MULTIPLE REPRESENTATIONS IN 8TH GRADE ALGEBRA LESSONS: ARE LEARNERS REALLY GETTING IT?

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The potential benefits to be gained from multiple representations in mathematics education, both where the representations are constructed by learners and where learners use standard representations, have long been recognized. In this paper, qualitative data from 8th grade lessons on linear equations are produced questioning how well this potential, in the case of standard representations, is realized in a real learning environment.

INTRODUCTION

The general case for multiple representations in mathematics education hardly needs defending anymore—most of us have long been persuaded of the central place of multiple representations in problem solving and in the understanding of mathematical ideas (thorough discussions can be found in, for example, Goldin, 2002; Schultz & Waters, 2000; Kaput, 1999; Greeno & Hall, 1997; Janvier, 1987). This paper, therefore, does not aim to adduce further evidence for the importance of multiple representations, nor to challenge it. Rather, we wish to look at the practical question of how ideas about multiple representations are realized in real classrooms. Do teachers succeed in creating learning environments in which they and their students share an understanding of why multiple representations of mathematical ideas and problems ought to be entertained? Are students truly reaping the potential benefits from lessons explicitly designed with multiple representations in mind?

These questions are in fact quite complex for they concern not only students' ability to work with multiple representations as prescribed in documents such as the NCTM *Principles and Standards* (NCTM, 2000), but also their *interpretations* of the meaning and value of what they are doing when they use multiple representations. In the present paper, we can only hope to leave readers with the sense that they ought not be complacent about these practical questions and their complexities, even while they are thoroughly convinced of the correctness of the theory. To this end, based on data from an 8th grade classroom studying systems of linear equations, we shall show that it can happen that in a learning environment where multiple representations have been fully taken into account by a well-informed teacher learners may, nevertheless, fail to grasp the idea of multiple representations and why they are important. Given the allowable length of the paper, we shall give most of our attention to one particular interview, though others could have served as well.

But before we present this data and discuss their significance, we need to circumscribe our treatment of multiple representations. For the word 'representation' itself has multiple meanings in mathematics education, a fact that made discussions

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 57-64. Melbourne: PME.

in the original 1990-1993 PME working sessions on representations at once difficult and rich (see Goldin, 1997). But with respect to classroom practice, it is possible to distinguish two main tendencies concerning multiple representations. One points towards students constructing their own representations, both in pure mathematical contexts and in situations where mathematics is applied to non-mathematical or real-life situations. The other points towards students using or adapting standard representations, particularly, algebraic, graphic, tabular, and verbal representations. Of course, these tendencies are not exclusive. Both tendencies are evident in the NCTM 'representation standard', which stipulates that "Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record , and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena" (NCTM, 2000, p.67)

In many classroom situations, especially where standard material is taught, it is the second tendency, that is, towards multiple representations of a standard kind, that naturally dominates (this is true even where the means of presenting the representations are not entirely standard as in Schultz & Waters (2000)). In this case, what the teacher aims towards is chiefly the ability to select, apply, and translate among different representations; this, in turn, demands that learners understand the meaning and value of representations. In this paper, we shall be concerned only with this second tendency.

