ORCHESTRATING MATHEMATICAL PROOF THROUGH THE USE OF DIGITAL TOOLS

Rosamund Sutherland*, Federica Olivero*, Marnie Weeden**

*Graduate School of Education, University of Bristol

**The City Academy, Bristol

This paper explores the role of the teacher in the orchestration of common knowledge and in the interplay between intuitive/empirical and formal aspects within the context of students learning mathematical proof using a dynamic geometry software. The case of a secondary school teacher is discussed through the analysis of her learning initiative which involved the introduction of proof in geometry by using a dynamic geometry software. The analysis shows Marnie's focus on the relationship between construction and mathematical properties and the difference between proof and demonstration, and the way she orchestrates the work of the whole class in the construction of mathematical proofs.

INTRODUCTION

The research presented in this paper is framed within a socio-cultural approach, which suggests that mental functioning has its origins in social life and stresses the crucial role which communication through language and other semiotic systems plays in learning (Mercer et al, 1999; Wertsch, 1998). An important aspect of socio-cultural theory is the claim that human action is mediated by 'cognitive tools'. The notion of 'tools' includes a wide range of artefacts and semiotic systems where "cultural artefacts are both material and symbolic; they regulate interactions with one's environment and oneself. In this respect they are 'tools' broadly conceived, and the master tool is language" (Cole & Engestrom, 1993, p. 9).

This theoretical perspective also emphasises the fact that students actively construct knowledge drawing on what they already know and believe (Vygotsky, 1978). From this point of view students bring their implicit theories to any new learning situation and these influence what they pay attention to and thus new knowledge construction. Within this context the teacher has an important role in that "appropriately arranged contrasts can help people notice new features that previously escaped their attention and learn which features are relevant or irrelevant to a new concept" (Bransford et al 1999, p. 48). The teacher also has an important role in opportunistically and contingently drawing students' conversations and actions into increasingly elaborated and sophisticated mathematical domains.

In the context of mathematics, it is now widely accepted that the dynamic and symbolic nature of computer environments can provoke students to make links between their intuitive notions and more formal aspects of mathematical knowledge (Hoyles & Sutherland, 1989; Sutherland, 1998). It is also accepted that mathematical understandings do not develop spontaneously and that there is a need for a teacher to support students to move between informal mathematical knowing and the virtual world of mathematics (Balacheff & Sutherland, 1994). As discussed in Sutherland & Balacheff (1999) there is often a tension between students' individual and personal intellectual constructions and the collective and common knowledge which the teacher intends to teach.

This paper explores the role of the teacher in the construction of common knowledge and in the interplay between intuitive/empirical and formal aspects within the context of students learning mathematical proof using a dynamic geometry software.

PROVING

Research shows (e.g. Balacheff, 1988; Hoyles, 1997; Olivero, 2002) that the major difficulties of students' construction and understanding of proofs are represented by the coexistence of formal and intuitive aspects, which materialise for example in the transitions from empirical to theoretical practices, from intuition to deduction, etc.

The representations of geometric objects in a dynamic geometry software, as for example The Geometer's Sketchpad (GSP), are a way of bringing together formal and intuitive elements. GSP figures are midway between empirical and generic objects: they can be manipulated as empirical objects and the effect of this manipulation can be seen on the screen as it happens, but at the same time they incorporate geometric properties and as such represent generic mathematical objects. A range of tools is available to manipulate dynamic objects in GSP; dragging and measurements are two of the most commonly used. Within the context of proving in geometry, the possibility of using a measuring tool implies a need for scaffolding a 'good' use of measures, which does not get in the way of the development of formal proofs. Measures can be exploited differently according to different phases of the proving process (Olivero & Robutti, 2002) and the role of the teacher is key in making the students aware of this.

THE CASE OF MARNIE

This paper centres around an analysis of a learning initiative which Marnie Weeden developed through a process of working within the mathematics design team of the InterActive Education Project[i]. Marnie chose to work on geometry and proof with 13-14 year old students who were in the top-set of an inner city multi-ethnic comprehensive school (proof had recently re-entered the English mathematics curriculum). The design of the learning initiative was informed in an iterative way by theories of teaching and learning, research-based evidence on the use of ICT for learning mathematics, teacher's craft knowledge and the research team's expertise. The lessons which constituted Marnie's learning initiative were: 1- Introducing dynamic geometry and the construction process, 2- Proof or demonstration: identifying the difference, 3 & 4 - Proving that the sum of the angles in any triangle equal 180 degrees, 5 & 6 - Students presenting their proofs to the whole class (Weeden, 2002).

