# UNDERSTANDING HOW THE CONCEPT OF FRACTIONS DEVELOPS: A VYGOTSKIAN PERSPECTIVE

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*Vygotsky posed a variety of meaningful ideas for education in his short life. This paper focuses on* everyday concepts *and* mathematical concepts *or* scientific concepts *from his theory, reorganizing these ideas according to a new idea of* sublated concepts*. Using a series of interviews from a third grade fraction class in Japan, the paper discusses how everyday and mathematical concepts arise out of discussions among children and a teacher, and how they develop into sublated concepts.*

## DEVELOPMENT OF CONCEPTS ACCORDING TO VYGOTSKY

### "Everyday concepts" and "mathematical concepts"

Vygotsky (1934/1987) thought that concepts consist of two types, i.e. *everyday concepts* and *scientific concepts*, and pointed out that the greatest difference between these two is whether they are based on a system. According to Vygotsky (1933/ 1975; 1934/1987), everyday concepts are not based on a system; rather they are based in rich daily contexts and, therefore, might be used incorrectly by children. For instance, in series of one-on-one interviews with second graders in Japan, when asked the meaning of a word *half*, a child described that half meant to share something equally among *three* people. This thought was derived from her daily experience of sharing sweets with the other two brothers. For her, therefore, halving always meant dividing something equally among three based on the meaningful daily contexts at home.

On the other hand, scientific concepts are defined according to a system that has developed in human history, and therefore they lack concrete contexts (Vygotsky, 1933/ 1975; 1934/1987). According to Kozulin (1990), Vygotsky's scientific concepts are based on "formal, logical, and decontextualized structures" (p.168). Therefore the concepts themselves are independent of any scientific subjects. *Mathematical concepts* are within a system and are characterized by formal and decontextualized structures. For example, 1/2 is a number constituting a number system, which consists of rational and irrational numbers. As a result, in this paper mathematical concepts will be used as an example of what Vygotsky called scientific concepts.

According to Vygotsky (1933/ 1975; 1934/1987) and the other researchers who discussed Vygotsky's everyday and scientific concepts (Panofsky, John-Steiner, and Blackwell, 1990; Kozulin, 1990; Davydov, 1972/1990; and Schmittau, 1993), everyday and mathematical concepts are defined as follows.

*Everyday concepts* are concepts that originate from children's daily lives through communication with their family, friends, or community; and thus are closely connected to concrete personal contexts. Children express them through their own words and use them in their ways of thinking without conscious awareness. As a result, everyday concepts are not systems; rather they are based on the subjectivity.

*Mathematical concept*s are scientific concepts that are connected to mathematics. They are based on a system, and therefore they have logic and objectivity. They are expressed in a mathematical language and introduced to children in a formal, highly organized education. Mathematical concepts make children to develop mathematical thinking and to require conscious awareness and voluntary behavior for concepts. The development of mathematical concepts as children's psychological developments, however, depends on their everyday concepts.

**The relation between the two concepts: mutual dependence or dichotomy**

As Zack (1999) discussed, the relation between spontaneous and scientific concepts might be regarded as either mutually dependent or dichotomous. On the one hand, the relation could be though of as dichotomous or mutually exclusive on the surface because of their conflicting characteristics. Yet Vygotsky's words suggest he may have taken another position: "There is a mutually dependence in the relation between the processes of development of children's concepts in daily lives and in school. It enables such relation that the processes of these two concepts pass in the different ways" (Vygotsky, 1933/1975, p.122). The following sections describe in detail the relation between the two kinds of concepts and their development. However, care must be taken when inferring what Vygotsky said about the relation of scientific and everyday concepts, for while mathematical concepts are here equated with scientific concepts, Vygotsky himself did not deal specifically with *mathematical concepts.*

**The development of concepts in Vygotsky's theory**

Vygotsky (1934/1987) thought of *concepts* as follows:

> We know that the concept is … a "complex and true act of thinking" that cannot be mastered through simple memorization. … At any stage of its development, the concept is an "act of generalization." The most important finding of all research in this field is that the concept – represented psychologically as word meaning – develops. The essence of the development of the concept lies in the transition from one structure of generalization to another. … This process is completed with the formation of true concepts (pp.169-170).

Accordingly, a concept is not static or unchangeable, but a complex and dynamic act of thinking and an act of generalization. In addition, psychologically this concept develops from one structure of generalization to another. The process of this concept development could be paraphrased as an identical process to the development of word meaning (Vygotsky, 1934/1987).

It is apparent that this notion of concepts applies to everyday concepts but does not apply to mathematical concepts because mathematical concepts have been developed through history. Thus, the distinction between the two types of mathematical concepts which have developed in human history and which develop based on children's everyday concepts as their psychological development must be made clearer than it is in some of Vygotsky's writing.

