THE USE OF MODELS IN TEACHING PROOF BY MATHEMATICAL INDUCTION

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Abstract: *Proof by mathematical induction is known to be conceptually difficult for high school students. This paper presents results from interviews with six experienced high school teachers, concerning the use of models in teaching mathematical induction. Along with creative and adequate use of models, we found explanations, models and examples that distort the underlying mathematical ideas and show teachers' conceptual difficulties.*

INTRODUCTION

The Israeli high school curriculum includes proof by mathematical induction for high and intermediate level classes. Usually in grade 11, students are taught to prove algebraic relationships such as equations, inequalities and divisibility properties by mathematical induction.

Proof by mathematical induction is a method to prove statements that are true for every natural number. In order to prove by mathematical induction that a statement is true for every natural number n, one has to establish the validity of two conditions:

That the statement is true for n=1;

That if a statement is true for any natural number k, then it is also true for its successor k+1.

In this paper the validation of the first condition will sometimes be called the induction basis and the validation of the second condition will sometimes be called the induction step. Moreover, the assumption used in the second condition, namely that the statement is true for k, will be called the induction hypothesis.

The research literature shows that high school students and prospective teachers are facing difficulties in understanding the idea of mathematical induction in depth. For example, Fischbein & Engel (1989) showed that for many students it is difficult to build the induction step on the basis of a statement that has not been proven, namely the induction hypothesis. As a consequence, many students adopt wrong attitudes like "The validity of the induction basis confirms the induction hypothesis", "the validity of the induction step confirms the induction hypothesis" or "the validity of the induction hypothesis is limited and should be considered true unless the contrary has been proved".

Avital and Libeskind (1978) also relate to students' difficulty to understand the implication in the induction step. They recommend to hold preparatory discussions about the nature of implications and to give students opportunities to explore and formulate their conjectures. They also recommend to give students the opportunity to

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gain confidence in the induction step by introducing a naive approach to mathematical induction, namely to show how the truth of the statement for n=2 follows from its truth for n=1, the truth for n=3 from the truth for n=2, and so on.

Movshovitz-Hadar (1993a) showed that many prospective teachers lack conceptual understanding of proof by mathematical induction and it is therefore easy to put them in situations of cognitive conflict and thus to shake their confidence that proof by mathematical induction works. She also suggests (Movshovitz-Hadar, 1993b) holding discussions about conceptual aspects of mathematical induction with students, and proposes tasks that can be used as trigger for such discussions.

In summary, the research literature shows that meaningful understanding of proof by mathematical induction requires complex knowledge. In this paper, we shall refer to three aspects of that complex knowledge:

Understanding the structure of proof by mathematical induction, namely

that the two conditions are independent of each other and that both of them are necessary;

how these two conditions are integrated to result in the overall proof.

Understanding the stage of the induction basis, namely

that checking the validity of the initial case is an integral part of the proof – not a preliminary activity that is intended to shed light on the statement or to give confidence that the statement to be proved is true;

that one has to check only for n=1 and that other checking activities, if conducted, are not necessary parts of the proof. (We disregard here more complicated cases such induction steps from k to k+2.)

Understanding the stage of the induction step, namely

that this is a separate statement and proof, nested in the overall proof;

that the induction hypothesis is different from the overall statement, and what the essence of the difference is;

that the variable k used at this stage can take any natural value.

THE USE OF MODELS FOR MATHEMATICAL INDUCTION

In this paper, we shall deal with some aspects of the use of models for teaching mathematical induction. Hereby, we use the notion of model in a wide sense, not necessarily limited to physical models.

A role for models can be to demonstrate, illustrate and interpret the method of proof by mathematical induction, and thus to support understanding by use of pictorial language that might be more accessible to learners than the formal language commonly used in teaching mathematical induction. According to Fischbein (1987), if the original and the model belong to different systems, the model provides an analogy. On the other hand, if the original is a certain category of objects and the model is provided by an example in this category, then the model is paradigmatic.