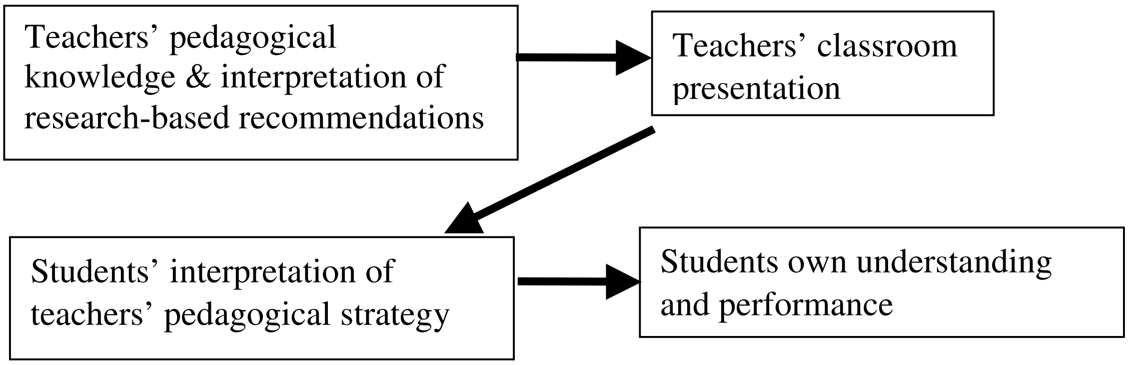
RESEARCH SETTING AND METHODOLOGY

The research setting for the results to be presented here is the *Learners' Perspective Study* (LPS), which is an international effort involving nine countries (Clarke, 1998; Fried & Amit, 2004). The LPS expands on the work done in the TIMSS video study, which exclusively examined teachers and only one lesson per teacher (see Stigler & Hiebert, 1999), by focusing on *student actions* within the context of whole-class mathematics practice and by adopting a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

As specified in Clark (1998), classroom sessions were videotaped using an integrated system of three video cameras: one viewing the class as a whole, one on the teacher, and one on a "focus group" of two or three students. In general, every lesson over the course of three weeks was videotaped, that is, a period comprising fifteen consecutive lessons. The extended videotaping period allowed every student at one point of another to be a member of a focus group.

The researchers were present in every lesson, took field notes, collected relevant class material, and conducted interviews with each student focus group. Teachers were interviewed once a week. Although a basic set of questions was constructed beforehand, in practice, the interview protocol was kept flexible so that particular classroom events could be pursued. In this respect, our methodology was along the lines of Ginsburg (1997).

This qualitative methodology was chosen in general because the overall goal of LPS is not so much to test hypothesized student practices as it is to discover them in the first place. In this particular instance, however, a qualitative methodology was also necessary because, as we remarked above, our investigation of multiple representations in the classroom involved to a great degree teachers' and students' *interpretations* of the meaning and intent of the classroom activity, as can be seen schematically in the following figure:



The specific case that formed the basis for this paper was a sequence of 15 lessons on systems of linear equations taught by a dedicated and experienced teacher, whom we shall call Danit. Danit teaches in a comprehensive high school whose direction in mathematics education is along the lines of the NCTM standards approach; Danit herself is well-informed about the educational issues involved. The 8th grade class Danit teaches is heterogeneous in ability and multiethnic.

DATA

As mentioned above, Danit is a teacher who, partly through her own interest and partly through the educational framework embodied in the national curriculum, is familiar with new developments in mathematics education. Thus, in constructing her lessons on systems of linear equations, she quite consciously introduces different representations relevant to them. Danit goes back and forth between representations in a way that keeps them always in play and in a way that gives her lessons a flow describable as 'turbulent', albeit carefully controlled turbulence (Fried & Amit, 2004). Her desire that students think about the idea of different representations, that they suggest to the students different approaches to mathematical problems and different ways of conceiving mathematical ideas, that the students do not see them merely "...as though they are ends in themselves" as Greeno & Hall put it (1997, p. 362), is evident in the way Danit makes a transition from the symbolic representation of an equation in two unknowns to a graphical representation. Referring to the equation x+y=6 written on the board, she begins as follows:

> [*min 35*]Who is willing to tell me what is written here in Hebrew? I want a translation [with emphasis] into Hebrew, not just “x plus y equals six”!...You’ve seen this [*i.e.* an equation like this] in your book, and you know to do with them [referring to exercises given in the last lesson]—now translate it into Hebrew [*i.e.* into your spoken language].

