CONSTRUCTION OF MATHEMATICAL DEFINITIONS: AN EPISTEMOLOGICAL AND DIDACTICAL STUDY

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*The definition-construction process is central to mathematics. The aim of this paper is to propose a few **S**ituations of **D**efinition-**C**onstruction (called SDC) and to study them. Our main objectives are to describe the definition-construction process and to design SDC for classroom. A SDC on "discrete straight line" and its mathematical and didactical analysis (with students' productions) will be presented too.*

INTRODUCTION

This paper would like to show that "definition" is not only a meta-mathematical term. Actually, mathematical definitions can be approached from two different standpoints. The first one consists in taking for granted that definition is not a problem and that the definitions provide mathematical concepts: "le premier piège est de croire facile à acquérir ce qui est simple à énoncer" (Kahane-1999-p12) [1]. When we are constructing a concept, a dialectical process involving both the construction of the definitions and the construction of the concept is at work: "A definitional procedure is a procedure of concept formation" (Lakatos-1961-p54). We shall start from those two points of view to introduce our research topic: *definition-construction*, in other words: what type of defining situations are designed and analysed in mathematics education research? Does the analysis of those situations give specific results concerning concept-formation? Can they really help towards the assimilation of mathematical concepts? Which theoretical background can we use for the analysis of such situations?

EXISTING RESEARCH ON DEFINING ACTIVITIES

Some researchers in mathematics education stress the need for a learner to be an apprentice-mathematician. Freudenthal (1973), in particular, tackles the case of mathematical definitions, which don't have to be considered as arbitrary rules by pupils. To illustrate his point of view, he gives the example of the classification of quadrilaterals in geometry, and thus underlines the nature of the exploration of several properties of different quadrilaterals; his theoretical approach rests on the Van Hiele levels. Freudenthal specifies two different types of defining activities: *descriptive (a posteriori) defining* and *constructive (a priori) defining*. There are a systematisation of existing knowledge and a production of new knowledge. This kind of defining activity is visited again by De Villiers (1998) who underlines that students are active learners in such situations. In this connection, Vinner emphasizes the importance of constructing definitions: "the ability to construct a formal definition is for us a possible indication of deep understanding" (Vinner-1991-p79). Within his theoretical framework, Vinner suggests to expose a flaw in the students' *concept image* of a mathematical concept, in order to induce students to enter into a process

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of reconstruction of the *concept definition*. Can we imagine other kinds of situations involving definition-construction? This is precisely what Borasi proposes, insisting on mathematical inquiry, and more specifically on the role of mathematical definitions. She proposes three instructional heuristics for the design of defining activities: the in-depth analysis of a list of incorrect definitions of a given concept; the use of definitions in specific mathematical problems and proofs, the exploration of what happens when a familiar definition is interpreted in a different context (Borasi – 1992 – p155), and she underlines the difficulties in building defining activities in which unfamiliar concepts are at stake (she uses the notion of "à la Lakatos"). Duchet (1997) proposes such problem-situations, called *research situations*, inspired by ongoing mathematical research, in which a definition-construction process may appear: but Duchet's analysis is not specifically turned to **S**ituations of **D**efinition **C**onstruction (called SDC).

Let us underline the major objectives of these previous authors (and others) in order to point out their common denominator i.e. the definition-construction.

Type of situations (*definition-construction*)	Mathematical concept	Mathematical field
Classification (starting from representations, examples and counterexamples)	Quadrilaterals (Freudenthal/De Villiers) Convexity (Fletcher)	Geometry
Redefining		
- starting from representations, ex/cex (familiar concept)	- Function / triangles (Vinner)	Analysis /Geometry
- starting from a list of incorrect definitions (familiar concept)	- Circle (Borasi)	Geometry
- redefining in a different context	- Taxicab metric (Borasi) /Square on a sphere (Duchet)	Geometry
- extending definitions	- Exponentiation beyond the whole numbers	Algebra
Problem-situation	- Generator, minimality (Displacements on a regular grid map) (Duchet)	Geometry-algebra
	- Polygon (Borasi)	Geometry