The most common model for mathematical induction appears to be the domino model: Domino tiles are standing in a row. The dominoes are arranged in such a way that if we push the first one it falls and causes a chain reaction - every domino tile knocks down the next one and as a consequence all the dominos in the row fall. In Table 1 (see next page), we present our analysis of how elements of proof by mathematical induction can be represented by the domino model, and how teachers can potentially use the model's language in their explanations.

Another model that teachers sometimes use for illustrating mathematical induction is presented by the story of the Hanoi towers (see Figure 1)

THE STORY OF THE HANOI TOWERS

In the temple of Brahma in Benares there are three diamond needles. At the creation, 64 golden rings of different sizes formed a tower on one of the needles, the largest at the basis, the others stacked upon it according to decreasing size. The priests in the temple transfer rings between the needles, day and night, according to Brahma's two rules: (1) Only one ring can be carried at a time, (2) No ring may be placed on top of a smaller one. They believe that when the 64 rings will form the same tower on another needle the world will vanish.

Is there a reason for panic?

Figure 1: The story of the Hanoi towers

There are several ways to demonstrate the idea of proof by mathematical induction using the story of the Hanoi towers. One possibility is to concentrate on the proof that it is possible to transfer any number of rings according to the rules: It is trivial to transfer one ring; and if we know how to transfer k rings from one needle to another, than we can transfer k rings to the third needle, put the $k+1^{st}$ ring at the bottom of the target needle and transfer the k rings on top of it using the same method as before. Another possibility is to concentrate on the proof that the number of single ring transfers needed to transfer a stack of n rings is 2^n-1.

A table similar to table 1 can be built for demonstrating the use of the Hanoi towers in teaching mathematical induction. For example, the legitimacy of the induction hypothesis, or the similarity between the statement to be proved and the induction hypothesis, can be dealt with by reference to the local scope of the hypothesis. In the language of the model, one can say: The hypothesis is that we can transfer k rings from one needle to another; on this basis we only prove that we can transfer k+1 rings from one needle to another. We are not talking about the whole tower now. That we shall do later…".

THE POTENTIAL CONTRIBUTION OF THE DOMINO MODEL TO UNDERSTANDING PROOF BY MATHEMATICAL INDUCTION	ELEMENTS OF PROOF BY MATHEMATICAL INDUCTION AND KNOWLEDGE RELATED TO PROOF BY MATHEMATICAL INDUCTION	
The pushing of the first domino tile so that it falls represents the basis stage.	The **representation** of the induction basis	The induction basis
In order to stress the necessity of this stage the teacher can demonstrate that no tile falls if the first one doesn't, even though the tiles are arranged in a row such that every tile would knock down the next one, if it were to fall.	The **necessity** of the induction basis	
Any falling tile knocks down the next one.	The representation of the induction step	The induction step
If a tile falls then it knocks down the next one.	**What** do we prove in the induction step?	
Any one of the domino tiles falls.	The representation of the induction hypothesis	
Questions about the **legitimacy** of the induction hypothesis, or about the similarity between the statement to be proved and the induction hypothesis, can be dealt with by reference to the model, and specifically to the local scope of the hypothesis. The teacher can say, for example: "On the basis of the hypothesis we only prove that the statement is true for k+1. It is like saying that if the tile **k** falls, then the tile **k+1** also falls. We don't talk about the all the tiles but only about **one tile and its successor**."	The induction hypothesis is **a part of the induction step** What is the **role** of the induction hypothesis? Why is it **legitimate** to use the hypothesis? What is the **difference** between the hypothesis and the original statement?	
Every tile knocks down the next one.	The **generality** of the variable k that is used in the induction hypothesis.	
A domino row with a sufficiently big distance somewhere	Situations, where the induction step cannot be applied at some place	

THE STUDY

Aim

Our aim in the present study was to learn about teachers' use of models for the teaching of proof by mathematical induction. What are the models that teachers use? What do they intend to clarify by means of the models, and how? In particular, we wanted to know whether the teachers' use of models is adequate in the sense that it suits the mathematical ideas they want to represent, demonstrate or illustrate.