After some discussion, she finally lets the students know what she is up to:

> [writes: ‘Two numbers whose sum is six’] Find me two numbers whose sum is six. In the language of algebra, we say, ‘x plus y equals six’. [*min. 37*] Today, we’re going to learn to *translate this into another language* [our emphasis]; we’re going to sketch this, that is, what is written here, x+y=6, I don’t have write in the language of algebra, I don’t have to say it in words: I can sketch it.

Thus, besides referring to different kinds of representation, Danit uses words such as ‘language’ and ‘translation’ which refer to the meaning of representation and to moving between representations. She wants the students to know what representations and the act of representing are all about.

In our focus group for that lesson were two boys, Oren and Yuri. By asking them simply “What was the lesson about?” we hoped to find out in the interview whether they grasped Danit’s message as well as her words. Yuri answered “How to solve equations with a number line” [both Yuri and Oren, as well as many of the other students we interviewed, tended to refer to the coordinate system as ‘the number line’—an interesting fact in itself!]. Oren’s answer was somewhat more revealing:

> Oren: We learned [*min 2*] [glances at the whiteboard] about, um, well, equation exercises [*sic*] with two unknowns we started to learn and how to solve them. And, also we learned about the number line and we connect that with equations.

Two observation can be made here. First, Both boys spoke about using the ‘number line’ to *solve* equations. Danit did speak about *solutions* of equations in two unknowns and used yet another representation, a tabular representation, to bring out the pairs of numbers that solve the equation; however, at this point she did not present the graphical representation *as a means* of solving the equation but as a way of seeing the equation in a different light. This tendency was strikingly illustrated in the next lesson, where in the video of the lesson, two girls (that day’s focus group), are seen to continue carrying out only the arithmetical calculations of finding y for a given x, without ever mentioning the graphic representation of the linear relation—and that, just when Danit has been emphasizing aspects of the graphic representation to the whole class!

The second observation is that Oren described the content of the lesson by means of simple conjunctions—this *and* this *and* this *and* this—it is a fragmented account containing the facts of the story but not its theme. Even where he does use the word ‘connect’ (*lechaber*), he is still only reporting factually what Danit has said: indeed, in the previous lesson she said that she would ‘link’ (*lekasher*—which is synonymous to *lechaber*) everything together in the lesson we are looking at now. Here, Oren’s glance at the whiteboard is telling: he has to remind himself what the lesson was about by looking at what *Danit wrote* rather than looking at his own thoughts. We shall return to this point in a moment.

Pushing our original question a little further, we asked what Danit tried to accomplish in her lesson and whether she succeeded in achieving her goals. Oren again emphasized the word ‘connect’, and both Oren and Yuri agreed heartily that Danit did truly achieve her lesson aims:

Interviewer:	What do you think the teacher tried to accomplish in this lesson?
Oren:	She tried to connect [for] us, because before we studied the number line in a separate lesson and equations in the second lesson, so in my opinion she tried to connect [for] us, how the number line is connected to equations.
Yuri:	To equations.
Interviewer:	Do you think she succeeded in her goal?
Yuri:	Yeah, I think so.
Oren:	In my opinion, yes [*min 5*].

They *seem* to have grasped what Danit was trying to do, at least they know the right words to use. But just a couple of minutes later, while discussing one of the exercises they worked on in the class and for which they had asked Danit for help, Yuri described the general procedure, which involved substituting a value for x in the equation, say, x+y=6 (Danit’s example), solving for y, finding the point (x,y) on the ‘number line’, and then repeating the process for another value of x. Yuri describes the procedure in a very disjointed way, and soon afterward, both Yuri and Oren admit that they did *not* understand the point of the lesson:

Yuri:	...Solve it a few times so that the numbers, the unknowns, will be different and afterwards see it on the number line—so it will be a straight line, sort of, that it will be correct—that I didn’t understand—she explained it to me.
Interviewer:	[to Oren] Did you have the same question?
Oren:	Yeah, exactly. I also was a bit mixed up about the teaching, because I understood, but I didn’t understand, it was hard for me to connect with [*sic*] the number line and the equations.
Yuri:	Yeah.
Oren:	The teaching [presumably, “The teaching wasn’t clear to me”].
Interviewer:	[to Yuri] Why didn’t you ask him [Oren]?
Yuri:	Because he asked [Danit] too.
Oren:	I really [with emphasis] didn’t understand either.

