A CONCEPTUAL FRAMEWORK FOR STUDYING TEACHER PREPARATION: THE PIRIE-KIEREN MODEL, COLLECTIVE UNDERSTANDING, AND METAPHOR

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This theoretical paper describes a conceptual framework for studying metaphoric mechanisms of the growth of collective understanding among prospective middle and high school mathematics teachers. The framework draws upon a growth of mathematical understanding model, studies of metaphor, and research on collective understanding. Researchers whose studies contributed to this conceptual framework include Pirie and Kieren, English, Lakoff and Nunez, Sfard, Davis, and Simmt. The framework is further defined and illustrated with examples from a teaching experiment in a first mathematics methods class.

FOCUS

This paper develops a conceptual framework to study growth in pedagogical content knowledge (PCK) of prospective teachers. We assume that PCK is grounded in mathematical understanding. Based on this assumption, we use theories on learning *mathematics* as a basis for the framework. These theories include the Pirie-Kieren model, metaphor analysis, and a collective understanding perspective.

THEORETICAL BACKGROUND

We consider three theoretical perspectives as background frameworks to study the growth of understanding among prospective teachers. Shulman (1986), in defining the knowledge base of teaching, initiated the notion of pedagogical content knowledge (PCK) as one of the fundamental categories of teacher knowledge. This type of knowledge is unique within each subject matter domain, and mathematics PCK is accessed by teachers, in concert with their knowledge of mathematics, to communicate mathematical ideas. The Pirie-Kieren model for the growth of mathematical understanding (Pirie & Kieren, 1994b) is fundamental to our studies of prospective teachers. The Pirie-Kieren model is enhanced by coordinating it with metaphoric mechanisms of movement across the layers of the model. Metaphor studies (English, 1997; Lakoff & Nunez, 2000; Sfard, 1997) provide the second theoretical perspective for our framework. Finally, we selected Collective Understanding (Davis & Simmt, 2003; Kieren & Simmt, 2002) to accommodate for the contexts where PCK growth occurs. The Collective Understanding perspective provides the view of learning within a social endeavour, such as a methods class, a school class, or work with others on projects outside of class. Next, we give backgrounds of the three perspectives that provide the theoretical foundation for our research.

Pirie-Kieren Model for the Growth of Mathematical Understanding

The first of the theoretical perspectives we consider is a description of the growth of understanding as a dynamic, levelled but non-linear and recursive process (Pirie & Kieren, 1994b). The Pirie-Kieren model was originally designed as a perspective to study students' changing mathematical ideas. The model provides a framework to map student actions in a variety of contexts, tracing the back and forth movement among eight levels of understanding activities. Within these activities, learners build, search, and collect ideas. The innermost level is *Primitive Knowing*, consisting of one's previous knowledge brought to the learning context. This level serves as a source of materials to build subsequent understanding. Moving outward within the model, *Image Making* and *Image Having* are learner activities for making a new image or revising an existing image, and then for manipulating that image in the mind. These two levels of activity play a prominent role in growth of prospective teachers' understanding (Berenson, Cavey, Clark, & Staley, 2001). The next level, *Property Noticing*, is an action of identifying properties of the constructed image. A method, rule, or property is generalized from the properties in the level of *Formalising*. Beyond are levels of *Observing*, *Structuring*, and *Inventising*. Pirie and Kieren (1994b) describe the process of *folding back* to inner levels of understanding to retrieve primitive knowledge, make or have new images, or notice new properties.

Metaphor

Defined in the past as an embellishment or a figure of speech, metaphor is now seen as a primary mechanism of thinking (Lakoff & Nunez, 2000). The process of metaphoric projection involves a source, consisting of more concrete, better understood images, and a target, which is the new, more formal concept being constructed (English, 1997). This process is a recursive, zig-zag movement between the source and the target, where the target is created and the source is modified (Sfard, 1997).

