STUDYING THE MATHEMATICAL CONCEPT OF IMPLICATION THROUGH A PROBLEM ON WRITTEN PROOFS

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In this paper, we present a didactic analysis of the mathematical concept of implication under three points of view : sets, formal logic, deductive reasoning. For this study, our hypothesis is that most of the difficulties and mistakes, as well in the use of implication as in its understanding, are due to the lack of links in education between those three points of view. This article is in the continuation of one previously published in the acts of PME 26. We present here the analysis of another problem from our experimentation. We want to show how a work on written proofs can allow a work on implication. Then we conclude with some transcripts.

INTRODUCTION

The existence of the implication as an object of natural logic, leads to confuse it with the mathematical object. As a result, the implication seems to be a clear object. Yet, students have difficulties related to this concept until the end of university, especially with regard to necessary conditions and sufficient conditions. Moreover, though it is in the heart of any mathematical activity, it is hardly ever taught in French teaching.

Our theoretical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is based on the work of V. Durand-Guerrier [Durand-Guerrier, 1999] on the one hand and of J. Rolland [Rolland, 1998] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. Rolland, as for him, was interested in the distinction between sufficient condition and necessary condition.

This study is a part of our thesis on the mathematical concept of implication. It follows and supplements the study presented at PME 26. We present three points of view on the implication, a mathematical and didactical analysis of a problem on written proofs and conclude with some transcripts.

THREE POINTS OF VIEW ON THE IMPLICATION

This paragraph was detailed in our previous research report in PME 26. Yet, we think this part of our research is necessary for the reader to understand the following problem and the aim of our research hypothesis.

The mathematical implication seems to be a model of the natural logic implication we use in our everyday life. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the natural one leads to obstacles in the use of the

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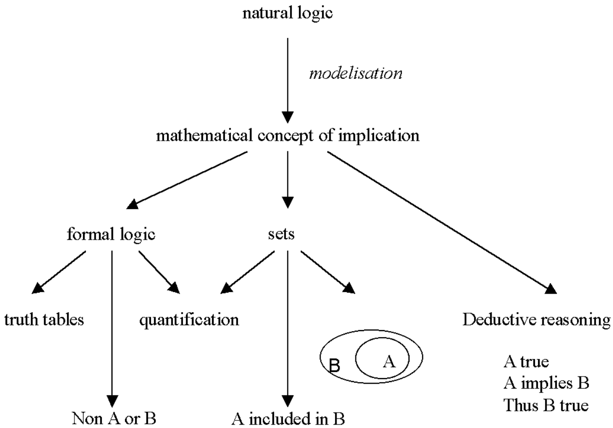
mathematical concept. An epistemological analysis [Deloustal, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.



Of course, these three points of view are linked and their intersections are not empty. We will not develop here the formal logic point of view (for example truth tables or formal writing of the implication).

We call "deductive reasoning" the structure of an inference step : "A is true ; A implies B is true ; Thus B is true". Its ternary structure includes a premise "A is true", the reference to an established knowledge "A $\Rightarrow$ B" and a conclusion "B is true" [Duval, 1993, p 44]. The reference statement may be a theorem, a property, a definition, etc. One thus builds a chain of inference steps : the proposition obtained as the conclusion of a given step is "recycled" as the entrance proposition of the following step. Therefore, in the deductive reasoning, the implication object is used only as a tool. However, in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects : to each property corresponds a set, the set of the objects which satisfy this property. The sets point of view on the implication can then be expressed as follows : in the set ***E***, if ***A*** and ***B*** are respectively the set of objects satisfying the property A and the set of objects satisfying the property B. Then, the implication of B by A (i.e. A $\Rightarrow$ B) is satisfied by all the objects of the set ***E*** excluded those which are in ***A*** without being in ***B***, i.e. by all the objects located in the area shaded here after.

