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# HYPERHOLOMORPHIC FUNCTIONS IN $\mathbb{R}^4$

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ABSTRACT. Let $\mathbb{H}$ be the algebra of quaternions generated by $e_1, e_2$ and $e_{12}$ satisfying $e_1e_2 = e_{12}$ and $e_ie_j+e_je_i = -2\delta_{ij}$ for $i = 1, 2, 12$. Any element $x$ in $\mathbb{H}\oplus\mathbb{H}$ may be decomposed as $x = Px+Qxe_3$ for quaternions $Px$ and $Qx$. The generalized Cauchy-Riemann operator in $\mathbb{R}^4$ is defined by $D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3$. Leutwiler noticed that the power function $(x_0 + x_1e_1 + x_2e_2 + x_3e_3)^m$ is the solution of the generalized Cauchy-Riemann system $x_3Df + 2f_3 = 0$ which has connections to the hyperbolic metric. We study solutions of the equation $x_3Df + 2Q'(f) = 0$ (the prime $'$ is the main involution) called hyperholomorhic functions. If $f = f_0 + f_1e_1 + f_2e_2 + f_3e_3$ for some real functions $f_0, f_1, f_2,\ f_3$ then $f$ is the solution of the generalized Cauchy-Riemann system stated earlier.

## 1. Introduction

Let $\mathbb{H}$ be the associative algebra of quaternions generated by $e_1$, $e_2$ and $e_{12}$ satisfying the usual relations

$$e_1^2 = e_2^2 = e_{12}^2 = -1$$
$$e_1e_2 = e_{12} = -e_{12}.$$

Set $e_1e_2 = e_{12}$. The conjugation $\overline{q}$ of the quaternion $q = t+xe_1+ye_2+ze_{12}$ is defined by

$$\bar{q} = t - xe_1 - ye_2 - ze_{12}.$$

The involution $' : \mathbb{H} \to \mathbb{H}$ is the isomorphism defined by

$$q' = t - xe_1 - ye_2 + ze_{12}.$$

The second involution $^* : \mathbb{H} \to \mathbb{H}$, called **reversion**, is the anti-isomorphism defined by

$$q^* = t + xe_1 + ye_2 - ze_{12}.$$

We consider the set $\mathbb{H}\oplus\mathbb{H}$ with the usual addition and the multiplication defined by

$$(u_1, u_2)(v_1, v_2) = (u_1v_1 - u_2v_2', u_1v_2 + u_2v_1'). \tag{1}$$

It is known that

$$\mathbb{H}\oplus\mathbb{H} \cong C\ell_{0,3},$$

where $C\ell_{0,3}$ is the Clifford algebra generated by the elements $e_1$, $e_2$ and $e_3$ satisfying the relation $e_ie_j + e_je_i = -2\delta_{ij}$ for the usual Kronecker delta $\delta_{ij}$. The isomorphism $\varphi : \mathbb{H}\oplus\mathbb{H} \to Cl_{0,3}$ is given by $\varphi(q_1, q_2) = q_1 + q_2e_3$. We identify the space $\mathbb{H}\oplus\mathbb{H}$ with $Cl_{0,3}$.

The elements $x = t + xe_1 + ye_2 + we_3$ for $t, x, y, w \in \mathbb{R}$ are called *paravectors* in $\mathbb{H}\oplus\mathbb{H}$. The space $\mathbb{R}^4$ is identified with the set of paravectors. We also denote $e_0 = 1$.

The involution $()'$ is extended to an isomorphism in $\mathbb{H}\oplus\mathbb{H}$ by

$$(q_1 + q_2e_3)' = q_1' - q_2'e_3 \quad (q_i \in \mathbb{H}). \tag{2}$$

Note that

$$e_3q = q'e_3 \tag{3}$$

for any $q \in \mathbb{H}$.