In this paper we shall present findings concerning the use of models for the induction basis, specifically checking for n=1, and findings concerning the use of models for the induction step, specifically the use of the induction hypothesis in the proof.

Population

Six experienced high school mathematics teachers were interviewed for 30 - 60 minutes each. Teachers were considered experienced if they had at least 10 years of teaching experience. Two of the teachers have a Master's degree in mathematics education, and three are presently studying toward such a degree. One teacher (T6) is a graduate in economics and statistics who has taken a long-term course in order to become a mathematics teacher. This teacher also participated in three-year professional development program for mathematics teachers. All the teachers taught proof by mathematical induction at least twice, of which at least once during the two years immediately prior to the interviews. The interviews were semi-structured. The teachers were not informed that our main interest was the use of models. The interviewer referred to the use of models only if the teachers didn't bring up the issue by themselves in the course of the natural flow of the interview.

Use of models

Five out of the six interviewed teachers do use models when teaching proof by mathematical induction. All five use the domino model when they introduce the concept. Usually they arrange a set of domino tiles (or wafers) in an appropriate row and ask the students about the conditions that ensure that all the tiles in the row will fall. Three of the teachers said that they go back to the model later in the instruction, when students forget to check for n=1, or when they fail to show that the induction step is valid for all k.

Two of the teachers also use the Hanoi towers in the introductory stage of teaching mathematical induction. T2 always introduces the subject in an exploratory lesson, in which the students are asked to work out the number of steps needed for transferring n rings from one needle to another. Usually, some of her students discover a recursive law and most of the others discover the explicit formula. They are then asked to show that the two solutions are equivalent. This leads them to construct proofs by mathematical induction, without being explicitly aware of this. Her teaching design includes conducting the induction step for particular examples before generalizing it, following the naive approach as proposed by Avital and Libeskind (1978).

> “ …and then I say: let’s assume that one of the students went home and checked that it [the equivalence of the two laws] is valid for all the particular cases until the up to n=10. Can we show now that it is also true for n=11? … and then, during exercising they discover the depth of proving by induction. That someone checked until a certain place and we can prove it for the next place”.

We shall refer to this quote again in another context.

The explanations of two teachers, T1 and T3, always reflected the mathematical idea properly. However their use of the models was limited to stressing the necessity of the two conditions and their combined role. The explanations of the other teachers included occasional misuse of models.

Misuse of models for the induction basis

When talking about checking for n=1, T4 related:

> “We want to prove something so we start by checking case by case an example or two or three, and then we are convinced that it starts to work. And then I give them an example of an electric tool. I tell them that I want to buy a used tool. Before I bargain about the price, I plug it in to see if it’s worth bargaining.”

In another place, when relating to checking, she said:

> “If someone wants to get into the water, let’s say to a lake. Then, before a normal person jumps into the water, she decides to check the water temperature.”

Already the first sentence by T4 quoted above raises the suspicion that she presents the action of checking as a motivation for the proof rather than as a first step of the proof. This suspicion is then more than confirmed: Buying an electrical tool or getting into the water cannot possibly reflect the idea of checking as a part of a proof by mathematical induction because of two reasons. The first reason is that in these two examples the claim to be proved does not have the structure that makes proof by mathematical induction feasible: These claims do not include a natural variable, i.e. a variable that takes the natural numbers as values. The second reason is that the examples don’t stress the unique role of the checking as an integral part of the proof.

Misuse of models for the induction step

During the interviews the teachers related to the induction step, to the legitimacy of using a hypothesis and to the pertinent explanations that they give to their students. However, none of them mentioned that the assumption is a part of the induction step, and that at this stage there is no need to know whether the assumption is true or not. Nor did they mention that in the induction step we don’t prove that the statement is true for any “k+1”, but we prove that if the statement is true for a particular k then it is also true for it’s successor.