**“Sublated concepts”**

Once clearly distinguished from one another, we now need to consider the relationship between these two kinds of concepts in mathematics education. In the cognitive development of children, everyday concepts arise from below to above in a certain sense when children learn formal and systematic mathematical concepts in school (Vygotsky, 1933/ 1975; 1934/1987). In other words, everyday concepts are reorganized and raised to a higher level by the appearance of mathematical concepts. However, the everyday concepts that are raised to higher level might not be called everyday concepts after this elevation because they now include elements of more systemic thinking. Likewise, mathematical concepts also change from their proper definition, for after this elevation they now include notions derived from experience in concrete contexts in addition to their systematic characteristics.

Though Vygotsky (1974) did not give detailed accounts of how these two mutually dependent concepts develop in children, he pointed out that the relation between higher and lower forms could be expressed well using the idea of a dialectic, where from the interrelating of everyday and mathematical concepts a new form- the *sublated* concept- gradually develops. Brushlinsky (1968) described this idea as follows:

> In his [Vygotsky’s] words, any higher stage of developments does not replace [lower stages] but *subordinate* them as their parts. That is, the higher stages contradict the lower. However, they do not eliminate the lower; but the higher stages include the lower as their *sublated* parts. Categories become components within a system, contradicting one another in the system. (L. S. Vygotsky reminds us of the duality of Hegelian meaning of *sublate* – elimination [of original form] together with preservation [of crucial features of the category]) (p.12).

A dictionary in philosophy (Shimonaka, 1971) explains *sublation* as follows:

> Sublation is the German word for *aufheben* that has the meaning Hegel defined. It has a sequence of meanings as follows: 1) contradict, 2) lift and 3) preserve. … This word is used for unifying contradiction, because development is based on contradiction or conflict. ‘Some elements were subordinated by means of an unification ’ of a sequence of processes and the results are as follows: some split elements contradict and fight against each other, permeated internally by each other, are unified through such process and as a result highly developed situations are formed (pp.705-6).

Shimonaka (1971) described that *sublated* is used for unifying things that contradict each other. As Vygotsky (1933/1975) recognized, contradictions between everyday and mathematical concepts are factors that promote the intellectual development and bring new possibilities for their development of children. Therefore, it is useful to use the idea *sublated* to explain how the concepts develop. These are defined as follows.

<u>Sublated process and sublated concepts:</u>

(1) Mathematical concepts *contradict* a part of the children’s everyday concepts; (2-A) The everyday concepts are *lifted* to a higher level, based on a system in mathematics; (2-B) The mathematical concepts are *lifted* to a higher level in which the daily contexts according to children’s everyday concepts are accompanied with

them; (3) They are *preserved* as a unified concept, that is, a *sublated concept*. Consequently, sublated concepts are defined as follows: Concepts developed through the sublated process, having both a system in mathematics and rich daily contexts, wherein children are free to move back and forth between the everyday and the mathematical world. Furthermore, children should be able to use the sublated concepts with conscious awareness and voluntary behavior.

## A SERIES OF FRACTION CLASSES

### The outline of a series of fraction classes

A series of seven fraction classes for 40 third graders were observed for five days in Hiroshima, Japan, in March 2001. It was the first time the students had encountered fractions at school. The teacher was a mathematics teacher at the school. Before the classes, he already recognized that children have difficulty dividing equally an object into three, and that they are unconscious of a *unit-whole*, or a *one-whole*. His teaching purpose was to let children overcome these difficulties. Therefore, in the first of five classes he provided situations based on the three elements: (1) multiple meanings of fractions – *partition fractions* and *quantity fractions*; (2) multiple objects for fractions – rectangle and square papers, 1 *l* measure cup, 1 *m* and 4/3 *m* tapes; and (3) multiple ways of dividing equally – into three and into four.

### Traditional discussion in Japan: partition vs. quantity

Before describing the classes, it is important to provide a brief history of the discussion on the first class of fractions in Japan. Traditionally, teachers have taken into consideration what kinds of fractions should be introduced to children in the first fractions class: partition fractions OR quantity fractions. When partitioning objects into *b* parts equally and picking up *a* out of *b*, the amount of *a/b* is defined as partition fractions. Therefore, 1/2 (of a whole pizza) is a type of partition fractions. Moreover, fractions without universal units are called partition fractions nowadays. On the other hand, the prevailing meaning of quantity fractions is defined as fractions that have a universal unit; and traditionally this was led by the particular method according to the Euclidean algorithm (the EA method). As a result, the most remarkable characteristic is the concept of a unique unit-whole, which is independent of any situations. For instance, 1/2 *m* is an example and the unit-whole for this corresponds to 1 *m*.