The involution $*$ is extended to $\mathbb{H}\oplus\mathbb{H}$ as follows

$$e_3^* = e_3,\ (ab)^* = b^*a^*$$
$$(a+b)^* = a^* + b^*$$

and the conjugation by $\overline{a} = (a^*)' = (a')^*$. Note that $\overline{ab} = \overline{b}\,\overline{a}$.

Paravectors can be characterized as follows.

**Lemma 1.** *An element $x \in \mathbb{H}\oplus\mathbb{H}$ is a paravector if and only if*

$$\sum_{i=0}^{3} e_ixe_i = -2x'. \tag{4}$$

*Proof.* It is easy to see that the equation (4) holds for all $e_i$ with $i = 0, ..., 3$. Using the linearity we infer that it holds for all paravectors. Conversely, calculate first

$$\sum_{i=0}^{3} e_ie_je_ke_i = 2e_je_k \quad \text{for } 0 < j < k \leq 3$$
$$\sum_{i=0}^{3} e_ie_1e_2e_3e_i = -2e_1e_2e_3.$$

Hence comparing the components of the left and right side of the equality (4) we note that the equality (4) implies that $x$ has to be a paravector. □

The projection operators $P : \mathbb{H}\oplus\mathbb{H}\to\mathbb{H}$ and $Q : \mathbb{H}\oplus\mathbb{H}\to\mathbb{H}$ are defined by equations $P(q_1 + q_2e_3) = q_1$ and $Q(q_1 + q_2e_3) = q_2$ for $q_i \in \mathbb{H}$. Applying (2) we note that $P(a') = (Pa)'$ and $Q(a') = -(Qa)'$. By virtue of (1) we obtain

$$P(ab) = PaPb - Qa(Qb)', \tag{5}$$

$$Q(ab) = PaQb + Qa(Pb)'. \tag{6}$$

The notation $f \in \mathcal{C}^k(\Omega)$ for a function $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ means that the real coordinate functions of $f$ are $k$-times continuously differentiable on $\Omega \subset \mathbb{R}^4$.

## 2. Hyperholomorphic functions

The Cauchy Riemann operator is defined by

$$Df = \sum_{i=0}^{3} e_i \frac{\partial f}{\partial x_i}$$

for a mapping $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$, $\Omega \subset \mathbb{R}^4$, whose components are partially differentiable. An operator $\overline{D}$ is defined by

$$\overline{D}f = \frac{\partial f}{\partial x_0} - \sum_{j=1}^{3} e_j \frac{\partial f}{\partial x_j}.$$

Note that $D\overline{D} = \overline{D}D = \triangle$, where $\triangle$ is the Laplace operator in $\mathbb{R}^4$. If $Df = 0$ the function $f$ is called (left) *monogenic*. For the reference on properties of monogenic functions in the general case see [1] and for quaternions [21].

Using (5) we obtain

$$P(Df) = \sum_{i=0}^{2} e_i \frac{\partial Pf}{\partial x_i} - \frac{\partial Q'f}{\partial x_3}. \tag{7}$$

Similarly, the property (6) implies that

$$Q(Df) = \sum_{i=0}^{2} e_i \frac{\partial Qf}{\partial x_i} + \frac{\partial P'f}{\partial x_3}. \tag{8}$$

The **modified Cauchy-Riemann operator** $M$ is defined by

$$(Mf)(x) = (Df)(x) + \frac{2}{x_3} Q'f$$

and the operator $\overline{M}$ by

$$\left(\overline{M}f\right)(x) = \left(\overline{D}f\right)(x) - \frac{2}{x_3} Q'f.$$

Note that

$$(Mf)(x) + \left(\overline{M}f\right)(x) = (Df)(x) + \left(\overline{D}f\right)(x) = 2\frac{\partial f}{\partial x_0}$$

and

$$\overline{D}f = \left(D\left(f'\right)\right)', \quad \overline{M}f = \left(M\left(f'\right)\right)'. \tag{9}$$

Let $\Omega \subset \mathbb{R}^4$ be open. If $f \in \mathcal{C}^2(\Omega)$ and $Mf(x) = 0$ for any $x \in \Omega \backslash \{x \mid x_3 = 0\}$, the function $f$ is called **hyperholomorphic** in $\Omega$. If $f$ is hyperholomorphic in $\Omega$ and $f = \sum_{i=0}^{3} f_i e_i$ for some real functions $f_i$ the function $f$ is called an **$H$-solution**.