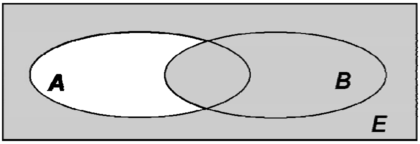


Figure 1

RESEARCH HYPOTHESIS

The experiments carried out for three years, within the framework of our research, have shown that the implication was not a clear object even for beginner teachers. Moreover, they showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of these mistakes and difficulties.

Following these comments, we formulate the research hypothesis : it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it.

In the following paragraph we show that a problem on written proofs, using only easy properties, may question the reasoning in a non obvious way.

CONDITIONS OF THE EXPERIMENTATION

The problem we present results from an experimentation carried out in 2001 with beginner teachers of mathematics. We worked with two groups of approximately 25 students at the IUFM[1] of Grenoble and Chambéry (France). This experimentation includes two three-hour-sessions on the proof and, in particular, on the implication. The first session contained two problems (one in geometry, one on pavings), the second one proposed a work on written proofs. For each meeting, a work by groups of three or four people was following an individual work to allow questionings and discussions. We presented, in PME 26, a problem of geometry resulting from the first session. We present, now, a problem on proofs, following the previous one.

Before beginning the analysis of this new problem, we want to remind the reader of the problem the beginner teachers had to solve in the previous session :

Let ABCD be a quadrilateral with two opposite sides having the same length. What conditions must diagonals satisfy to have : (P3) two same-lengthed other sides ?[2]

PRESENTATION OF THE PROBLEM

Here is the new problem as we gave it to the students:

Let us remind the previous problem :

Let ABCD be a quadrilateral with two opposite sides having the same length. What conditions must diagonals satisfy to have : two same-lengthed other sides[3] (P3) ?

A necessary and sufficient condition is : “the diagonals cut in their middle” (We call it C1)

What do you think about the following dialogue ?

X : Look, the condition “one of the diagonals cuts the other in its middle” is maybe also a necessary and sufficient condition ? (We call it C2)

Y : Impossible, since this condition C2 is strictly weaker[4] than the other (C1).

MATHEMATICAL ANALYSIS OF THE TASK

Let us call *Q*: the set of quadrilaterals ; *H*: the set of quadrilaterals with two same-lengthed sides (H: the property of having two same-lengthed sides) ; *NC*: the set of non crossed quadrilaterals ; *Cvx*: the set of convex quadrilaterals.

Discussion between X and Y

The argument of Y is not a valid argument. Indeed, two conditions can be equivalent on a subset even if they are not usually equivalent. For example, in the set of parallelograms, the condition “having one 90 degrees angle” is equivalent to the conditions “having four 90 degrees angles”.

In this problem, C1 could be equivalent to C2, to deny it one must prove it is false on the mathematical objects. Under a logical point of view Y is wrong.

Implications between C1 and C2

We can translate the first question (from X) by: “ In the subset *H*, is the equivalence C1$\Leftrightarrow$ C2 true ? ” or “ Is the equivalence [C1 and H] $\Leftrightarrow$ [C2 and H] true ? ”

In our problem, there is no equivalence between C1 and C2. We present two counter-examples, i.e. two quadrilaterals which satisfy H and C2 but not C1, figures 1 and 2.

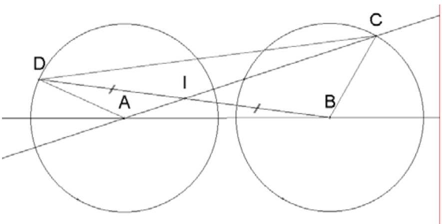


Figure 2

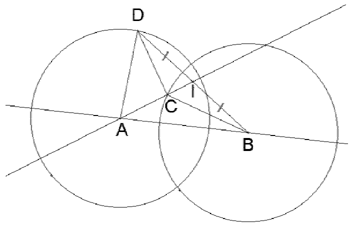


Figure 3

In this paper, we have no time to describe the set in which the two conditions are equivalent. But, we can notice that it is an interesting question for the beginner teachers who want to solve the problem.