Three of the teachers, T2, T4, and T6, stated that the hypothesis is based on the induction basis. When T2 explained the legitimacy of the use of the hypothesis she referred to the Hanoi towers:

> "Just a moment, do we have a problem with that one student went home and checked all the cases up to 10 rings and basing on this we succeeded to prove that it is true also for 11 rings?"

T2 used the story of the Hanoi towers in a creative way to explain the induction step. However, her explanation is problematic: When she stresses the fact that someone went home and checked all the cases up to n=10, she does *not* present this as an analogy to relying on a hypothesis. One of the main difficulties of the reasoning process of the induction step is that we rely on the hypothesis without knowing whether it is true or not. To stress that point a teacher might say to her students: "None of you checked for n =20. Can we nevertheless prove that *if a tower of k=20 rings can be transferred, then a tower of k=21 can also be transferred*?"

T4 tried to exemplify the induction step with an example that does not include a natural variable. She continued her earlier example of buying an electric tool:

> "Yes, yes. The checking stage is clear. Now we go to the hypothesis. OK. If it works why do we have to assume that it works up to k? Because I say, if I see that it works then I assume that it will work until a certain stage. But I can't assume that it will work forever because it can stop working like all the electric tools do. I assume that it works. My task is to prove that it will also work in the next stage".

Two of the teachers, T2 and T6 didn't distinguish between the induction hypothesis and other features of the problem situation. T2, for example, said:

> "What is the assumption that it is true up to the k^{th} place - a certain natural k? The assumption is that all the dominoes are with equal distances, if they are close enough to each other, of course."

The distance between the domino tiles represents features of the statement to be proved, while the induction hypothesis would be represented by a falling tile.

One teacher, T5, who expressed some anxiety about using a hypothesis in the proof referred in her explanation to the use of axioms in geometry and to assumptions that people make in everyday life:

> "… so sometimes I look for help from geometry and I say that in geometry we also use hypotheses and base all the theory on these hypotheses."

T5 also attributed the anxiety about using a hypothesis to the mathematical language:

> "One thing that mathematics reflects is language, and this language is not complete and not always precise, but it clarifies things."

Later on, T5 stated explicitly that even though she knows that the use of a hypothesis is correct, she does not know what to answer when her students ask her why they may use a hypothesis.

SUMMARY

Teaching proof by mathematical induction meaningfully requires teachers' awareness of the difficulties that student may encounter with respect to proof by mathematical induction as well as complex mathematical knowledge.

The interviewed teachers are aware of the complexity of the proof process and of some students' difficulties. Consequently they make efforts to overcome the difficulties, including the use of models during instruction. Alongside some adequate and sometimes creative use of models, we found models that don't reflect the mathematical idea properly.

Concerning the induction hypothesis we found several wrong explanations: Looking at the hypothesis as based on the checking; considering the hypothesis as an axiom that can be true or false (also after completion of the proof process); confusing between the induction hypothesis and other conditions of the problem situation; attribution of the difficulty in using a hypothesis to the mathematical language, and even declaration of anxiety with using a hypothesis.

Concerning the induction basis we found a teacher who considers the induction basis as an action that justifies the proving effort and not as an integral part of the proof. This teacher also used examples that don't contain a natural variable.

Where teachers' explanations reflected the mathematical ideas properly, their use of the models was limited to stressing the necessity of the two conditions and their combination in the overall proof.

Considering the potential contribution of models to explaining proof by mathematical induction, we have shown that teachers could make more profound use of the models, in particular with respect to delicate points concerning the induction hypothesis. We suggest models should be used in teacher education, and that student teachers should be asked to explicitly and in detail establish the connections between model and abstract idea of mathematical induction.

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