One of the university researchers observed and video recorded each of the lessons. The process of analysis involved viewing the video recordings of each lesson and progressively analysing the video data through the lens of our theoretical perspective.

FOCUSING ATTENTION: THE ROLE OF THE TEACHER

Analysis of the data shows that throughout the learning initiative Marnie continued to emphasise the relationship between construction and mathematical properties and the difference between proof and demonstration. Whereas there is no simple relationship between this focusing of attention and students' activity analysis of the whole teaching and learning initiative shows that across the series of lessons there was a convergence between students' and teacher's perspectives.

Construction and mathematical properties

The first introduction to a new tool is likely to influence students ongoing use of the tool and research has shown that students often start by drawing mathematical objects within a dynamic geometry environment as opposed to constructing objects from their properties. Marnie was aware of this literature and used the idea of the 'dragging test' (Healy et al, 1994) from the beginning of her work with students.

> Marnie [ ] basically this is a session in which you become familiar with the software you are using…[ ] so I am going to show you first of all what the tools do, what you can do [ ]. But we are going to do some construction of a variety of shapes. But using what we know about those shapes. Using their properties. (Lesson 1)

Within the first lesson, as Marnie demonstrated to the students how to use GSP to construct a square (through projecting her portable computer image on a screen at the front of the class) she explicitly modelled her own knowledge construction processes, emphasising that she was explicitly using the properties of a square and that "there's always right angles in it and the construction remains the same. [ ] just different sizes".

After this introductory phase the students worked in pairs on portable computers and tried for themselves to construct a square. Throughout this work as Marnie became aware that not all students were using mathematical properties she intervened again:

> Marnie Ok we've had something interesting here. Someone has just found out…they thought they were clever and drew a square, measured it, measured the angles, and guess what it didn't stay. Moved it about and suddenly it was a quadrilateral of all sorts of different dimensions. It has to stay…this one went all over the place because it wasn't constructed. You've got to use what you know are the properties and utilise them in this construction. Otherwise it will break. [ ] Just drawing lines will not work…you need to actually use your

knowledge of shapes in order to construct it. You need to use commands like the perpendicular bisectors, like parallel lines, that's what you need to do. Without actually using commands like that, using the constraints of a circle, circumscribing things, stuff that you know….. (Lesson 1).

This focus on properties continued throughout the whole design initiative. In the third lesson when the students had been asked to construct a rectangle Marnie again focused on mathematical properties:

Marnie And unlike the square you've got less constraints with that…so you know how to construct a pair of parallel lines… so you should be able to produce a proper rectangle…just to remind you because it's been a while….we didn't have the laptops last time (Lesson 3).

This focus on mathematical properties, a crucial part of constructing mathematical objects within GSP was also important when students began to construct their own mathematical proofs and was evident in the final proofs which students presented to the whole class (using PowerPoint) at the end of the learning initiative (see for example fig.1).

The difference between proof and demonstration.

Marnie started the second lesson by emphasising the difference between proof and demonstration.

Marnie If I say proof what do I mean

Rob Gathering evidence ..in order to back…..

Sarah Exploration

Marnie Gathering evidence to support a theory, conjecture? [ ] In science we repeat an experiment loads of times. Is that mathematics proof as we know it? There is a difference between proof and demonstration…are your eyes and the way your brain works enough for you…

Marnie then introduced a proof that the angles of a triangle add up to 180 degrees. She started by constructing a triangle in GSP, measuring the angles, finding they added up to 180 degrees and then asking if this was a proof. After soliciting a range of responses she again emphasised that measurement is not mathematical proof.

Within the third lesson students worked in pairs with GSP to develop their own mathematical proofs that sum of the angles of any triangle is 180 degrees[ii]. Despite Marnie's discussion about what constitutes a mathematical proof the majority of students started to use the measurement tools to construct a proof. This is likely to relate to their previous experiences of measurement in geometry and the types of empirical proofs (Balacheff, 1988) which they are likely to have been introduced to in primary and early secondary school. The following excerpt illustrates how Rachel and Joanna start to explore the possibility of measuring.

Rachel Is there some way we can calculate what J, K, L and M add up to

Rachel We'll just have look around (they start to look through the menus).