Table 1: summary of didactical problematics on *definition-construction*

From this table, different features appear: first, the nature of mathematical concept at stake in situations of *definition-construction* seems to be specific, because almost all of those concepts come from the geometry or combinatorial geometry. Second, the predominance of classification and redefining situations: it is perhaps directly link to

the real difficulty in designing a "problem-situation" whose resolution involves a *definition-construction*. Thirdly, there is no common theoretical background for the global analysis of SDC. In fact, the existing theoretical backgrounds used (*concept image*, *concept definition* -Vinner- & **T**heory of **D**idactical **S**ituations-Brousseau) are useful because they help us to grasp some key elements for the design (TDS) and the analysis of students' processes (*concept image*), so we have no choice but to model mathematical definition-construction process (present specifically in Lakatos's work). Thus, the mathematical and didactical study of processes of definition-construction both involves a description of these processes in mathematics and a typology of SDC including classification, redefining situations among others. Our challenge consists in building theoretical tools, efficient for the characterization and the analysis of processes of definition-construction, in modelling the dialectic between concept-formation, definition-construction and proof (cf. Lakatos). The didactical stakes lie in the fact that the SDC constitute a real challenge for concept construction, and for the evolution of students' conceptions about definitions.

THEORETICAL FRAMEWORK & AIMS

We need to consider the concept of "definition" through its main features i.e. language (a definition is a specific discourse), axiomatic (a definition is inscribed in a mathematical theory) and heuristic (a definition-construction process, which is heuristic, leads to concept formation). The main references we chose for describing conceptions about "definition" are Aristotle (language), Popper (axiomatic) and Lakatos (heuristic). We chose the cK¢ model (Balacheff-2003) in order to describe these three conceptions of the notion of definition, because it allows a recognition of definition-construction process, and thus, it brings elements for analyzing mathematical concept formation. Balacheff presents a *conception* as an "instantiation" of a subject's knowledge by a situation and stresses that conceptions and problems are dual identities. Starting from a psychological presentation of a concept (referent, signified and signifies – Vergnaud, 1991), the cK¢ model calls *conception* (C) a quadruplet (P, R, L, Σ) in which:

- P is a set of *problems*: this is the sphere of practice of C;
- R is a set of *operators* (to solve a problem 'means' to modify it with a sequence of operators);
- L is a *representation system* (it allows the explanation of the elements of P and R)
- Σ is a *control structure* (in control structure, there are strategic knowledge and meta-knowledge specific to a given class of mathematical problems).

We would like to underline that validation is a key aspect of conceptualisation (Vergnaud introduces the notion of *theorem-in-action*). That the reason why the cK¢ model proposes a control structure: a clear identification of a control structure and the related operators (indeed, a meta-level with respect to action) allows the conceptualisation process to occur through a complex interaction with action and representation. We will concentrate our attention in this paper on Lakatos's conception. For the description of the others conceptions, see Ouvrier-Buffet (2003).

The Lakatosian conception (key elements)

Starting from an "intriguing relation" discovered for some polyhedra [2], Lakatos's dissertation (1961) tries to test it in different ways and hence, three main focuses of interest appear: definition-construction, concept-formation and proof. This work deals with three viewpoints relating to the mathematical concept of definition: the linguistic (it is specific to the Aristotelian conception), the axiomatic (gets rid of by Lakatos; this standpoint is described in the Popperian conception) and the heuristic. The latter is the particularity and thus the interest of the Lakatosian approach. Three notions are present in this heuristic approach of definitions: *naïve* definition, *zero-definition*, *proof-generated definition*. Each one has a specific role and a place in the concept-formation. A *zero-definition* is a tentative definition emerging at the beginning of the research process. It may evolve into a *proof-generated definition* or just disappear. It is brought about by proof and stands out as the most important notion in Lakatos's view: the product of *proof-generated definition* is directly linked to the type of SDC (i.e. problem-situation, according to Lakatos). Logically, the zero-concepts may be naïve, but Lakatos concentrates his attention on the expansion of zero-concepts; according to him, this expansion is not possible from naïve concepts. Hence, we have "stages" in a definition process, but how can the evolution of the definition-construction incorporate them and describe the operators and controls between *zero-definition* and *proof-generated definition* for instance?

