# GENERALIZED DIAGRAMS AS A TOOL FOR YOUNG CHILDREN'S PROBLEM SOLVING

Barbara J. Dougherty and Hannah Slovin

University of Hawai'i

*Measure Up is a research and development project that uses findings from Davydov (1975) and others to introduce mathematics through measurement and algebra in grades 1–3. This paper illustrates the use of generalized diagrams and symbols in solving word problems for a group of 10 children selected from a grade 3 Measure Up classroom. Students use the diagrams to help solve word problems by focusing on the broader structure rather than seeing each problem as an entity in and of itself. The type and sophistication of the diagram can be linked to Sfard's theory (1991 1995) on mathematical development. The consistent use of the diagrams is related to students' experience with simultaneous presentations of physical, diagrammatic, and symbolic representations used in Measure Up.*

## INTRODUCTION AND THEORETICAL FRAMEWORK

Solving number sentences (equations) and word problems that involve number sentences is an area in early grades that often creates much difficulty for children. This may be related to children's misconceptions about and misapplications of equations, including the use of the equals sign (Kieran, 1981, 1985, Vergnaud, 1985) or an inability to think beyond the literal word translations of the problem to see the more general structure. This phenomenon is likened to an intraoperational period (Garcia & Piaget, 1989) in that initially, students are solely concerned with finding a solution to a specific equation or problem and the equation is treated as an object that exists only to solve a particular problem.

If children, however, are to make sense of generalized statements and classify problems into larger groups to create more efficient and robust methods of solving them, then a different approach to introduce problem solving might be considered. One such approach stems from the work of V. V. Davydov and B. Elkonin (Davydov, 1975) and is embodied in Measure Up (MU), a project of the Curriculum Research & Development Group (CRDG) of the University of Hawai'i.

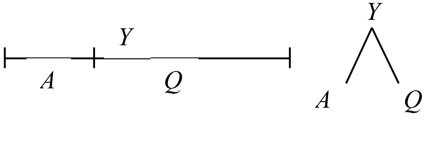
Davydov (1975a) believed that very young children should begin their mathematics learning with abstractions so that they could use formal abstractions in later school years and their thinking would develop in a way that could support and tolerate the capacity to deal with more complex mathematics. He (1975b) and others (Minskaya, 1975) felt that beginning with specific numbers (natural and counting) led to misconceptions and difficulties later on when students worked with rational and real numbers or algebra. He combined this idea with Vygotsky's distinction between spontaneous and scientific concepts (1978). Spontaneous or empirical concepts are developed when children can abstract properties from concrete experiences or instances. Scientific concepts, on the other hand, develop from formal experiences with properties themselves, progressing to identifying those properties in concrete instances. Spontaneous concepts progress from natural numbers to whole, rational, irrational, and finally real numbers, in a very specific sequence. Scientific concepts reverse this idea and focus on real numbers in the larger sense first, with specific cases found in natural, whole, rational, and irrational numbers at the same time. Davydov (1966) conjectured that a general-to-specific approach in the case of the scientific concept was much more conducive to student understanding than using the spontaneous concept approach.

Davydov (1966) wrote, “there is nothing about the intellectual capabilities of primary schoolchildren to hinder the algebraization of elementary mathematics. In fact, such an approach helps to bring and to increase these very capabilities children have for learning mathematics” (p. 202). Beginning with general quantities in an algebra context enhances children’s abilities to apply those concepts to specific examples that use numbers.

Davydov (1975a) proposed starting children’s mathematical experiences with basic conceptual ideas about mathematics and its structure, and then build number from there. Thus young children begin their mathematics program in grade 1 by describing and defining physical attributes of objects that can be compared. Davydov (1975a) advocated children begin in this way as a means of providing a context to explore relationships, both equal and unequal. Six-year-olds physically compare objects’ attributes (length, area, volume, and mass), and describe those comparisons with relational statements like $H < B$, where $H$ and $B$ represent unspecified quantities being compared, not objects. The physical context of these explorations and means by which they are recorded, link measurement and algebra so that children develop meaning for statements they write and do not see them as abstract.