Set of objects satisfying H and P3

To allow the reader to tackle the problem with the same knowledges as the beginner teachers, we give the solution of the previous problem. We do not detail it for lack of time but the reader can convince himself easily.

The objects satisfying both H and P3 have two same-lengthed sides and two other same-lengted sides, we have then the equivalence:

H and P3 $\Leftrightarrow$ Parallelogram OR Crossed Quadrilateral called CQ [figure 4]

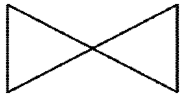


Figure 4

Implications between C1 and P3

In our problem, the assertion "A necessary and sufficient condition is that the diagonals cut in their middle " is false. Indeed, in the set *H*, C1 is not a necessary condition to P3, since C1 is equivalent to the condition "to be a paralellogramm".

$$\text{H and P3} \Leftrightarrow \begin{cases} \text{Parallelogram} \qquad\qquad \Leftrightarrow \text{C1 (diagonals cut in their middle)} \\ \text{or} \\ \text{Crossed Quadrilateral (CQ)} \end{cases}$$

That is to say that, in *H*, C1 implies P3 (C1 is sufficient for P3), but P3 does not imply C1 (C1 is not necessary for P3) as shows the counter-example (CQ).

On the other hand, if we place the problem in the set *NC* (non crossed quadrilaterals), there are then the equivalences : H and P3 $\Leftrightarrow$ Parallelogram $\Leftrightarrow$ C1. That is to say, in the set $H \cap NC$, C1 is a necessary and sufficient condition for P3.

Implications between C2 and P3

According to what we said previously, in $H \cap NC$, there are the implications:

H and P3 $\Leftrightarrow$ C1 $\Rightarrow$ C2, that is to say that C2 is then necessary to P3 but not sufficient as shown by the counter-examples figures 2 and 3.

DIDACTICAL ANALYSIS OF THE SITUATION

We present now the choices we made for this problem in terms of didactical variables. A didactical variable (DV) is a characteristic of a problem likely to involve, according to the values alloted to it, various strategies of resolution by students. All the variables of a problem are not didactical variables, the various strategies involved must be really different compared to the aimed learning. We want to show the variables wich allow a work on implication and especially those linked to the three points of view.

General choices

DV1: Mathematical framework for the problem

First of all, we choose, for our experimentations, very easily accessible mathematical concepts. Indeed, our hypothesis is that to see a work on the reasoning there must not be difficulties linked to a mathematical concept, to be able to distinguish difficulties due to the concept of implication. In this problem, there are only mathematical notions well known by students such as quadrilaterals, parallelograms, diagonals...

Moreover, we chose to place this problem within a geometrical framework. Another problem of our experimentation concerns pavings, it is easily accessible and is appropriate as for previous requirements. But we wanted to show them, since they are teachers, that even with a taught concept, like geometry, pupils can study the reasoning in a non usual way.

DV2: Practical organization of the session

Our hypothesis is that a research in groups is necessary for our problem. That allows a confrontation between the various points of view, in particular logical and deductive reasoning points of view. Furthermore, it stimulates discussions. Nevertheless, the first individual work gives to each one time to have his own idea about the problem. These various ideas will feed discussions.

Choices for this situation

DV3: point of view on the implication

Our hypothesis is that, to allow discussions, the problem must be in a mathematical context and not only in a logical one. Indeed, our previous experimentations with university students have shown that logical knowledges can coexist with false conceptions on implication. That is to say that, in case of a problem in an only logical context, all students could agree on the right solution without showing any difficulty. Therefore, we chose to place our question in a geometry context, in order to confront the formal logic point of view and the deductive reasoning point of view. However, since beginner teachers have studied the problem during the last session, we do the hypothesis that the difficulties of geometry should not be an obstacle to the discussion.

We want to know if, within the framework of a proof in geometry, the students are able to work only under one logical point of view without using mathematical properties. In response to our questioning, we will thus distinguish three types of answers:

The first one based on logical point of view: Y is wrong, this can be possible.