A metaphor has a certain "life cycle" (Figure 1). When a metaphor is born from its grounding, the target *is* the source. The two parts of the metaphor are inseparable; in other words, the target is not yet constructed. For example, a learner may work with the metaphor of "fair sharing," grounded in sharing actions. In the next stage, the target emerges from the source, which starts to fade, and the metaphor turns into a simile. Now the target is *like* the source; division is *like* fair sharing. Finally, the target disconnects from the source. The metaphor "dies" (Pirie & Kieren, 1994a; Sfard, 1997), and the target lives on as a self-sustained entity. In our example, the learner is able to think about the idea of division without referring to fair sharing.

The paradox of metaphor analysis is that a metaphor can only be studied when it dies, or at least turns into a simile. While the source and the target are inseparable, the metaphor is unnoticeable. Researchers can analyse this first stage in the metaphor's life *retrospectively*, from the vantage point of the future emergence and separation of the target. A participant observer can also influence a metaphor, thus helping metaphor's users learn.
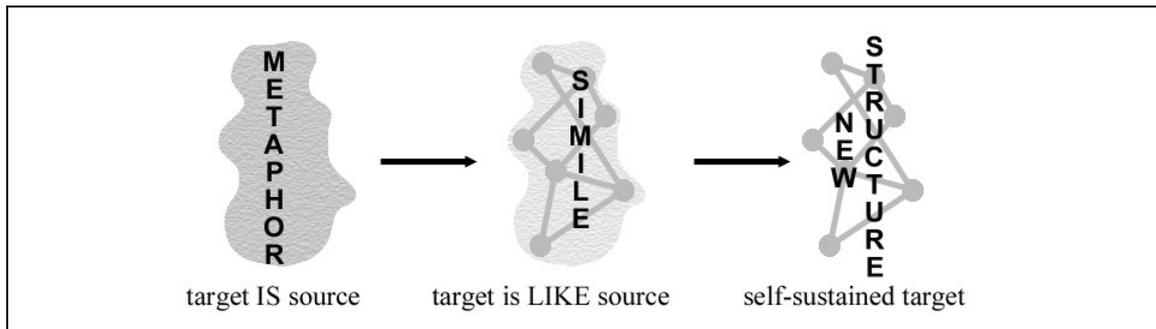
Figure 3: Metaphor life cycle

Collective understanding

The theory being developed by the Collective Understanding Research Collective (Davis & Simmt, 2003; Kieren & Simmt, 2002) focuses on emergent phenomena in groups seen as wholes, rather than simple sums of individuals. A key feature of the complexity of these emergent structures is that they cannot be caused, but may be occasioned, and analysed retrospectively. The authors claim five necessary conditions for such collective understanding: *internal diversity*, *redundancy*, *decentralized control, organized randomness*, and *neighbour interactions. Internal diversity* in response to a task means that members of the collective contribute in different ways. Diversity creates possibilities for new, varied paths toward understanding. *Redundancy* is defined as the occurrence of more actions or ideas than, seen retrospectively, would be necessary to complete the task. Redundancy serves the role of supporting communication and the feeling of "us" within the collective, and of helping to cope with perturbations. *Decentralized control* suggests no single organizing agent within the collective; in the classroom, it means that the teacher is not the only authority on correctness. *Organized randomness* is a set of proscriptions and within it, freedom from prescriptions. "*Neighbours*" interacting in the fifth conditions are not people, but ideas, metaphors, and other representations.

PROBLEM

The research problem this study addresses is constructing a conceptual framework for analysing learning of prospective teachers during an introductory methods class, and for designing ways to help them learn. Each of the three theoretical perspectives described above provided a necessary lens for the framework. The Pirie-Kieren model, initially focused on mathematical understanding and adapted to teacher preparation by Berenson et al. (2001), describes the actions of understanding. The model has been related to metaphor theories (Droujkova, 2004; Pirie & Kieren, 1994a), the lens for examining mechanisms of growth of understanding. The Pirie-Kieren model helps to map *what* is happening with understanding, and metaphor describes *how* it is happening. The Pirie-Kieren model, as well as metaphor theories, was initially developed for describing individuals. However, we looked at a group of prospective teachers interacting, connecting ideas, and building their understanding together. We used the third lens: work on collective understanding (Davis & Simmt,