When children are introduced to partition fractions before quantity fractions, it is difficult for them to be aware of a unit-whole because the unit-whole is implicit in everyday situations. Therefore, when they subsequently encounter quantity fractions, they may have difficulties. For example, when asked to cut "1/2 *m*" out of 2 *m* tape, children often cut "1/2 of the whole tape." It is difficult for them to recognize 1*m* as a unit-whole because the whole tape does not correspond to a unit-whole of the tape, yet in a daily pizza situation the whole pizza is usually identical to a unit-whole. Yet despite this difficulty, partition fractions are sometimes introduced first for when quantity fractions first children feel no necessity to use the EA method.

**Three settings consist of a series of seven fraction classes**

The first setting consisted of the first and the second classes in which the following problem were figured out: "How much does each person has when dividing a piece of paper equally (1) among four people, and (2) among three people?" The main discussion was on the possibility of dividing into *three* pieces and on the way of doing that. In the end, the teacher gave children the definition of fractions: Each part of an object divided into four or three parts is called 1/4 or 1/3. Secondly, the second setting consisted of the third to the fifth classes. This paper specifically focuses on the episodes in the fifth class. Then, the third setting consisting of the sixth and the seventh classes encouraged children to figure out problems of equivalent fractions.

## TWO EPISODES IN THE FIFTH CLASS

In the third class, after each child received a piece of blue tape, they decided the day's task as to measure the length of the tape by measuring, cutting, or folding, and to express it in (1) integers, and (2) fractions. Then, they reported that the tape was almost 133 *cm* long in integers. Furthermore, a pair of children (YN and TN) presented their own idea on how they expressed the length in fractions, in which they explained by folding the tape in half twice, and then measuring the length of the one part out of the four parts. In the fourth class, after the short discussion about the idea YN and TN gave in the previous class, each child struggled with the problem to express the length of the same blue tape in fractions. Finally, they presented four ideas. In the beginning of the fifth class, the teacher wrote five ideas down on a board that the children had presented in the previous classes:

(1) 1/4 (*m*) = 33 *cm*; (2) 33/25 *cm* (◊ *m*); (3) 4/4 *m* 33 *cm*; (4) 4/3 *m*; (5) 133 *cm* 4/4

### Episode 1 – The difference between 1/4 and 1/4 *m*

Because YN objected to the notation 1/4 without the unit *m* for # 1, the teacher asked the children, "What is the difference between 1/4 and 1/4 *m*?" The following is the discussion that took place among them. In the following transcripts, two capitalized letters express a child's name. In addition, words within parentheses make up meanings, and sentences within brackets means actions. Besides, three dots in a bracket abbreviate several people's words. The essentials are underlined.

1T (teacher): Well, HS said, "This (1/4 *m*) is 25 *cm*." Is this right?

2Cs (many children): Yes.

3 T: So, what kind of length 1/4 is?

4 Cs: It is 1/4 of a whole.

5 T: There are many different meanings for a "whole." [He shows various lengths with his hands.] This is a whole, or this is also whole and so on. <…>

6 T: So, which is correct, the upper answer (1/4 = 33 *cm*) or the lower answer (1/4 *m* = 33 *cm*)? <…>

7 MM: In the upper answer, 1/4, the unit is unclear, so I thought the lower answer is better.

8 T: The 1/4 by itself does not tell us how long it is. The measurement of "1/4" depends upon the item being measured, right?

9 HS: If you need any units, you can say 1/4 of "a blue tape."

10 T: Is it good to say 1/4 of a blue tape?

11TM: (1/4 of) "the blue tape": in the universe, in the earth, in the world, in Japan, in Hiroshima, in this school, in this room, on the board, where the blue tape is attached. <…>

12 T: This is right, isn't it? But, it's too long. [He points 1/4 *m* = 33 *cm*.] This does not include any redundant information. <…>

13 IH: I got it; 1/4 of a 1 *m* 33 *cm* tape is 33 (*cm*).

Through this discussion, children recognized that the unit-whole for 1/4 *m* is always 1 *m*, and furthermore the unit-whole for 1/4, in this situation, is 1 *m* 33 *cm*.

**Episode 2 – Two names for a remainder of a tape: 1/4 of a whole or 1/3 *m***

For #4, children discussed what 4/3 *m* meant because many children could not be convinced of the meaning i.e. "four out of three."

21KO: Mr., it means four pieces out of something divided by three pieces, right?

22 T: Exactly.

23KO: It is impossible.

24 T: You mean, this sounds strange as a fraction, right? There are numbers more than numbers divided. Is not it funny? <…>

25 T: Let's figure out this problem. The clue is that I did cut out (from the blue tape) the remainder of the tape. [He points out the 1 *m* tape after he cut the blue tape out into 1 *m* and 1/3 *m*.] This tape is 1 *m* long, so what is the length of the remainder of the tape in fractions?