The second one linked to the mathematical contents: Y is right, in the subset H of this problem, C2 is not a sufficient condition for P3.

The third one is a combination of both, the strategy can use the deductive reasoning point of view and the logical one. That can possibly call into question the formal logic point of view.

Besides, to refute the argument of Y, properties of sets might be used. Our hypothesis is that some easy counter-examples (like: in quadrilaterals, to have three 90 degrees angles is equivalent to have four 90 degrees angles) will convince more easily than theoretical speech which will thus be less likely to appear.

DV4: value of truth of the starting assertion

The assertion “A necessary and sufficient condition is that the diagonals cut in their middle” is false in the set of quadrilaterals but true in the subset of non crossed quadrilaterals. Our hypothesis is that it compells students to take into account crossed quadrilaterals as soon as they leave the strict framework of formal logic. Crossed quadrilaterals are hardly ever taught in France, and we have shown in a previous experimentation that their presence enhance strategies based on sets point of view.

DV5: value of truth of the assumption of X

Our hypothesis is that, in order to confront the formal logic point of view and the deductive reasoning point of view, it is necessary that they give conflicting answers. Indeed, if both points of view say that X is right, then no discussion can take place. This is why we chose that the assumption of X is false from the deductive point of view whereas this argument could be valid from the formal logic point of view.

SOME RESULTS

Whereas the study of this experimentation is not finished, we can present right now some results and transcripts.

First of all, we can assert that a work on reasoning and implication was done. Indeed, no group found the problem obvious, they have all studied it during a long time. But, on the other hand, no group was stopped by difficulties of a mathematical nature. There were discussions, even though the mathematical objects were very well known.

Robert: It is an exercise which, as a teacher, I would not give before university.

There was really a confrontation between the formal logic point of view and the deductive reasoning point of view in a lot of groups. In most of these groups, even when they agreed the logical answer, they used the mathematical properties to search whether X is right or not.

Paul : Y says that if a condition is removed, it is not necessary any more, I do say it can remain necessary if it is already checked in the hypotheses.

But the group do not agree this ensemblist argument and keep searching implications between properties. Few minutes later, Paul convince them with a counter-example:

Paul: In the parallelograms, the condition “four 90 degrees angles” is necessary to be a rectangle but the condition “one 90 degrees angle” is necessary too.

This argument accepted, the group wants to know if X is right. After a few counter examples, they conclude showing that they distinguished well the logical reflexion and the mathematical reflexion.

Robert: X is wrong but the argument from Y is false.

Finally, we have seen marks of the three points of view. The sets point of view was also used in this task as shows Armelle’s argument.

To show to her group that Y is wrong, Armelle draws three “potatoes” (i.e. sets), *P*, *C1*, *C2* so that, in the set *P*, the sets *C1* and *C2* are equal, whereas outside *P* they are different. This is a theoric counter-example to the affirmation from Y, yet the group accepts it only at the last minute.

CONCLUSION

The analysis of the students' answers is still in progress. However, we can already say that the exercise fulfiled its role, as for the work on the implication since all groups have worked at least one hour on this problem. In addition, the three different points of view appear, implicitly or explicitly, in most groups. In particular, the sets point of view, which is not taught in france, appears many times. This is why we can do the hypothesis that the work, on this point of view, made at the time of the preceding meeting was used again.

These results are to be placed among others. Indeed, this problem forms part of a six hour experimentation on implication and reasoning. It includes other stages of work, in particular, other studies of written proofs, one problem of geometry and one problem in discrete mathematics. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem is, thus, to consider as part of a broader context.

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[1] Institut Universitaire de Formation des Maîtres (University Institute for the Formation of the Teachers)

[2] There were two other questions : (P1) two other parallel sides ? (P2) two 90 degrees angles ?

[3] not necessarily the same one as the two precedents, this was specified orally.

[4]a condition A is weaker than a condition B if the implication B $\Rightarrow$ A is true, i.e. if the set linked to the property B is included in the set linked to the property A. This expression is commonly used in french mathematics.

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