The $H$-solutions in $\mathbb{R}^3$ were introduced by H. Leutwiler ([17]).They are notably studied in $\mathbb{R}^n$ by H. Leutwiler ([18], [19], [20]), H. Hempfling ([13], [14], [15]), J. Cnops ([4]), P. Cerejeiras ([3]) and S.-L. Eriksson-Bique ([6], [11], [8], [9]). In $\mathbb{R}^3$ the hyperholomorphic functions are researched by W. Hengartner and H. Leutwiler ([16]) and in $\mathbb{R}^n$ by H. Leutwiler and S.-L. Eriksson-Bique ([11]).

Applying (7) and (8) for $x_3 Mf = 0$ we obtain the following system.

**Proposition 2.** *Let $\Omega$ be an open subset of $\mathbb{R}^4$ and $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ be a mapping with continuous partial derivatives. The equation $Mf = 0$ is equivalent with the system of equations*

$$\begin{aligned} &x_3\left(D_2(Pf) - \tfrac{\partial (Qf)'}{\partial x_3}\right) + 2Q'f = 0, \\ &D_2(Qf) + \tfrac{\partial P'(f)}{\partial x_3} = 0, \end{aligned}$$

*where $D_2 = \sum_{i=0}^{2} e_i \frac{\partial f}{\partial x_i}$.*

Using the preceding result we infer the relation between monogenic and hyperholomorphic functions.

**Proposition 3.** *Let $\Omega$ be an open subset of $\mathbb{R}^4$. Then if $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ is monogenic and $Qf = 0$, then $f$ is hyperholomorphic.*

**Proposition 4.** *Let $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ be twice continuously differentiable on an open subset $\Omega$ of $\mathbb{R}^4$. Then*

$$\begin{aligned} \overline{M}Mf = M\overline{M}f = \triangle(Pf) - \frac{2}{x_3}\frac{\partial Pf}{\partial x_n} \\ + \left(\triangle(Qf) - \frac{2}{x_3}\frac{\partial Qf}{\partial x_3} + \frac{2}{x_3^2}Qf\right)e_3, \end{aligned}$$

*where $\triangle$ is the Laplacian in $\mathbb{R}^4$.*

*Proof.* The property (8) implies that

$$QMf = QDf + \frac{2}{x_3}Q\left(Q'f\right) = D_2Qf + \frac{\partial P'f}{\partial x_3}.$$

Hence we have

$$\overline{M}Mf = \overline{D}Df + 2\overline{D}\left(\frac{Q'f}{x_3}\right) - \frac{2}{x_3}\left((D_2Qf)' + \frac{\partial Pf}{\partial x_3}\right).$$

Since by (9) and (3)

$$\begin{aligned}\overline{D}\left(\frac{Q'f}{x_3}\right) &= \frac{\overline{D}Q'f}{x_3} + \frac{e_3Q'f}{x_3^2} = \frac{(DQf)'}{x_3} + \\ &= \frac{(D_2Qf)'}{x_3} - \frac{1}{x_3}\frac{\partial Qf}{\partial x_3}e_3 + \frac{Qf}{x_3^2}e_3,\end{aligned}$$

we obtain

$$\overline{M}Mf = \triangle Pf - \frac{2}{x_3}\frac{\partial Pf}{\partial x_3} + \left(\triangle Qf - \frac{2}{x_3}\frac{\partial Qf}{\partial x_3} + 2\frac{Qf}{x_3^2}\right)e_3.$$