In the prenumeric phase, children grapple with how to make 1) unequal quantities equal or 2) equal quantities unequal by adding or subtracting an amount. A statement representing the action is written: if $H < B$, students could add to volume $H$ or subtract from volume $B$. First graders observe that regardless of the action they choose, the amount added or subtracted is the same and is called the difference.

Number is introduced when students are presented with problematic situations that require quantification. It is then that a unit is presented within a measurement context. First graders re-examine some of the situations where they transformed two unequal quantities into equal amounts. For example, when working with mass, they may have written $Y = A + Q$. They notice now that mass $Y$ is the whole and masses $A$ and $Q$ are the parts that make up the whole.



Both diagrams represent the relationships among the parts and whole of any quantity. From these diagrams children can write equations in a more formal way.

$$Y = A + Q \quad Q + A = Y \quad Y - Q = A \quad Y - A = Q$$

The part-whole concept and related diagrams help young children organize and structure their thinking when they are working with word or contextual problems. The scheme supports children writing equations and identifying which ones are helpful in solving for an unknown amount, without forcing a particular solution method. For example, students are given the following problem:

> Jarod's father gave him 14 pencils. Jarod lost some of those pencils, but still has 9 left. How many pencils did Jarod lose? (Dougherty et al., 2003)

In this case 14 is the whole, 9 is a part and the lost pencils ($x$) are a part. There are at least four equations (a fact team) that can be written to describe this relationship.

(1) $14 = 9 + x$ (2) $14 = x + 9$ (3) $14 - 9 = x$ (4) $14 - x = 9$

The third equation, $14 - 9 = x$, could be an appropriate choice to solve for the unknown. Some of the students in the MU research study use that method. However, other students choose to use the first or second equation to solve for the unknown amount. Their reasoning follows the compensation method for solving an equation by asking the question, "What do I add to 9 that gives 14?"

## DESCRIPTION OF STUDY AND DATA COLLECTION

Measure Up uses design research (Shavelson, Phillips, Towne, & Feuer, 2003) as a means of linking research with the intricacies found in classrooms. The design research includes two sites 1) Education Laboratory School (ELS) Honolulu, HI and 2) Connections Public Charter School, (CPCS), Hilo, HI. These sites were carefully selected to provide a diverse student group representative of larger student populations in regard to 1) student performance levels, 2) socio-economic status, and 3) ethnicity. Student achievement levels range from the 5th to 99th percentile, with students from low to high socio-economic status and ethnicities including, but not limited to, Native Hawaiian, Pacific Islanders, African-American, Asian, Hispanic, and Caucasian. Students at ELS are chosen through a stratified random sampling approach based on achievement, ethnicity, and SES. No segregation or tracking of students is done at either site; all special education students are part of an inclusion program. Both sites have a stable student population for longitudinal study.

The MU project team is in the classroom observing and/or co-teaching daily. Three project staff members record observation notes in three separate formats. One person is responsible for documenting the mathematical development, one scripts the lesson, and one records instructional strategies used. Observation notes form a microgenetic study of student learning related to the materials under development. Videotapes are made of critical point lessons where the complexity of the mathematics shifts. Indicative of the design research approach, the MU project team discusses the types of problems or tasks that were used, the discourse that evolved from them, the expectations about participation across the broad range of students, the role(s) played by representations and tools in the learning process, and the mathematics itself in debriefing sessions where analyses of data and interpretations of such are done.

Two types of individual student interviews are also conducted. One type, “teaching experiments,” is adapted from a similar technique used in developing an algebra program at CRDG (Rachlin, Matsumoto, & Wada, 1987). As tasks are presented, students are asked to “think aloud” as they attempt the them. If students stop explaining, the interviewer prompts them by asking ‘What are you trying to find (do)?’ or ‘What’s giving you a problem?’ If difficulty persists, the interviewer gives more specific clues, increasing the specificity until the child can complete it. The clues range from noting a particular error to more direct instruction.

