ARITHMETIC/ALGEBRAIC PROBLEM-SOLVING AND THE REPRESENTATION OF TWO UNKNOWN QUANTITIES

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We deal with the study of the senses and the meanings generated in the representation of the unknowns in the resolution of word problems involving two unknown quantities. The discussed cases show the difficulties that the students beginning the algebra learning have to deal with when using the equality between "unknown things". For them, applying the equality transitivity property between different (but equivalent) algebraic expressions, or replacing an unknown quantity with its representation in terms of another one is not derived from an extension of the transitivity between numerical equalities or from the numerical substitution. This may have important implications in the algebra problem-solving teaching domain, in which it is usual to take for granted that students spontaneously transfer these numerical issues to the algebraic realm.

Previous research has been undertaken to probe cognitive processes that take place in solving word problems in the transition from arithmetic to algebraic thinking. Bernardz, Radford, Janvier, & Leparge (Bernardz, Radford, Janvier, & Leparge 1992; Bernardz, 2001) have substantially contributed to this research area. Puig and Cerdan (1990) have formulated criteria to determine when a word problem can be considered as algebraic. From a different perspective, A. Bell (1996) has approached this matter by showing through examples how generic problems can provide algebraic experiences that develop manipulative algebraic abilities. Rojano and Sutherland (2001) have studied how technological environment can help students to represent and solve word problems without having to take on board with the algebra symbolic code, from the very beginning.

The present paper addresses the theme of arithmetic/algebraic problem-solving from a different point of view. The results presented are part of the research program "The Acquisition of Algebraic Language " (Filloy and Rojano, 1989; Rojano, 1994), which intends to throw light on the uses of **Mathematical Sign Systems** (**MSSs**) (Filloy, 1990) which will culminate in the competent use of the System of Signs of Symbolic Algebra. In previous reports (Filloy, Rojano and Solares; 2003), we have dealt with the problems of the *significations* and *senses* generated in the acquisition of the syntactic abilities needed for the manipulation of what is "unknown" in solving equation processes. In the study we report here , we approach the same theme in the context of the arithmetic/algebraic world problem resolution. We emphasize the role of teaching interventions that promote the use of *Signs Systems* in which suitable strategies for the solution process may be developed.

THEORETICAL FRAMEWORK

In this study, we adopt the theoretical perspective of *Mathematical Sign Systems* (*MSSs*) developed by Filloy, Rojano, Puig (see Filloy, 1990) to analyze the different ways in which pupils specify what is unknown and identify the central relationships of a word problem during the solution process. In previous papers, we have referred to this actions as *logical/semiotic outline* (see for example Filloy y Rubio, 1993). Specifically, we use the theoretical elements that are define below.

The ways to manage what is "unknown" and the resolution methods.- We call *Method of Successive Analytic Inferences (MSAI)* the classic analytic method, consisting of the conception of the problem formulation as descriptions of "possible states of the world", and the transformation of such descriptions through logical inferences. What is "unknown" in the problematic situation is interpreted as an *unknown quantity* (a particular number, the age of a person or a measurement of a figure, etc.) The solution is obtained as a "possible state of the world" in which the *unknown quantity* value is described (Filloy and Rubio, 1993).

We call *Cartesian Method* (*CM*) the one usually presented in the algebra text books. In our theoretical perspective it consists of the representation of the relationships between the known and unknown elements using a Sign System (that of algebra) "more abstract" than the one in which the problem is formulated, and which allows to transform these relationships into one or several *mathematical texts* the decoding of which gives the solution when returning to the original Sign System.

Mathematical Texts and Teaching Models.- From this theoretical perspective, a *Teaching Model* is understood as a problem situation sequence, that is, a *mathematical text* sequence T_n (Filloy, 1990), which production and decoding enable to interpret, finally, all the texts T_n is an MSS more abstract, which code makes it possible to decode the texts T_n as messages with a socially well-established mathematical code: the one proposed by the educational aims.

Sense of a method.- During the learning process, the competent use of the method is achieved through the concatenation of the actions triggered by the solution process (Filloy, 1990; Filloy, Rojano and Solares; 2003). Such concatenation, carried out at each stage of the *Teaching Model*, produces what we call the *sense of the method*.