Jess Oh…angle bisector….that looks fun

Rachel So I guess we'll have to highlight an angle.

By experimenting they discovered how to measure an angle. They then discovered the calculator tool and started to sum the angle measures. At this point Marnie, becoming aware of their activity, intervened to the whole class.

Marnie ……before you go off on a tangent which is where you seem to be going...[ ] …you need construction but the other important thing is don't get het up and caught up in the measuring………measuring is not proof…you've already said that…measuring is not proof…for a start computers can make mistakes…also for the particular computer program it tends to measure to the nearest point. Zero point something…so what you'll end up with is something which doesn't equal 180 degrees….when you've measured it will add up to 181. So you cannot rely on that software. And the reason we are here doing this now is proof…so don't get muddled up with the measuring…measuring is not proof…it is being able to apply what we know about our angle laws to a situation in order to come out with some kind of reasoning, mathematical reasoning as to why that may add up to 180 and I know some of you are nearly there…

Interestingly Marnie had shown the students how to measure lengths and angles within lesson 1, unwittingly drawing attention to the measurement tools. We believe that the use of measurements should not be discouraged because anyway students will use this tool, drawing on their work with paper and pencil. On the contrary we need to find ways of 'enculturating' students in giving the appropriate status to measurements, according to the different phases of the proving process: they cannot be used as a mathematical proof, but they can be very useful in the phase of exploring, conjecturing and validating a conjecture within a dynamic geometry environment. Certainly the need for measures comes from the perceptual level when students have the intuition that, for example, two sides of a figure are equal, or one equals the double of the other and so on; however, when they read measures on the screen, or on paper, they are no longer working at a purely empirical level: he is working at a higher level, because they are looking for an answer to the question *Is my intuition true or false?* Measures work as a tool which can provide an answer: *yes/no.* The quantitative side of the information linked to the use of measures makes students feel safe and certain about a result and can provide a solid starting point for the subsequent construction of a proof.

Analysis of the video data shows that Joanna and Rachel, eventually stopped measuring and started to construct proof statements on the screen.

> Angles A, B, C and D are all right angles, they are 90 degrees and are all in rectangle so all the angles in the rectangles add up to 360 degrees
>
> Angles j,k,l and m are an average of 45 degrees each.

Whereas these ‘proof statements’ could be criticised for being empirical and descriptive of the figure the students have on the screen, they seem to have provided an important starting point in terms of supporting these students to enter the world of mathematical proof.

All the students produced their final proofs in PowerPoint and presented them to the whole class. An analysis of the final proof produced by Joanna, Rachel and Rick shows that they have moved from a focus on measurement and tautologies to the production of a proof which contains logical justifications for what they observed on the figure.

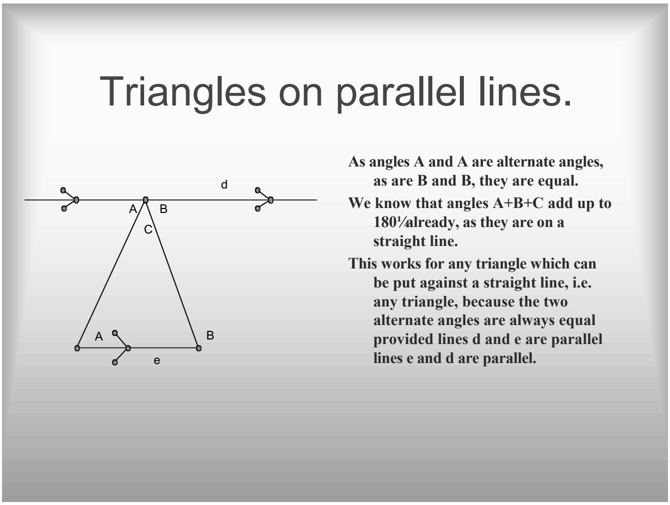

Figure 1. Excerpt from final PowerPoint proof for one group of students

All the students shifted from early uses of measurement to the construction of theoretical proofs and we argue that Marnie’s interventions played a crucial role in this respect. These interventions were based on Marnie’s observations of student activity in the class, her own a-priori analysis of what constitutes mathematical proof and her engagement with the research literature. We also argue that Marnie created a collaborative community which empowered the students to share their ideas and progressively refine their ideas about what constitutes a mathematical proof.

Marnie remember that there is no wrong or right here there is just ideas. There is just us coming together with ideas and that is us learning from each other about what we're doing and this is to do with working collaboratively together. OK learning to work together and come together with our ideas. (Lesson 1).