From the definitions we note that $Mf + \overline{M}f = 2\frac{\partial f}{\partial x_0}$. Hence we obtain

$$\begin{aligned}M\overline{M}f + M^2f &= 2M\left(\frac{\partial f}{\partial x_0}\right) = 2\frac{\partial Mf}{\partial x_0} \\ &= \left(M + \overline{M}\right)(Mf) = \overline{M}Mf + M^2f,\end{aligned}$$

which implies $M\ \overline{M}f = \overline{M}Mf$, completing the proof. □

**Corollary 5.** *If $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ is hyperholomorphic then*

$$x_3\triangle Pf - 2\frac{\partial Pf}{\partial x_3} = 0 \tag{10}$$

*and*

$$-2Qf = x_3^2\triangle(Qf) - 2x_3\frac{\partial Qf}{\partial x_3}. \tag{11}$$

*Conversely, if the equations (10) and (11) hold, then $\overline{M}f$ is hyperholomorphic.*

The equation (10) is the Laplace-Beltrami equation associated with the hyperbolic metric

$$ds^2 = x_3^{-2}\left(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^3\right).$$

The second equation (11) presents the eigenfunctions of the Laplace-Beltrami operator corresponding to the eigenvalue $-2$.

Using standard methods we obtain the following observation.

**Lemma 6.** *Let $u : \Omega \to \mathbb{R}$ be twice continuously differentiable on an open subset $\Omega$ of $\mathbb{R}^4$. If $u$ satisfies the equation*

$$-2u(x) = x_3^2\triangle u(x) - 2x_3\frac{\partial u}{\partial x_3}(x), \quad x = (x_0, x_1, x_2, x_3) \tag{12}$$

*then the mapping* $g:\Omega\to\mathbb{R}$ *defined by*

$$g(x)=\begin{cases}\frac{u(x)}{x_3}, & \text{if } x_3\neq 0,\\ \frac{\partial u}{\partial x_3}(x), & \text{if } x_3=0,\end{cases}$$

*is harmonic on* $\Omega\setminus\{x_3=0\}$. *Moreover, if* $u\in\mathcal{C}^3(\Omega)$, *then* $g$ *is harmonic on* $\Omega$.

*Proof.* If $x_3\neq 0$, then

$$\triangle g=\frac{\triangle u}{x_3}-\frac{2\partial u}{x_3^2\partial x_3}+\frac{2u}{x_3^3}.$$

Using (12) we note that $\triangle g=0$. Since by (12) $\frac{\partial g}{\partial x_3}=\frac{1}{2}\triangle u$, we see that $g\in\mathcal{C}^2(\Omega)$ and therefore $\triangle g=0$ on $\Omega$ provided that $u\in\mathcal{C}^3(\Omega)$. □

**Proposition 7.** *If* $f:\Omega\to\mathbb{H}\oplus\mathbb{H}$ *is hyperholomorphic and* $f\in\mathcal{C}^3(\Omega)$, *then* $\triangle f$ *is monogenic and therefore also harmonic.*

*Proof.* Assume that $f:\Omega\to\mathbb{H}\oplus\mathbb{H}$ is hyperholomorphic. Using preceding Lemma and $\triangle=D\overline{D}$ we obtain

$$0=D\overline{D}Mf=D\overline{D}Df+2\triangle\left(\frac{Q'f}{x_3}\right)=D(\triangle f).$$

Hence $\triangle f$ is monogenic and furthermore harmonic. □

**Example 8.** *Let* $f=u+iv$ *be holomorphic in an open set* $\Omega\subset\mathbf{C}$. *We define the mapping* $\pi:\mathbb{R}^4\to\mathbb{C}$ *by*

$$\pi(x_0,x_1,x_2,x_3)=x_0+x_1 i$$

*and the mapping* $\rho:\mathbb{R}^4\to\mathbb{C}$ *by*

$$\rho(x_0,x_1,x_2,x_3)=x_0+i\sqrt{x_1^2+x_2^2+x_3^2}.$$

*Then the function* $f\circ\pi$ *is hyperholomorphic in the set* $\{x\mid\pi(x)\in\Omega\}$. *Moreover the function* $\widetilde{f}$ *defined by*

$$\widetilde{f}(x)=u\circ\rho(x)+\frac{x_1e_1+x_2e_2+x_3e_3}{\sqrt{x_1^2+x_2^2+x_3^2}}v\circ\rho(x)$$