THE EXPERIMENTAL DESIGN

We use the methodological framework of the *Local Theoretical Frameworks* (Filloy, 1990) in which each specific object of study is analyzed through four interrelated components: (1) a *Teaching Model*, (2) a *Cognitive Processes Model,* (3) a *Formal Competence Model,* that simulate competent performance of the ideal user of a *MSS*, and (4) a *Communication Model.*

We analyzed protocols from videotaped clinical interviews carried out with 15 13-14 year old pupils from "Centro Escolar Hermanos Revueltas", in Mexico City. These students had already been taught on how to solve linear equations with one unknown

and related word problems. At the beginning of the study, these pupils showed to have different ability degrees in these algebra topics.

The script of the interview was designed on the basis of a *Teaching Model* that introduces the resolution of word problems involving two unknown quantities through a sequence of *mathematical texts* divided into two blocks:

1. *Abaco Problems* (problems of the form "Find a number that…"), and

2. *Problems in context* (formulated in different contexts, such as geometry and ages calculation)

Based on the **Formal Model** analysis the items for blocks (1) and (2) were designed as follows: We designed *Abaco problems* that are associated with two equations systems formed by an equation of one unknown and another involving two unknowns. The *problems in context* designed are associated to systems formed by two equations, both involving two unknowns. Items in both blocks were presented to pupils according to the syntactic complexity (from less to more complex).

RESULTS FROM THE INTERVIEW ANALYSIS.

We analyze interview excerpts from three cases. Two of them, An and Mt are highly competent students in relation to operating one mathematical unknown, and the third one, L, is an average- student in the same terms. These cases allow us to analyze the difficulties that subjects have to cope with problems involving two *unknown quantities*. We shall focus on the *problems in context* block, for in this part of the interview, pupils are required to develop new solution methods as well as new semantics and syntax associated to such methods. On the basis on these three cases, we discuss children's productions in the **perimeter problem** and in the **ages problem**.

Regarding the results found in the *Abaco problems* block, it is important to mention that pupils could identify the two unknown quantities and recognize the two restrictions on their numerical values. In the same block all the subjects deal with the solution of the *Abaco problems* following a route of three steps: (1) finding the unknown value in the one-unknown "equation"[i], (2) replacing the found value in the two-unknown "equation", and (3) finally, finding the value of the second unknown. This route coincides in appearance with the **Formal Model** steps to solve this sort of problems. Nevertheless, what children do is to follow the route using a variety of MSS strata, such as natural language, arithmetic or algebra, and not necessarily that of the manipulative algebra. Due to the regularity with which this spontaneous strategy manifests itself and to the importance it attaches later, we will call it strategy **S(1,2)**.

The perimeter problem:

> *The perimeter of a rectangle is 5 times its width, the length measures 12 meters ¿How much does the width measure?*

This is the first problem in the interview in which the two equations include both unknowns. The subjects translate with no difficulties the problem into the algebraic

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language and find and equation system of the form: **P = 5a** and **P = 24 + 5a** They interpret these expressions as texts about **unknown quantities**, the measurement of the width "**a**" and the measurement of the perimeter "**P**". After solving the *Abaco problems block*, they realize that if they find the value of one of theses **unknown quantities**, they will be able to find the value of the other one and solve the problem.

The strategy S(1,2) and the two symbols.- L intends to apply strategy S(1,2), but after a number of syntactic transformations performed on the equations she notices that it is not possible to apply such strategy:

L: I want to, I mean I want to solve the... I want to solve this equation.	She points out the equation she previously wrote: *24 + 2a = P*.
L: This is going to change from this side to the other side, subtracting. But <u>I have two symbols</u>: "*P*" and "*a*".	She points out *2a*.
L: Look, I want to find the value of "*a*", but... ¡Ups! No, I don't know….	She points out *2a* and then writes: *24 = -2a*

The appearance of the two **unknown quantities** in both equations does not allow her to apply strategy **S(1,2)**, which previously allow her to solve all the previous problems. This phenomenon appears in all the cases analyzed. Two main algebra concepts that children will build up later on will allow them to apply this strategy. The concepts in question are: ***algebraic transitivity*** and ***equality between algebraic expressions***.

Algebraic transitivity and numerical transitivity.- Unlike what is taken for granted in algebra traditional texts books, the ***transitivity between algebraic expressions*** is not derived from a simple extension or generalization of the ***numerical transitivity***. For the cases we analyze, applying the transitivity property to the equations of this problem makes sense only when such property is referred to the problem context. This issue is illustrated with the following excerpt.

Mt wants to find the value of the width (represented by "***x***" in her case). She has written these set of equations: **P = 5x** and **P = 24 + 5x**.