The operators in fact are specifically related to the proof and the heuristic perspective in which the definitional procedure is inscribed. The generation of examples and counter-examples, in a refutative view, is certainly the most important operator, adding to the functions ascribed to a definition. These functions implicate specific stages in the definitional process: for instance, the functions of communication and denomination generate *zero-definitions*, and the catalysis of the proof brings *proof-generated definitions*. The main control structure refers to the proof i.e. the validity of the studied proof. The other is directly linked to the lack of counter-examples for refutation, but how can we stop the refutation-process? Lakatos informs us that we may stop the expansion of concepts, where it stops being a fertilizer to become a total weedkiller and underlines that a scientific research starts and ends with problems. Hence, this excerpt testifies to the implicit operators and controls existing in a definition-construction process.

Typology of SDC

Our theoretical work, by modelling conceptions of definition, brings a typology of SDC: we distinguish three types of SDC, called *Classification*, *Mathematisation/Modelling*, *Problem-situation*. The first one includes Fletcher's and Freudenthal's proposals, with mathematical objects accessible by theirs representations. The characterisation of the second is initiated by its name; let us give an example of such a situation we have not given yet: "define a mathematical object, which can represent the set of plants" (i.e. the elements of the whole vegetable kingdom) [3]. The third one is called *Problem-Situation* with reference to Lakatos's

situation; it includes *research-situations* (Duchet-1994&1997). According to Lakatos, starting from a vague idea of a mathematical concept (such as Euler's formula) can be enough for marking the beginning of a definitional procedure (Lakatos-1961,p69).

PRESENTATION OF A SDC – METHODOLOGY

We have experimented a SDC (*Classification*) with the mathematical concept "tree" before. We have given an analysis of the students' definition-construction processes by *zero-definitions* in Ouvrier-Buffet (2002).

We choose here the mathematical object "discrete straight line", for two main reasons. First, it is of the core of current problem in present mathematical research. This object is accessible by its representation, it is non-institutionalised, and thus allows a re-problematisation of the axiomatic problematics (it cannot be done with Euclidian geometry because the latter is too institutionalised). Second, it permits two types of SDC: the situation *Classification* starts from about ten representations of discrete lines (which are or resemble discrete straight lines). The text of the *Problem-situation* is: "draw discrete triangles and explain your construction". This is problematically close to Lakatos's *Problem-situation* because it involves a search for objects, mathematically still unknown but being dependent on a referent (real straight line). Our objectives are to analyse students' definition-construction process, to explore the influence of the possible explicit demand of definition on the latter and to determine the feasibility of different types of SDC with a common object.

Hence the methodology we chose: three groups of three or two students (from the first university year, scientific section but not especially from the mathematical sections) have taken part in this experimentation (2 or 3 hours required). There were videotape recorded; an observer was present for recalling the instructions (if necessary). Two situations were conceived with two different starting points: the first one (groupA) did not include an explicit request of definition and referred to an axiomatic problematic; neither examples nor counter-examples of discrete straight lines were given to students. The second one (groupB) is a *Classification*-situation, starting from examples and counter-examples of discrete straight lines non-identified as such, and it includes an explicit request of definition. We will use the description of the Lakatosian conception (via cK¢) in order to present the potentiality of the chosen mathematical object for SDC, on the one hand, and to analyse students' definitional processes on the other.

Presentation of the mathematical object "discrete straight line"

This presentation will allow both a mathematical explanation of this concept and its potentialities from the point of view of definition-construction (we will present this aspect with the notion of *zero-definitions* and the possible evolution of them).

To consider a discrete straight line (on a regular grid map, while colouring pixels) can generate a reference to real straight line and thus a use of properties of the last in order to define the same object in a discrete context. We call these problematics "real straight line". If one draws a real straight line, and chooses some pixels crossed by it

for forming a discrete straight line, the criteria of choice are requested (see figure 1 below). Thus, different *zero-definitions* are conceivable (the words in bold type mark the orientation of the evolution of the *zero-definitions*: see Zdef1,2,3).

The second problematics is called "regularity": it consists in researching a regularity in the sequence of stages of pixels (see figure 2 below: how to modify a sequence in order to obtain a better regularity?). This problematic brings us two potentially evolutionary *zero-definitions*: Zdef4,5.

There is a third more complicated approach, called "axiomatic". It consists in questioning the mathematical object "discrete straight line", in connection with our knowledge of Euclidian geometry, and thus, for instance, studying the intersection of two discrete straight lines, the number of discrete straight lines by two given pixels etc. This approach is more difficult.