It turns out that Oren and Yuri do not truly see the point of the graphic representation. For them, drawing the graphs does not show them equations from a different perspective; for them, drawing the graphs is a redundant exercise. At one point during the video of the lesson, which we watched together with Oren and Yuri, Yuri says 'Boring!'. We asked what he was referring to. He said drawing the axes. Oren agreed and added, "It takes time." Asked if the problem was that they had to use a ruler, Yuri expanded and said, "Yeah, drawing the numbers and checking and putting down the points—it's easy but it takes time. Because of that." From their point of view, 'solving' the equations has weight; drawing the lines was just another task given to them by Danit, a task which could just be easy or hard. This was a typical attitude in Danit's class. For instance, when on another day we asked Annette and Chanita about why they need the 'number line', the 'axis', neither could say why. And when we pressed them, and asked why they didn't ask the teacher, the exchange was as follows:

Interviewer:	Chanita, tell me, why didn't you ask the teacher why? [*min 14*]
Annette:	[Answering for Chanita] It wasn't ( ) interesting.
Interviewer:	Sorry?
Annette:	Because it's no so interesting why you need the axis—we just solve, and, that's it, we go home [everyone chuckles]

Later in the interview, both Chanita and Annette answered that that lesson contained just exercises. And when asked what *they thought* would be in the next lesson, Chanita answered, laughing, that "*She [Danit] said* [our emphasis] in the next lesson we would stop drawing [graphs]." So, like Yuri and Oren, Chanita and Annette see no intrinsic value in pursuing the graphical representation of the linear relations. If Danit decides they should do it or not do it, so be it—but better if she decides not to!

This brings us back to Oren's glance at the whiteboard to answer what the lesson was about. Although Danit is at pains to make the students *themselves* think about the notion of representation, they take their cues from *her*; her authority is enormous (see Amit & Fried, in press). This could be seen when we asked the students about why the points representing the solutions of a linear equation lie on a straight line. The end of that exchange was as follows:

Interviewer:	[Referring to the equation (x-y)/7=(2y-x)/2, which was similar to an equation Danit had written on the board earlier just as an example of an equation in two unknowns] Is it possible, in your opinion, that this won't be a straight line?
Yuri:	I don't know...to check I need to get [lit. *do*] some results [of calculations] [*min 60*] but I think that it will come out a straight line.
Interviewer:	Why?
Yuri:	If the results are right then, well, I don't know exactly, sort of *that's what the teacher said, so it has to come out a straight line* [our emphasis].
Interviewer:	It's because the teacher said so?
Yuri:	I don't know, no—I can't explain it—I don't know.
Interviewer:	I see, she said it is a straight line, and you believe her? [the boys laugh]

Oren: Yes.
Yuri: [Sarcastically] No, she's lying.

CONCLUDING DISCUSSION

To summarize, what we see in the case of Yuri and Oren—and, as we suggested in the introduction, they were not atypical—is that despite Danit's conscious attempts to organize her lessons with eye to representations, Yuri and Oren do not appear to understand them as showing different mutually reinforcing views of linear equations; they do not see the line *as* a representation but as a *solution* method, which for them at this stage only means finding the value of y for a given x.

It may be because they expect the graphic representation to be a solution method, rather than a *bona fide* representation, that they think of the graphic representation as redundant. But whatever the reason, in none of our interviews did we find indications that students appreciated the graphic representation as complementary to the algebraic representation of linear relation. On the other hand, they seem to grasp that *Danit* attached importance to the different representations, and, accordingly, they produce statements in line with her approach. These statements, moreover, are sometimes convincing enough to deceive and, therefore, can mask the students' lack of true sympathy with and understanding of what the teacher tries to instil in them.