2003), which arose out of the work of Pirie and Kieren, and was developed, in part, within the context of teacher preparation. The resulting conceptual framework helps to map the growth of collective pedagogical content knowledge of a group of prospective teachers, and to analyse mechanisms of this growth. Applying metaphor analysis to preparation of teachers and tracing collective metaphors are two theory bridges constructed in the teaching experiment that is the source of examples for this work (Figure 2).
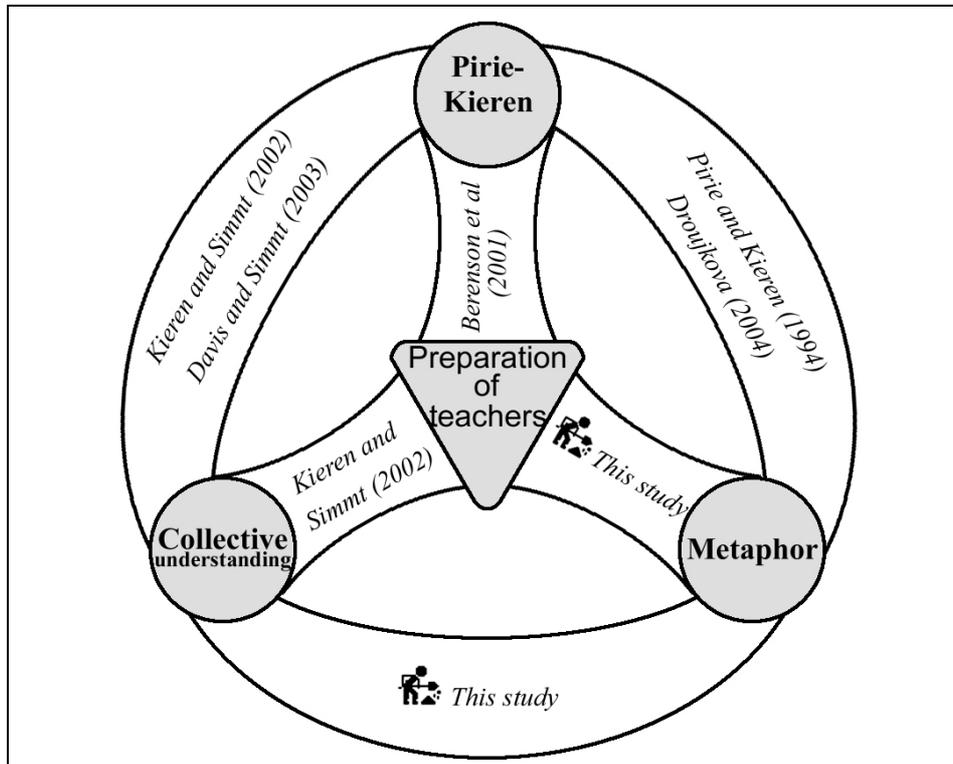

Figure 4: Theories and bridges in our conceptual framework

TEACHING EXPERIMENT METHODOLOGY

Examples in this paper come from a teaching experiment conducted during an Introduction to Teaching class at a large South-Eastern US university. The class met for two hours, one day a week, for fourteen weeks. Paper authors collaborated on planning, teaching and observing during the class. Teaching experiments are defined by the role of researchers as teachers and co-learners, and "environments that are explicitly designed to optimize the changes that relevant developments will occur in forms that can be observed" (Kelly & Lesh, 2000, p. 192). Since the background theories of our conceptual framework use recursive models of learning, it was especially important to allow data from each week to enter the interpretation cycle, and to influence the future data collection. The data include primary artefacts such as home and in-class assignments, teaching portfolios, and videotapes of lessons conducted by prospective teachers; and secondary artefacts such as field notes taken by researchers during class observations and planning meetings.

PEDAGOGICAL CONTENT KNOWLEDGE

In this part, we trace growth of collective PCK in the group of prospective teachers and researchers. Conditions for the emergence of collective understanding help to analyse the features of the class which supported growth.