In the second type of student interviews, project staff gives students tasks that would represent the level of mathematics that students would have experienced in a more conventional program or that focus on bigger ideas. Responses from students are compared to students from other grade levels that have not been part of MU. The purpose of these interviews is to determine how the mathematical understandings of students in the MU project compare to those students in other programs.

During student interviews, a group of mathematics educators, mathematicians, and psychologists has the opportunity to watch and participate in the interview live. The interviewer wears an earphone, and a video camera projects the interview into another room where the group is seated. The group can suggest additional questions through a microphone for the interviewer to pursue, enhancing the information gathered from the interview.

**Types of problems presented and solutions used**

As students move through their mathematical experiences, they use a variety of means of representing problem situations. They follow a progression similar to what Sfard (1991, 1995) described as a historical perspective (interiorization, condensation, and reification). As students work in each mathematical concept, *every* new idea is introduced with a physical model that is simultaneously represented with an intermediate model (like a diagram) and a symbolization (equation or inequality). This does not follow the typical approach where students work with physical models or manipulatives, *then* move to iconic or pictorial representations, and lastly, work only with symbols. The advantage of simultaneous representations is that students form a cohesive mapping of what the symbols represent and can see patterns and generalizations beyond the specific problem.

What follows are three snapshots of ten grade 3 students, taken from the 2003–04 school year. These students have been part of the Measure Up project since grade 1 with the exception of one student (Student S) who joined the class this year. In the first and last snapshots presented the data were taken from the classroom. The second example is from individual student interviews of these children. The tasks show the bridging from grade 1 and 2 mathematics to a more sophisticated level. Their responses are linked to Sfard's progression (1991, 1995) such that an interiorization response is indicative of problem-specific characteristics; a condesation response indicates a transition from a specific to general solution approach, and a reification response embodies multiples forms of representation as a means of generalizing the structure of the problem.

*Sample 1*

Ama caught *k* fishes and Chris caught *e* fishes less. How many fishes did Chris catch?

Dusty had *b* fishes in a bucket. When Anthony added his fish, there were *g* fishes in the bucket. How many fishes did Anthony catch?

Sarah caught *p* fishes, and Tara caught *c* fishes. How many more fish did Sarah catch?

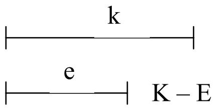
Students were asked to 'show the parts and whole in a diagram or on line segments' (Dougherty et al., 2003).

Students' representations varied with their ability to handle complex cognitive tasks. A sample of each type in response to the first problem follows.

Interiorization

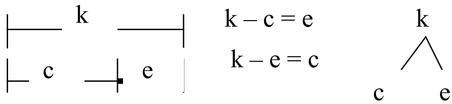

(Student M, 2003)

Condensation




(Student S, 2003)

Reification




(Student RI, 2003)

*Sample 2*

Reed gave Jackie a strip of paper *w* length-units long. He gave Macy another strip of paper 9 length-units shorter than Jackie's. How long are Macy's and Jackie's lengths altogether?

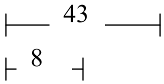
Jason has *v* mass-units of rice. Jon has *k* mass-units less than Jason. How many mass units of rice do Jason and Jon have together?

Karyn had 43 volume-units of water in one container. In another container, she had 8 volume-units less than in the first container. If she pours all the water into one large container, how many volume-units of water will she have?

Students were asked to decide in what order they would like to solve the problems. Then, solve the problems in any way they wanted.

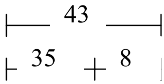
Consistently, regardless of cognitive level, students opted to do the numerical problem first. The mixed non-specified/numerical problem was chosen second, with the non-specified problem chosen last. Four students indicated that it did not matter what order was chosen, they were all solved the same way. They did, however, proceed to solve the numerical one first. In all cases, students, including those who had used the interiorization method at the onset of the school year, used either a condensation or reification approach to solve the problems.