I: ¿Don't you figure out how could we get the value of "x".	
Mt: Well, yes. Let's see ... "8" plus "24"... makes "16"…No! Okay, I have already found out how!	Mt does some calculations verbally. Mt writes down: *8*. This is the value of the width that solves the problem.
Mt: So, now we have two "*x*" plus 24.	Mt writes the equation: *2x + 24 = 5x.* Here Mt equalize the two expressions she has for the perimeter.
Mt: Here we have "*x*" two times and here we have "*x*" five times.	Mt points out *2x* and *5x*.

For Mt establish apply the ***transitivity*** to the expressions for the perimeter only has sense if there is a number that makes the equation true. In fact, Mt refuses to write the equation among the expressions and only does it once the width numerical value is found.

In short, to solve this problem, pupils carry out a series of *analytic inferences* (MSAI) that give sense to the algebraic expressions equalization. For them, the **unknown**

quantities have their referents, whether in the rectangle drawing corresponding to the problem or in the number that makes true the corresponding numerical equation. This tendency to be attached to the problem context makes in these cases that the *transitivity* is carried out for quantities (not for algebraic expressions).

The ages problem:

> *The difference between the ages of Juan and Carlos is of 12 years. In 4 years time Juan, who is the oldest, will double Carlos' age. Which are the current ages of both?*

This problem solution represented new difficulties for the interviewees. Its syntactic complexity demands not only high translation skills, but also to apply the *sense* of the actions recently generated: the use of one unknown expression given in terms of another unknown factor.

Manipulating the unknown through the MSAI.- An resorts to his high logical analysis competencies to solve this problem. Although he is highly competent in manipulating one unknown, he does not have yet the necessary syntactical competencies to algebraically manipulate the two unknowns of the problem. An uses the algebraic code to represent the relationship between data and unknowns, and understand them as numerical relationships between *unknown quantities* (Juan and Carlos' ages). These *mathematical texts* allow him to analyze the relationships by applying MSAI. He spontaneously carries out the following reasoning:

An: The difference between Juan and Carlos' ages is of 12 years. Juan, I'll name him "*J*", then you have that "*J − C = 12*" and we also have that "*J + 4*" will be equal... yes!, "*J + 4 = (C + 4) 2*".	Here he finds the associated system of equations: *J − C = 12* and *J + 4 = (C + 4) 2*.	*Observations.* *An's reasoning for finding the current ages is the following: After translating the problem into the algebraic language, he realizes (as an obvious implication of the problem context) that the*
An: Then, lets first see which we can get..., we get..., I've got it, wait, Juan's age..., no here.	He points out the equation: *J + 4 = (C + 4) 2*.	*difference between Juan and Carlos' ages will be always of 12 years. Specifically, in 4 years time it will be of 12 years. But he also*
An: I have to get that 12 will be the difference, even, of these two.	He points out *J + 4* and *C + 4* in the equation: *J + 4 = (C + 4) 2*.	*knows that in 4 years time, Juan's age will be the double of Carlos' age (up to that date). So, he have*
An: So Carlos' age up to that date would be 12…	He points out *J + 4* and *C + 4*.	*that in 4 years time Juan's age will be: (1) Carlos' age up to that date plus 12 years (because the ages*
An:…and the age of Juan would be 16 in 4 years time. No!, it wouldn't be 16. The age of Juan would be 24 in 4 years time.		*difference is always 12 years), but it will also be (2) two times the age of Carlos(up to that date).* *Then, he deduces that one time the age of Carlos(up to that date) has be equal with 12 years.*
An: So, from it I get that Juan is now 20 years-old, and Carlos is 8 years-old, the difference is 12… and this other thing I have would be 20+4 equal to (8+4)2, 8+4 is equal to 12, 2 times 12 is equal.	He refers to the equation: *J − C = 12*. He points out the equation: *J + 4 = (C + 4) 2*. He makes the calculations verbally.	*Then, up to that date, Juan's age will be 24.From that, he obtains that the current ages are 20 and 8.*

An carries out the series of *analytic inferences* from the Natural Language Signs System. He does not need to use any strictly algebraic transformation nor signification. His solution requires a complicated logical analysis and an intense use of the working memory.

Actions sense and the need for the mathematical unknown notion[ii].- L and Mt did not show to have the same logical analysis level as An in the ages problem. They require the interviewer's intervention to overcome the difficulties encountered in this problem. Introduction of the ***mathematical unknown*** notion was necessary. Lets see how it is done in L's case.