The division between the teacher's intention of what she was doing and the students' interpretation of what was expected of them (see the figure in the second section) might, then, be one reason why the students in this class did not seem to get the idea that representations are to be selected, applied, and translated. But, of course, this only begs the question. We need to ask *why*, in the first place, there was this gap, why these students seemed able only to give lip service to Danit's emphasis on 'connecting' and 'translating' representations.

One possibility may be the absence of *mediating elements*, that is, not just the presence of different representations said to be connected but 'connectors' as well. Formally, such connectors between representations are isomorphisms, and Powell and Maher (2003) have suggested that students can themselves discover isomorphisms. But in fact what allows learners to connect representations may have much more variety. Thus, Even (1998), speaking about multiple representations of functions, argues that the flexibility and ease with which we hope students will move from representation to representation depends on what general strategy students bring to mathematical situations, what context students place a problem, what previous underlying knowledge students possess, and perhaps other things as well. In other words, the efficacy of multiple representations in the classroom needs more than the multiple representations themselves. Thus, Even writes:

> "...concluding that the subjects who participated in this study had difficulties in working with different representations of these functions is not enough. Much more important is to understand how these subjects think when they work with different representations of functions" (p.119)

It might be then that we need to be more willing to treat multiple representations as a *terminus ad quem* than as a *terminus a quo*, that is, it may be that we have to challenge a multiple representations approach as a framework to begin with in teaching and think of as a distant goal that may not be achieved until the learner has had considerable experience in kinds of thinking that potentially link representations. This conclusion, if it is valid, is sobering for educators who want to promote multiple representations by presenting many representations all at once. But a sobering message such as this may be what is needed for learners to begin and reap truly the potential benefits of multiple representations.

References

Amit, M. & Fried, M. N. (In press). Authority and Authority Relations in Mathematics Education: A View from an 8^{th} Grade Classroom. *Educational Studies in Mathematics*.

Clarke, D. J. (1998) Studying the Classroom Negotiation of Meaning: Complementary Accounts Methodology. In A. Teppo (Ed.) *Qualitative Research Methods in Mathematics Education* (pp. 98-111). Reston, VA: NCTM

Even, Ruhama (1998). Factors Involved in Linking Representations of Functions. *Journal of Mathematical Behavior, 17(1)*, 105-121.

Fried, M. N. & Amit, M. (2004). Time and Flow as parameters in international comparisons: a view from an Eighth Grade Algebra Lesson. In Marit Johnsen Høines, Anne Berit Fuglestad (eds.), *Proceedings of PME 28, 2*, pp. 33-40. Bergen, Norway: Bergen University College.

Ginsburg, H. P. (1997). *Entering the child's mind: the cognitive clinical interview in psychological research and practice*. New York: Cambridge University Press.

Goldin, G. A. (1997). The PME Working Group on Representations. *Journal of Mathematical Behavior, 17*(2), 283-301.

Goldin, G. A. (2002). Representation in Mathematical Learning and Problem Solving. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp.197-218). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

Greeno, James G. & Hall, Rogers P. (January 1997). Practicing Representation: Learning with and about Representational Forms. Phi *Delta Kappan, 78*(5), 361-367.

Janvier, C. (Ed.). (1987). *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. (1999). Representations, Inscriptions, Descriptions and Learning: A Kaleidoscope of Windows. *Journal of Mathematical Behavior, 17*(2), 265–281.

NCTM (2000). *Principles and Standards for School Mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Powell, A. B. & Maher, C. A. (2003). Heuristics of Twelfth Graders' Building Isomorphism. *Proceedings of the PME 27, 4,* pp. 23-30. Honolulu, Hawaii: University of Hawaii.

Schultz, J., & Waters, M. (2000). Why Representations? *Mathematics Teacher*, 93(6), 448-453.

Stigler, J. W. & Hiebert, J. (1999). *The Teaching Gap*. New York: Free Press.