Image Making: birth of metaphors

"Multiple instructional representations" was a theme included in class activities and homework assignments every day. In first classes and homework assignments, prospective teachers were creating collections of instructional representations of different kinds, such as pictures, definitions and symbols. These collections were based on their Primitive Knowledge of mathematics. The recurring homework of creating four different representations for introducing a mathematical concept, such as slope or ratio, invited the diversity of actions and ideas, and illustrated inter-personal redundancy in actions and ideas – two conditions for emergence of collective understanding. Small group discussions and sharing their results with the whole group supported idea exchange. For example, small groups discussed Pi representations from homework, and sorted them into visual, manipulative, numerical and symbolic and so on to display to the whole class on bulletin boards. An example of a collective image made by the class is "multitude of instructional representations." This image included the ideas that there are many different ways to learn each concept, and that these choices can be sorted into categories. We, as a part of the group, were folding back to Image Making as well. An example of a change in our image at the time is the focus on overlaps, links and interactions between different types of representations, which we describe in a separate paper (Reference withheld).

Image Having and beyond

Assignments on mathematical and pedagogical connections among representations show that the class was using the idea of multiple representations without the actions of creating them, which indicates Image Having. Class assignments continued to invite individual ideas, either from homework or from a period of contemplation in class, into small group tasks. Each group then reported to the whole class, answered questions, and participated in whole-class discussions. This supported interactions of neighbour ideas within small groups and the whole class, which is another condition for emergence of collective understanding. Such "bumping of ideas" (Davis & Simmt, 2003) looked like animated discussions, and sometimes emotional arguments, within groups, between a group reporter and the rest of the class, and between individuals. In these discussions, we observed Image Having and actions from the further levels.

Property Noticing: metaphor turns into simile

Property Noticing actions are manipulations of images to construct their relevant properties. A transformation of metaphor corresponds to these actions. The newly

constructed properties are the metaphor target; they are built on, and separated from, images, which are the source. A plethora of potential properties can be constructed from images. Emphatically, properties are not embedded *in* images, but constructed *from* images, and co-determined by images and actions during Property Noticing. In the case of prospective teachers, mathematical and pedagogical sides intertwine in Property Noticing actions, and metaphor targets are PCK concepts.

An example of Property Noticing comes from the class task of making representations for connections between scaling and ratio, slope, and proportion. In an example of idea redundancy, two groups, working with maps and similar triangles, independently commented that the same representation can be used for scaling-ratio and scaling-proportion connections. The mathematical properties they noticed were connections between ratios and proportions. There were also pedagogical comments: prospective teachers noticed that they can help students learn relationships between ratio and proportion through one activity on scaling, and found two examples of such activities.

During the same task, the organized randomness feature of the collective understanding was expressed as the discussion turned toward the question of what kinds of representations are better as a starting point for students. Prospective teachers used the same image of multiple representations, but were now noticing other pedagogical properties. For example, they determined what representation was more abstract, or what representation was easier for learners to understand. Initially, prospective teachers were talking about particular representations, claiming that maps are easier than similar triangles, or vice versa. When the target of the metaphor began to separate, and the metaphor turned into a simile, prospective teachers started to name noticed properties directly. The sources of metaphors were still present at this point in the learning. A prospective teacher said she would start her lesson from a concrete representation, *like* a map, because it is easier. However, the sources were fading, and the targets became more self-sustained, as the group moved to formalising.

Formalising: death of metaphor, and self-sustained target

During the activities when prospective teachers noticed properties of instructional representations, they also noticed differences in pedagogical uses of representations. Disagreements about uses of representations arose on many counts, such as which representation is better as a starting point, or whether students would have enough prerequisite knowledge to handle a particular representation. The formalising actions were evidenced by abstracting this noticed idea of differences and disagreements about representations as the concept of learner diversity. As evidenced by comments, sometimes emotional, contradictions between ideas were frustrating, and it was imperative for the group to come to an overarching understanding resolving the contradictions. The formalization of the concept of learner diversity was greeted by the group members as a relief of this cognitive tension, and appeared, in many

different formalised forms, in the remaining homework assignments, lesson plans, and teaching philosophies of most students.