SOME CONCLUDING REMARKS

As we have discussed already all the students produced final proofs in PowerPoint and as a digital tool this seems to offer considerable potential in terms of supporting students to focus on the importance of linking together a set of deductive statements to be presented to a 'community' (the classroom in this case). Students imported their geometrical diagrams from GSP. The work with GSP is likely to have supported them to focus on the mathematical properties which became key aspects of their proofs. Each proof presentation was slightly different and some were more mathematically rigorous than others, but all students had started the process of producing mathematical proofs. Students when interviewed explicitly said that they valued the use of ICT tools, which allowed them to progressively develop their mathematical proofs. Within this context writing draft proofs on the screen in GPS enable them to begin to externally represent their proto-proofs which gradually evolved to become more formal and theoretically informed PowerPoint proofs. As Rick explained, constructing and undoing were an important part of this process:

Rick The thing was, much of our project was wrong; it wasn't wrong but large amounts of it were quite bad. So had we been doing it on paper it would have taken us longer to get nowhere, so it meant we could just delete it and start again. We used undo a lot. (final interview)

References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children* (pp. 216-238). London: Hodder and Soughton.

Balacheff, N. & Sutherland, R. (1994) Epistemological Domain of Validity of Microworlds: The Case of Logo and Cabri-géomètre, in Lewis R & Mendelsohn P (ed.) *Lessons from Learning*, IFIP Conference TC3WG3.3 North Holland, pp 137-150.

Bransford, J., Brown, A. & Cocking, R. (1999) *How People Learn, Brain, Mind, Experience, and School*, National Academy press, Washington.

Cole, M., & Engestrom, Y., (1993) A Cultural-historical approach to distributed cognition, Salomon, G. (ed) *Distributed Cognition*, CUP, Cambridge.

Healy, L., Hölzl, R., Hoyles, C., & Noss, R. (1994). Messing up. *Micromath, 10*(1), 14-16.

Hoyles, C. (1997). The Curricular Shaping of Students' Approaches to Proof. *For the Learning of Mathematics, 17*(1), 7-16.

Hoyles, C. & Sutherland, R. (1989) *Logo Mathematics in the Classroom,* Routledge, London.

Mercer, N., Wegerif, N. & Dawes, L. (1999) Children's talk and the development of reasoning in the classroom, *British Educational Research Journal*, Vol 25, No 1, 1999.

Olivero, F. (2002). *The proving process within a dynamic geometry environment*. Unpublished PhD thesis, Bristol: University of Bristol.

Olivero, F., & Robutti, O. (2002). An exploratory study of students' measurement activity in a dynamic geometry environment, *Proceedings of CERME2* (Vol. 1, pp. 215-226). Marianske Lazne, CZ

Sutherland, R. (1998) Teachers and Technology: the role of mathematical learning, in D., Tinsley, & D., Johnson, (eds) *Information and Communication Technolgies in School Mathematics*, Chapman & Hall, London. pp 151-160

Sutherland, R. & Balacheff, N. (1999) Didactical Complexity of Computational Environments for the Learning of Mathematics, the *International Journal of Computers for Mathematical Learning,* Vol. 4, pp 1-26

Vygotsky, L.S. (1978) *Mind in Society: The Development of Higher Psychological Processes*, Harvard University Press, Cambridge, Massachusetts

Weeden, M. (2002) Proof, proof and more proof, in *Micromath* Autumn 2002, pp.29-32

Wertsch, J. (1998). *Mind as Action*. New York: Oxford University Press.

---

[i] Interactive Education: Teaching and Learning in the Information Age, project directed by Rosamund Sutherland, Peter John and Susan Robertson (www.interactiveeduction.ac.uk) and funded by the ESRC Teaching and Learning Programme (Award No. L139251060). The mathematics team consisted of 11 teachers and 3 university researchers and worked together through a series of meetings of the whole team at the University and meetings of a teacher-researcher pair within the teacher's school. Each teacher chose an area of mathematics which they normally found difficult to teach and for which a particular use of ICT seemed to be a potentially valuable learning tool. The methodology of the project included the use of digital video as a research tool, together with interviews with the teacher and students and collections of the students' digital and non digital work.

[ii] From lesson 1 students had been encouraged to write on the screen and within lesson 3 they were asked to write their proofs on the screen.