26 C: That is 1/4 (of a whole).

27 T: This is 1/4. Wait a second. I am going to mark the 1 *m* tape with chalk. [He measures the 1 *m* tape stuck on the board using the remainder of the tape as a unit of measuring.] (This is the EA method.)

28 Cs: It is 1/3!

29 T: This part disappeared. [He draws the figure of the remainder of the tape in a dotted line next to the 1 *m* tape.] Well, I will ask you once again. Is this (remainder of the tape) 1/4 long or 1/3 long?

30 Cs: It is 1/3! <…>

31TN: [She points out the remainder of the tape.] This 33 *cm* tape is 1/4 of the whole (tape), right? Then, which expressions of fractions do you need, a fraction for 1 *m* (i.e. quantity fractions) or a fraction for the whole (i.e. partition fractions)? … If you do not decide it, we have two answers. <…>

32 T: (NM meant that) 1/3 is better (to express the length of the remainder), because the length of 1 *m* is clear, therefore the remainder of the tape is 1/3 of 1 *m*.

33YN: That is, [she points at the 2/3 width of the 1 *m* tape on the board] this longs 2/3, and then 3/3 until there. (In the first piece of the blue tape) there are four parts of the lengths of the rest of the tape, so this (whole tape) longs 4/3 (*m*).

Through this discussion, children recognized that the remainder of the tape was equal to 1/3 *m* as well as 1/4 for the whole through the teacher's demonstration.

## DISCUSSION

Episodes 1 and 2 express the process how children's concepts are sublated. In episode 1, line 4 expresses the children's typical everyday concepts for fractions.

That is, the length of 1/4 means 1/4 of a whole. Through their everyday experience, they recognize that 1/4 means the action of dividing an object equally into four. Therefore, while children focus on the action, they do not pay attention to the unit-whole of the object because the unit-whole is often identical with a whole of the object in their daily life, and accordingly the unit-whole is implicit for them. However, through the discussion on ambiguities of the expression 1/4, they were convinced that they should have a conscious awareness of a unit-whole for the partition fraction as in the lines 11 or 13.

In episode 2, children exposed their everyday concepts in which fractions should be less than 1 in lines 21, 23, and 24. Based on their everyday experience, children are thinking that fractions show actions of dividing or the result of them. Hence, they have no idea that 4/3 *m* means a quantity. Moreover, teacher's demonstration in line 27 led them to recognize that a unit-whole for a fraction is not necessarily a whole of something given, contrasting with their everyday concepts. The remainder of the blue tape cut off from 1 *m* can be named 1/4 when regarding the whole tape as the unit-whole, and at the same time it can be also named 1/3 *m* when regarding 1 *m* as the unit-whole (line 26, 28, and 31). And also, through this new finding, children acquire the concepts of fractions as quantity (line 33). In conclusion, for each episode, everyday, mathematical, and sublated concepts are given in table 1.

| Everyday Concepts | Mathematical Concepts | Sublated Concepts |
|---|---|---|
| A partition fraction, 1/4, expresses the action of dividing an object into four.<br>A unit-whole of fraction always corresponds with the whole implicitly. | The unit-whole for 1/4 is always 1 or the whole, and for 1/4 *m* it is 1 *m*.<br>The quantity of 1/4 changes depending on the quantity of the unit-whole. | The length of 1/4 of 1 *m* 33 *cm* tape is always 33 *cm*, and in this context, the unit-whole is explicitly 1 *m* 33 *cm* or the blue tape stacked on the board. |
| Fractions must be less than 1 because fractions express actions of dividing or the result of the actions.<br>"1/4 of a whole" is only way to express the remainder of the blue tape. | Fractions can express quantity, number etc., and therefore are not necessarily less than 1.<br>A unit-whole does not always correspond to a whole of an object. | When the whole of the blue tape is a unit-whole, the remainder is expressed as 1/4 of the whole; when 1 *m* is a unit-whole, it is also expressed as 1/3 *m*, and therefore the whole is 4/3 *m* |

**Table 1: Everyday, mathematical, and sublated concepts in each episode**

As exemplified in these episodes, conflict between the everyday and mathematical concepts leads the development of concepts for fractions in children. That is, children's everyday concepts are exposed in the class and these contradict the system of the mathematical concepts; and the everyday and mathematical concepts are lifted up to the higher levels; then they are preserved as unified concepts having both system and concrete contexts. Specifically in these episodes, their everyday concepts are finally subordinated to the view in which fractions themselves can express quantities (the importance of quantity) and the view of the conscious awareness that what is a unit-whole for a fraction (the importance of a unit-whole) as well as the view in which fractions correspond to two integers (the part-whole, or partition fraction).

In addition, the intentional settings by the teacher make these concepts develop possible. In other words, because he gave two types of fractions, partition and quantity fractions, at the same time, children could be confronted with problems or contradictions that they have never held in their daily lives.

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