Problematics	Figures	*Zero-definitions*
"Real straight line" Function of def.: to build the object	… or … ? Figure 1	Zdef1: set of the pixels crossed by a real line. Zdef2: set of the pixels "**the nearest**" of a real line. Zdef3: set of the pixels "**inside**" a band
"Regularity" Function of def.: to recognize, to build the object	Figure 2	Zdef4: sequence of stages of pixels with specific **properties**. Zdef5: sequence of pixels' stages with a uniform repartition, **non-improved** from the regularity viewpoint.
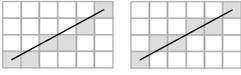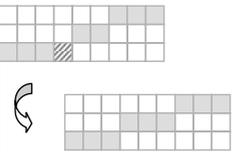
Table 2: two problematics and the *zero-definitions* of "discrete straight line"

To characterize the pixels "the nearest" of a real line, searching a property relating to the sequence of stages (called chaincode string), leads to a theorem. This approach to the discretization of a straight line by checking linearity conditions is directly related to number theoretical issues in the approximation of real numbers by rational numbers. These linearity conditions can be checked incrementally, leading to a decomposition of arbitrary strings into straight substrings (cf. Wu-1982).

RESULTS

Two problematics were tackled by students: they bring several *zero-definitions*.

Group	*Zero-definitions* produced	Operators, controls	Final Statement
Group A (*Problem-*	- Zdef 1,2,3 (abandon because problem of the	Perceptive controls	*Arithmetical rule* involving slope

situation)	criteria of choice) - Zdef5 (search of regularity by modification of stages)		
Group B (*Classific ation*)	- Zdef1 (abandon because of the non appropriate use of a ruler for a mathematical definition) - Zdef4	Counter-examples, linguistic and logical operators; Perceptive controls	*Repetition of a sequence, in the same direction. The difference between two stages does not exceed 1.*

Table 3: students' statements and their evolution

For these two groups, the first approach consists in using the "real straight line", and thus, some aspects of the *concept image* of straight line appear among what follows: perceptive regularity, slope and infinity of the points. This *concept image* is here insufficient in view of the difficulty of the discrete straight line concept, but still present for students' perceptive controls. The final statements produced by students are very close to actual mathematical definitions. It is noteworthy that these two groups abandon the "real straight line" approach (because of the problem of the choice criteria or the use of a ruler) for the benefit of the "regularity" approach. The students change their point of view relating to the mathematical object and thus abandon the external referent "real straight line". In that way, they define really the "discrete straight line", now fully considered as a mathematical concept.

We can formulate a hypothesis about the influence of the explicit demand of definition on the process of definition-construction, in particular on the evolution of *zero-definitions*: groupB mobilizes more operators taking part in the definition-process than groupA. In this case, the explicit request of definition seems to be profitable for the definition-construction process because it seems to favour a connection progress between different definitions and mobilizes specific operators (linguistic and logical) which contribute to the reflexivity on definition and questions about the presence of new counter-examples (refutation-process). The lack of any form of control concerning the function of definition is conspicuous. Clearly, there has been no simultaneous treatment of the two functions of definition (i.e. drawing a discrete straight line and recognizing it).

CONCLUDING REMARKS

Different markers attest a students' process of definition-construction: the presence of *zero-definition(s)* underlines its beginning, and the mathematical treatment of these potential definitions consists in studying the lacks of these "working definitions", analysing the different implications between them. All this process involves specific Lakatosian operators such as testing a definition with a research of refutation by a counter-example, but also Aristotelian operators, linguistically and logically

orientated, such as searching a minimal, non-redundant definition (that implies a reflexivity on the definition as a statement but also as a characterisation of a concept). This points to great potentialities of SDC for revising students' conceptions on the notion of "definition", but also for the exploration and the understanding of mathematical concepts above all. To explore and capitalize on these potentialities implies the designing of a set of SDC, with different kinds of SDC, including *Classification* and *Problem-situation*. We have to analyse more precisely the *Problem-situation* phenomenon, as it is presented by Lakatos (in connection with proof): that is what the study of *research-situations* (Duchet) seems to be promising.

NOTES

1. We do not delude ourselves into thinking that what can be easily expounded can be easily assimilated.

2. Euler's formula: V-E+F=2, where V is the number of vertices, E the number of edges and F the number of faces. Notice that Euler had defined the concepts of vertex and edge.

3. It can be the mathematical concept of "tree".

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