We discussed two ways:

- by using a prepotential function
- by symplectic vectors and constraints

In *rigid* special Kähler geometry, these are completely equivalent. In the *local* theory, all special Kähler manifolds can be obtained from a prepotential, but in some cases that involves a duality transformation. Therefore not all actions can be described by the prepotential.

Rigid special Kähler geometry is realised by moduli spaces of certain Riemann manifold. That construction is not straightforward, and involves a choice of cohomology subspace and moduli. The *local* special Kähler geometry appears in the moduli space of Calabi–Yau threefolds. In this case the construction is straightforward. For a particular Calabi–Yau manifold one includes all the moduli. In this way a clear geometrical interpretation of the building blocks of special geometry is obtained. Duality transformations correspond then to a change of the basis of cycles. A prepotential does exist at least for a suitable choice of basis of the cycles.

Note, however, that not all special Kähler manifolds can be obtained as realizations in moduli spaces. E.g. the homogeneous manifolds, treated in [1, 2] are not obtained in this way.

In the First Meeting on Quaternionic Structures in Mathematics and Physics, 5 years ago, we have shown [1, 2] how homogeneous special Kähler spaces are related by the c-map to homogeneous quaternionic spaces and by the r-map to homogeneous 'very special' real spaces. The construction of special Kähler geometry that we have outlined here can be used as well for the quaternionic spaces (and for the real ones). In a recent work [10] it has been shown how the conformal tensor calculus can be applied to obtain the actions based on the quaternionic spaces (actions for 'hypermultiplets'). The scalars are the lowest components of superfields (or superconformal multiplets) A_i^α, with $i = 1, 2$ and $\alpha = 1, \ldots, 2(r+1)$ with a reality condition. The A_i^α can be considered as $Sp(1) \times Sp(r+1)$ sections. Again the number of multiplets $(r+1)$ is one more than the number of physical multiplets (r) that we will obtain. We thus start with $4(r+1)$ scalars. One of those will be a scale degree of freedom[3], three are $SU(2)$ degrees of freedom, the second part of the R-symmetry as was mentioned after (4.1), and the remaining ones form r quaternions. As in the metric of the vector multiplets, there is a connection to Sasakian manifolds. Putting the gauge fields of the $SU(2)$ invariance to zero, rather than using their field equations, one obtains a 3-Sasakian manifold. This is related to the talk of Galicki in the meeting 5 years ago [23].

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References

1. B. de Wit and A. Van Proeyen, *Isometries of special manifolds*, in the proceedings of the Meeting on Quaternionic Structures in Mathematics and Physics, Trieste, September 1994; ILAS/FM-6/1996, p.109; available on `http://www.emis.de/proceedings/QSMP94/` and hep-th/9505097.
2. B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces,* Commun. Math. Phys. **149** (1992) 307 [hep-th/9112027];
   B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries,* Nucl. Phys. **B400** (1993) 463 [hep-th/9210068];
   B. de Wit and A. Van Proeyen, *Hidden symmetries, special geometry and quaternionic manifolds,* Int. J. Mod. Phys. **D3** (1994) 31 [hep-th/9310067].
3. V. Cortés, *Homogeneous special geometry,* Transform. Groups **1** (1996) 337 [math.DG/9602213].
4. N. Seiberg, E. Witten, *Electric–magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang–Mills theory*, Nucl. Phys. **B426** (1994) 19 [hep-th/9407087];
   N. Seiberg, E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484 [hep-th/9408099].

[3] When vector multiplets and hypermultiplets are simultaneously coupled, there is one overall dilatational gauge degree of freedom. An auxiliary field of the superconformal gauge multiplet gives a second relation, such that as well the compensating field of the vector multiplet as that of the hypermultiplet are fixed.