She has not acquired the competencies to interpret Carlos and Juan's ages as ***mathematical unknowns***, which would allow to syntactically manipulate them. At the outset, she interprets them as ***unknown quantities***, which she cannot operate because she do not known its value.

With the intervention of I, L has translated the problem to the system of equations: *J – C = 12* and *J + 4 = 2 (C + 4)*.	
I: Here you have that Juan's age minus Carlos' age equals 12. If we had here a number, for example 10, could you find Juan's ages value?	I points out the equation: *J – C = 12* I writes *10* below *C* in that equation.
L: Yes! Adding these two.	L points out *10* and *12*.
I: Ok. So, if we knew Carlos' age, could you calculate Juan's age?	
L: Yes!	
I: How?	
L: Adding Carlos' age to 12.	
I: Then, if I represent Carlos' age with C, how much is the value of Juan's age?	I points out *C* and *J* in the equation: *J – C = 12*.
L: I don't know it yet.	

At this moment, the teaching intervention was necessary to favor the strategies and interpretations that allow to solve the problem:

I: How much would "*J*" value? It is a number, a number you don't know, but it is a number. Suppose that I already know what number is it, but I do not want to tell you. I'll call it "*C*". I tell you: it is "*C*". Then, how much is the value of Juan's age?	
L: "*C*" plus 12.	L writes the equation: *J = C + 12*

Once the ***mathematical unknown*** notion was introduced, L solves the problem picking up the ***sense*** learned in the previous problem (the ***perimeter problem***):

L already got the equations: *J = C + 12* and *J + 4 = 2 C + 8*. Below them she writes the equation: *(C + 12) + 4 = 2C + 8*.	
I: What did you do? Why did you put that here?	I points out *C + 12* in the equation: *(C + 12) + 4 = 2C + 8*.
L: So that... Do you remember when "*P*"... when here... we replaced it with "*5a*"? Well, here I substituted "*J*" by "*C* plus *12*", so that there are only "*C's*".	L points out to the perimeter problem. She points out the *J* in the equation: *J + 4 = 2C + 8*, and *C + 12* in the equation: *(C + 12) + 4 = 2C + 8*.

L makes a substitution in accordance with the (intermediate) sense learnt in the **perimeter problem**: "*I substituted "J" by "C" plus 12 so that there are only "C's"*". L's actions are addressed by the application of the strategy S(1,2).

Although, in the cases we analyzed, introducing the notion of **mathematical unknown** was crucial, this notion had to be interpreted by the subjects in terms of concrete referents (those of the arithmetic MSS), in order to **manipulate an unknown factor in terms of the other** or to apply the **transitivity** property to "algebraic expressions". In fact, in this new transition step (when students for the first time have to deal with problems and systems of equations involving two unknowns), building up more abstract notions involves interpretations of such notion at a more concrete level.

FINAL REMARKS AND TEACHING IMPLICATIONS

Issues arising from this study suggest that dealing with problems and equations involving two unknowns quantities imply the necessity to re-elaborate: (a) the notion of *mathematical unknown*, (b) the notion of *algebraic equality*, and (c) the notion of *algebraic representation of the unknown*. Nevertheless, traditional teaching models introduce the problems with several unknown quantities, assuming that, once the first elements of the algebraic language are acquired, the further development of algebra will be carried out because of a simple extension of what was previously learned: *one unknown representation and manipulation*.

Specifically, some of these results show that applying the equality transitivity property between different (but equivalent) algebraic expressions, or representing an unknown in terms of another one , are not derived from a generalization of the numeric equality transitivity property, nor from the numerical substitution.

Additionally, advantages and difficulties of the MC are shown. The application of the MC allows to translate the sentences given in the Natural Language Signs System into *mathematical texts* given in the algebraic MSS and then to apply syntactic transformations in said SS . In this way, it is prevented that the significations specific of the problem context interfere the final analysis stage of the resolution process and the operation of the unknown is enabled.

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[i] With **_"equation"_** we refer to a relationship between known and unknown quantities expressed in natural language and not to the algebraic equality called **_equation_**, as it is the case of the Abaco's problems which are entirely rhetoric.

[ii] Here we make the difference between the terms **_unknown quantity_** and **_mathematical unknown_**. With **_mathematical unknown_** we refer to a value which, once substituted in an equation or set of equations, makes true the restricted equalities. That is, the notion **_mathematical unknown_** belongs to the realm of symbolic algebra.