*is hyperholomorphic in the set* $\{x\mid\rho(x)\in\Omega\}$ *(see* [17, p.157]*).*

**Proposition 9.** *The space of hyperholomorphic functions in an open subset* $\Omega$ *of* $\mathbb{R}^4$ *forms a right quaternionic vector space.*

*Proof.* Let $q\in\mathbb{H}$. Using (6) we notice that

$$M(fq)=(Df)q+2\frac{Q'(fq)}{x_3}=(Mf)q,$$

implying the assertion. □

The following result is easy to see.

**Lemma 10.** *Let $\Omega$ be an open subset of $\mathbb{R}^4$ and $f:\Omega\to\mathbb{H}\oplus\mathbb{H}$ be hyperholomorphic. Then $\frac{\partial f}{\partial x_l}$ is hyperholomorphic for $l=0,1,2$.*

Product of hyperholomorphic functions is not necessarily hyperholomorphic.

**Theorem 11.** *Let $\Omega$ be an open subset of $\mathbb{R}^4$ and $f:\Omega\to\mathbb{H}\oplus\mathbb{H}$ be hyperholomorphic. Then the product $f(x)\,x$ is hyperholomorphic if and only if $f$ is an $H$-solution.*

*Proof.* Assume that $f:\Omega\to\mathbb{H}\oplus\mathbb{H}$ and $f(x)\,x$ are hyperholomorphic. Then by (6) we have

$$\begin{aligned}0=M(fx)&=(Df)\,x+\sum_{i=0}^{3}e_ife_i+2\frac{Q'(fx)}{x_3}\\&=(Df)\,x+2\frac{Q'f(Px)}{x_3}+2\frac{P'(f)\,Q'x}{x_3}+\sum_{i=0}^{3}e_ife_i.\end{aligned}$$

Since $Px=x-x_3e_3$ and $Qx=x_3$ we obtain

$$0=(Mf)\,x+\sum_{i=0}^{3}e_ife_i-2Q'fe_3+2P'(f)=\sum_{i=0}^{3}e_ife_i+2f'.$$

By virtue of Lemma 1 we find out that $f$ is para vector valued. The converse statement is proved similarly. □

**Corollary 12.** *The function $x^m$ is an $H$-solution.*

**Theorem 13.** *Let $\Omega$ be an open subset of $\mathbb{R}^4$ and $F:\Omega\to\mathbb{H}\oplus\mathbb{H}$ be hyperholomorphic. Then the function $f(x)=F(x)\,x^{-1}$ is hyperholomorphic in $\Omega\backslash\{0\}$ if and only if it is paravector valued.*

*Proof.* Assume that $F:\Omega\to\mathbb{H}\oplus\mathbb{H}$ and $f(x)=F(x)\,x^{-1}$ are hyperholomorphic. Since $F(x)=\left(F(x)\,x^{-1}\right)x$ we obtain from the preceding theorem that $\left(F(x)\,x^{-1}\right)$ is para vector valued. On the other hand, assume that $f$ is vector valued and $F$ is hyperholomorphic. Then by (6) and Lemma 1, we conclude

$$0=MF=(Mf)\,x+\sum_{j=0}^{n}e_jfe_j+2f'=(Mf)\,x$$

and therefore $f$ is also hyperholomorphic. □

**Corollary 14.** *The function $x^{-m}$ is an $H$-solution.*

**Theorem 15.** *Let $\Omega$ be an open subset of $\mathbb{R}^4$ and $f:\Omega\to\mathbb{H}\oplus\mathbb{H}$ twice continuously differentiable. Then $f$ is hyperholomorphic if and only if for any $a\in\Omega$ and any ball*