5. B. de Wit and A. Van Proeyen, *Potential and symmetries of general gauged $N = 2$ supergravity–Yang–Mills models*, Nucl. Phys. **B245** (1984) 89.
6. G. Sierra and P.K. Townsend, *An introduction to $N = 2$ rigid supersymmetry*, in *Supersymmetry and Supergravity 1983*, Proceedings of the XIXth winter school and workshop of theoretical physics Karpacz, Poland, ed. B. Milewski (World Scientific, Singapore 1983);
   S. J. Gates, *Superspace formulation of new nonlinear sigma models*, Nucl. Phys. **B238** (1984) 349.
7. A. Strominger, *Special geometry*, Commun. Math. Phys. **133** (1990), 163.
8. A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang–Mills theories coupled to supergravity*, Nucl. Phys. **B444** (1995) 92 [hep-th/9502072].
9. B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special Kähler geometry?,* Nucl. Phys. **B503** (1997) 565 [hep-th/9703082].
10. B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets,* hep-th/9909228.
11. S. Ferrara, J. Scherk and B. Zumino, *Algebraic properties of extended supergravity theories*, Nucl. Phys. **B121** (1977) 393;
    B. de Wit, *Properties of $SO(8)$ extended supergravity*, Nucl. Phys. **B158** (1979) 189;
    E. Cremmer and B. Julia, *The $SO(8)$ supergravity*, Nucl. Phys. **B159** (1979) 141;
    M.K. Gaillard and B. Zumino, *Duality rotations for interacting fields*, Nucl. Phys. **B193** (1981) 221.
12. L. Castellani, R. D'Auria and S. Ferrara, *Special Kähler geometry: an intrinsic formulation from $N = 2$ space–time supersymmetry,* Phys. Lett. **B241** (1990) 57;
    L. Castellani, R. D' Auria and S. Ferrara, *Special geometry without special coordinates,* Class. Quant. Grav. **7** (1990) 1767;
    R. D'Auria, S. Ferrara and P. Frè, *Special and quaternionic isometries: General couplings in $N = 2$ supergravity and the scalar potential,* Nucl. Phys. **B359** (1991) 705.
13. G.W. Gibbons and P. Rychenkova, *Cones, tri-Sasakian structures and superconformal invariance*, Phys. Lett. **B443** (1998) 138 [hep-th/9809158].
14. P. Claus, K. Van Hoof and A. Van Proeyen, *A symplectic covariant formulation of special Kähler geometry in superconformal calculus*, Class. Quantum Grav. **16** (1999) 2625 [hep-th/9904066].
15. E. Witten and J. Bagger, *Quantization of Newton's constant in certain supergravity theories*, Phys. Lett. **115B** (1982) 202;
    J. Bagger, *Supersymmetric sigma models*, in *Supersymmetry*, (NATO Advanced Study Institute, Series B: Physics, v. 125), ed. K. Dietz et al., (Plenum Press, 1985).
16. A. Das, *$SO(4)$ invariant extended supergravity,* Phys. Rev. **D15** (1977) 2805;
    E. Cremmer, J. Scherk and S. Ferrara, *$U(N)$ invariance in extended supergravity,* Phys. Lett. **B68** (1977) 234;
    E. Cremmer and J. Scherk, *Algebraic simplifications in supergravity theories,* Nucl. Phys. **B127** (1977) 259.
17. E. Cremmer, J. Scherk and S. Ferrara, *$SU(4)$ invariant supergravity theory,* Phys. Lett. **B74** (1978) 61.
18. N. Seiberg, *Observations on the moduli space of superconformal field theories,* Nucl. Phys. **B303** (1988) 286;
    S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories,* Int. J. Mod. Phys. **A4** (1989) 2475;
    S. Ferrara and A. Strominger, *$N = 2$ space–time supersymmetry and Calabi–Yau moduli space,* in *Strings '89*, eds. R. Arnowitt, R. Bryan, M.J. Duff, D.V. Nanopoulos and C.N. Pope (World Scientific, Singapore, 1989), p. 245;

P. Candelas and X. de la Ossa, *Moduli space of Calabi–Yau manifolds,* Nucl. Phys. **B355** (1991) 455;
A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, Nucl. Phys. **B359** (1991) 21;
P. Candelas, X. C. De la Ossa, P. S. Green and L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds,* Phys. Lett. **B258** (1991) 118.

19. P. Frè and P. Soriani, *The $N = 2$ wonderland, from Calabi–Yau manifolds to topological field–theories,* (World Scientific, Singapore, 1995).
20. S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative results on the point particle limit of $N = 2$ heterotic string compactifications,* Nucl. Phys. **B459** (1996) 537 [hep-th/9508155];
A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory,* Nucl. Phys. **B477** (1996) 746 [hep-th/9604034];
W. Lerche, *Introduction to Seiberg–Witten theory and its stringy origin,* in *Gauge theories, applied supersymmetry, quantum gravity*, eds. B. de Wit et al., Leuven notes in mathematical physics, B6, 1996, p. 53; Nucl. Phys. Proc. Suppl. **55B** (1997) 83; Fortsch. Phys. 45 (1997) [hep-th/9611190];
A. Klemm, *On the geometry behind $N = 2$ supersymmetric effective actions in four dimensions,* in "1996 summer school in High Energy Physics and cosmology", Trieste 1996, World Scientific, p. 120 [hep-th/9705131].
21. M. Billo, F. Denef, P. Fre, I. Pesando, W. Troost, A. Van Proeyen and D. Zanon, *The rigid limit in special Kähler geometry: From K3-fibrations to special Riemann surfaces: A detailed case study,* Class. Quant. Grav. **15** (1998) 2083 [hep-th/9803228];
M. Billo, F. Denef, P. Fre, I. Pesando, W. Troost, A. Van Proeyen and D. Zanon, *Special geometry of Calabi–Yau compactifications near a rigid limit,* Fortsch. Phys. **47** (1999) 133 [hep-th/9801140].
22. S. Cacciatori and D. Zanon, *Gravitational corrections to $N = 2$ supersymmetric Lagrangians,* Phys. Lett. **B444** (1998) 332 [hep-th/9809202];
F. Denef, *Attractors at weak gravity,* Nucl. Phys. **B547** (1999) 201 [hep-th/9812049].
23. C. P. Boyer, K. Galicki and B. M. Mann, *Quaternionic geometry and 3-Sasakian manifolds*, in the proceedings of the Meeting on Quaternionic Structures in Mathematics and Physics, Trieste, September 1994; ILAS/FM-6/1996; available on `http://www.emis.de/proceedings/QSMP94/`;
C. P. Boyer and K. Galicki, *3-Sasakian manifolds,* to appear in '*Essays on Einstein manifolds*', eds. M. Wang and C. Lebrun [hep-th/9810250].

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