From the perspective of this formalising we retrospectively see that metaphor started from the images of multiple representations during Image Making and Image Having actions. It turned into a simile while prospective teachers were noticing different, oftentimes contradictory, instructional uses of representations among group members. This property was formalised as the idea of learner diversity. Here are examples of individual expressions of collective formalised understanding, taken from teaching philosophy statements of three prospective teachers:

> Using different types of representations to explain a concept will give students a chance to see the lesson from a different perspective.

> Sometimes students lose motivation when they don't understand the material. I plan to use a variety of instructional representations in order to give students who come from different backgrounds and learn differently the opportunity to learn the material.

> Students learn best when they are presented with several different ways of looking at a topic. A lecture can only do so much but in combination with hands-on activities and group work students can learn much more effectively.

These examples show variability in individual threads of meaning in the collectively understood idea: one class member focused on learning of a particular concept, another on motivation, and another on class format. Individually and as a collective, prospective teachers started to develop a crucial area of their PCK: understanding of how a particular mathematical concept can be represented in multiple ways, and how differences in learners inform choices, decisions, and development of these instructional representations.

CONCLUSIONS AND FUTURE RESEARCH

Using data from a teaching experiment, we developed a conceptual framework for studying growth of prospective teachers' collective understanding. Our data illustrate metaphoric mechanisms of the growth of understanding, and correspondences between stages in metaphor development, and understanding actions observed at the onset of each stage and mapped by the Pirie-Kieren model (Table 1).

Stage	Metaphor birth from grounding	Inseparably, target *is* source	Metaphor into simile: target is *like* source	Metaphor death: self-sustained target
Actions	Image Making	Image Having	Property Noticing	Formalising
Example	"Multitude of representations"		"Learning differences are like contradicting instructional uses of representations"	"Learner diversity"

Table 1: Co-occurrence of stages in metaphor development, and understanding actions.

Within a metaphor, there is a recursive movement between the emerging target and the source. Growth of understanding, as describe by the Pirie-Kieren model, is also a recursive process where folding back to inner levels is prominent. However, the development of metaphor: birth, turning into simile, and death into the self-sustained target, is unidirectional. This leads us to questions for the future studies: "What is the role of metaphor in folding back? In particular, how does a simile inform folding back from Property Noticing? What is the role of self-sustained targets in folding back from Formalising?" Answers to these questions can help us study ways prospective teachers learn, and ways to prepare teachers.

References

Berenson, S., Cavey, L., Clark, M., & Staley, K. (2001). Adapting Pirie and Kieren's model of mathematical understanding to teacher preparation. Paper presented at the 25th Annual Meeting of the International Group for the Psychology of Mathematics Education, Utrecht, NL.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. Journal for Research in Mathematics Education, 34(2), 137-167.

Droujkova, M. (2004). Roles of metaphor in the growth of mathematical understanding. Unpublished Doctoral Dissertation, North Carolina State University, Raleigh.

English, L. (1997). Analogies, metaphors and images: Vehicles for mathematical reasoning. In L. English (Ed.), Mathematical reasoning: Analogies, metaphors and images (pp. 3-20). Mahwah, NJ: Lawrence Erlbaum.

Kelly, A., & Lesh, R. (2000). Handbook of research design in mathematics and science education. Mahwah, NJ: Lawrence Erlbaum.

Kieren, T., & Simmt, E. (2002). Fractal filaments: A simile for observing collective mathematical understanding. Paper presented at the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA.

Lakoff, G., & Nunez, R. (2000). Where mathematics comes from: How the embodied mind brings mathematics into being (1st ed.). New York, NY: Basic Books.

Pirie, S., & Kieren, T. (1994a). Beyond metaphor: Formalising in mathematical understanding within constructivist environments. For the Learning of Mathematics, 14(1), 39-43.

Pirie, S., & Kieren, T. (1994b). Growth in mathematical understanding: How can we characterize it and how can we represent it? Educational Studies in Mathematics, 26, 165-190.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. English (Ed.), Mathematical reasoning: Analogies, metaphors and images (pp. 339-372). Mahwah, NJ: Lawrence Erlbaum.

Shulman, L. (1986). Those who understand: The knowledge growth in teaching. Educational Researcher, 15(2), 4-14.