*$B(a,r)$ with $B(a,r) \subset \Omega$ there exists a continuous differentiable mapping $H$ from $B(a,r)$ into $\mathbb{H}$ satisfying the equations*

$$f = \overline{D}H \tag{13}$$

*and*

$$x_n \triangle H - 2\frac{\partial H}{\partial x_n} = 0 \tag{14}$$

*on $B(a,r)$.*

*Proof.* Assume that a mapping $H : B(x,r) \to \mathbb{H}$ satisfies (14) and $f = \overline{D}H$. Since $D\overline{D} = \triangle$ we have $x_3 Df = x_3 \triangle H$. The equality $-2\frac{\partial H}{\partial x_n} = Q'f$ follows from

$$Qfe_3 = Q\overline{D}H = -e_3\frac{\partial H}{\partial x_n}.$$

Hence $Mf = 0$ and therefore $f$ is hyperholomorphic.

Conversely assume that $f : \Omega \to \mathbb{H}\oplus\mathbb{H}$ is hyperholomorphic. Set $Pf = f_0 + f_1e_1 + f_2e_2 + f_{12}e_{12}$. Let $B(a,r)$ be a ball in $\mathbb{R}^4$ centered at $a = (a_0, ..., a_n)$ satisfying $B(a,r) \subset \Omega$. Let $s_i : (B(a,r) \cap \{x \mid x_3 = a_3\})^{\sim} \to \mathbb{R}$ be a twice continuously differentiable solution of the Poisson equation (which exists for example by [2, p.171] )

$$\triangle s_i(\tilde{x}) = f_i(\tilde{x}, a_3).$$

for $i = 0, 1, 2, 12$ and a ball $B((a_0, a_1, a_2), r)$. Set $s = \sum_{i \subset \{0,1,2,12\}} s_i e_i$ and define a mapping $H : B(a,r) \to \mathbb{H}$ by

$$H(x) = -\int_{a_3}^{x_3} Q'f(\tilde{x}, t)\,dt + D_2 s.$$

Then we have by Proposition 2

$$\begin{aligned}
\overline{D}H(x) &= e_3 2Q'f(x) - \int_{a_n}^{x_n} \overline{D_2}Q'f(\tilde{x}, t)\,dt + \overline{D_2}D_2 s \\
&= Q_2 f(x)\,e_n + \int_{a_n}^{x_n} \frac{\partial Pf}{\partial x_n}(\tilde{x}, t)\,dt + (Pf)(\tilde{x}, a_n) \\
&= f(x).
\end{aligned}$$

Since the image space of $H$ is $\mathbb{H}$ we note that

$$Qfe_3 = -e_3\frac{\partial H}{\partial x_3}.$$

Using the assumption $f$ is hyperholomorphic we obtain

$$0 = x_3 Df + 2Q'f = x_3 D\overline{D}H - 2\frac{\partial H}{\partial x_3}.$$

Hence the mapping $H$ satisfies (14), completing the proof. □

Hyperholomorphic functions may be obtained from $H$-solutions as follows.

**Theorem 16.** *A mapping $f$ is hyperholomorphic on a ball $B(a,r) \subset \mathbb{R}^4$ if and only if there exist $H$-solutions $g_i$ such that*

$$f = g_0 + g_1 e_1 + g_2 e_2 + g_{12} e_{12}.$$

*Proof.* Assume that $f$ is hyperholomorphic on a ball $B(a,r) \subset \mathbb{R}^4$. Applying Theorem 15 we find a mapping $H$ from $B(a,r)$ into $\mathbb{H}$ satisfying the equations (14) and $f = \overline{D}H$. Denote $H = h_0 + h_1 e_1 + h_2 e_2 + h_{12} e_{12}$ for real functions $h_i$. Then the mapping $g_i = \overline{D} h_i$ is vector valued and therefore an $H$-solution. Clearly we have $f = g_0 + g_1 e_1 + g_2 e_2 + g_{12} e_{12}$. □

**Corollary 17.** *If $f$ is hyperholomorphic on $\Omega$, then it is real analytic on $\Omega \backslash \{x_3 = 0\}$. Moreover, if $f \in C^3(\Omega)$is hyperholomorphic on $\Omega$, then it is real analytic on $\Omega$.*

*Proof.* The $H$-solutions are real-analytic by [8, Theorem 4]. Hence the preceding theorem implies the statement. □

The fundamental homogeneous polynomial $H$-solutions are defined as follows.

**Definition 18.** *The homogeneous polynomials $L_m^\alpha$ are defined for any multi-index $\alpha \in \mathbb{N}_0^2$ and a non-negative integer $m$ by*

$$L_m^\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} x^{m+|\alpha|}}{\partial x^\alpha}.$$

*The homogeneous polynomial $H$-solution $T_m^\alpha$ of degree $m$ is defined by*

$$\frac{\partial T_m^\alpha}{\partial x^\alpha} = \alpha! x_3^2 e_3$$

*and $T_m^\alpha(x) = 0$ for any $x$ with $x_3 = 0$.*

**Theorem 19.** *The set $\{L_m^\alpha \mid |\alpha| \le m\} \cup \{T_m^\alpha \mid |\alpha| = m - n + 1\}$ is a basis of the right $\mathbb{H}$-module of homogeneous hyperholomorphic polynomials of degree $m$.*

*Proof.* Using [8, Theorem 4] we obtain that the set in question is a basis of the right $\mathbb{H}$-module generated by the homogeneous polynomial $H$-solutions of degree $m$. If $f$ is a homogenous hyperholomorphic polynomial of degree $m$, then by Theorem 16 there exists homogeneous polynomial $H$-solutions $p_i$ of degree $m$ satisfying $f = \sum_{i \in \{0,1,2,12\}} p_i e_i$. Hence it is also a basis of the right $\mathbb{H}$-module of homogeneous hyperholomorphic polynomials of degree $m$. □

**Theorem 20.** *Let $f \in C^3(\Omega)$ be hyperholomorphic in a neighborhood of a point $x = (x_0, x_1, x_2, 0)$. Then there exist constants $b_k(\alpha), c(\alpha) \in \mathbb{H}$ such that*

$$f = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=0}^{k} L_k^\alpha b_k(\alpha) + \sum_{|\alpha|=k-2} T_k^\alpha c_k(\alpha) \right)$$

*Proof.* Assume that $f \in \mathcal{C}^3(\Omega)$ is hyperholomorphic in a neighborhood of a point $x = (x_0, x_1, x_2, 0)$. If $T(y) = y + a$ for $a \in \mathbb{R}^4$ with the last coordinate $a_3 = 0$ then $f \circ T$ is also hyperholomorphic. Hence we may assume that $x = 0$. Since $f$ is hyperholomorphic, $f$ is real analytic and therefore admits the presentation

$$f(y) = \sum_{\beta \in \mathbb{N}_0^{n+1}} a(\beta) y^\beta.$$

in some neighborhood $B_r(0)$. Applying $M$ we obtain

$$0 = Mf(y) = \sum_{k=0}^{\infty} M\left(\sum_{|\beta|=k} a(\beta) y^\beta\right).$$

Since $M\left(\sum_{|\beta|=k} a(\beta) y^\beta\right)$ is a homogeneous polynomial of degree $k$ we infer

$$M\left(\sum_{|\beta|=k} a(\beta) y^\beta\right) = 0.$$

This implies that $\sum_{|\beta|=k} a(\beta) y^\beta$ is hyperholomorphic. Applying Theorem 19 we obtain the result. □

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