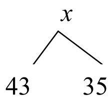
Condensation




43 – 8 = 35  35 + 43 = 78 (Student C, 2003)

Reification




$x = 43 + 35$ (Student RE, 2003)

$x = 78$

*Sample 3*

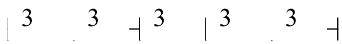
When his father takes 1 step, it takes Michael 3 steps to travel the same distance.

a. Michael's father walks 5 steps to get from the front door to the sidewalk. How many steps would Michael take to get from the front door to the sidewalk?

b. Michael took 27 steps to get from the front door to his neighbor's front door. How many steps would Michael's father take to walk the same distance?

These problems were used in the introduction of multiplication. Without direct instruction, students used similar diagrams to represent, and assist in solving, the problems. Samples from each type are as follows.

Interiorization

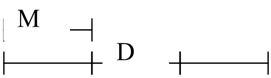


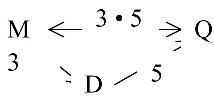

(Student C, 2003)

Condensation

3 + 3 + 3 + 3 + 3 = 15 (Student J, 2003)

Reification




3 • 5 = $x$  $x$ = 15  (Student M, 2003)

The generalization of the structure increases with each successive level. The diagram used in the reification approach is indicative of the general model of multiplication used in Measure Up. Unit *M* (in this case, Michael's step) is used three times to make intermediate unit *D* (the dad's step). Intermediate unit *D* is used five times to make quantity *Q*. If Unit *M* had been used by itself to create *Q*, *Q* is then represented as the product of 3 and 5. This last diagram embodies the use of a unit used to create a larger, intermediate unit as a means of creating a quantity. As MU students explore multiplication and division, this model helps them define the quantities they are working with and their relationships.

## IMPLICATIONS

Student solution methods strongly suggest that young children are capable of using algebraic symbols and generalized diagrams to solve problems. The diagrams and associated symbols can represent the structure of a mathematical situation and may be applied across a variety of settings. Students appear to utilize some form of the diagram and regardless of the sophistication of that model, students are developing a fluidity that allows them to attempt, and solve, word problems. The use of algebraic symbols and diagrams appears, at this stage of the research, to positively impact on students' mathematical development, especially when children develop their understanding of, and applications for, such diagrams through an approach that consistently and simultaneously links the physical model, intermediate representations, and symbolizations within each lesson, and not in a sequential manner.

### References


Davydov, V.V. (1975a). Logical and psychological problems of elementary mathematics as an academic subject. In L. P. Steffe, (Ed.), *Children's capacity for learning mathematics. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. VII* (pp. 55–107). Chicago: University of Chicago.

Davydov, V.V. (1975b). The psychological characteristics of the "prenumerical" period of mathematics instruction. In L. P. Steffe, (Ed.), *Children's capacity for learning mathematics. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. VII* (pp. 109–205). Chicago: University of Chicago.

Davydov, V. V., Gorbov, S., Mukulina, T., Savelyeva, M., & Tabachnikova, N. (1999). *Mathematics*. Moscow: Moscow Press.

Dougherty, B. J., Zenigami, F., Okazaki, C., & Thatcher, P. (2003). *Measure up, grade 3, draft*. Honolulu: Curriculum Research & Development Group, University of Hawai'i.

Garcia R., & Piaget, J. (1989) *Psychogenesis and the history of science*. New York, NY: Columbia University Press.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317–326.

Kieran, C. (1985). Constructing meaning for equations and equation-solving. In A. Bell, B. Low, & J. Kilpatrick (Eds.), *Theory,/ Research & Practice in Mathematical Education* (pp. 243–248). University of Nottingham, UK: Shell Center for Mathematical Education.

Minskaya, G. I. (1975). Developing the concept of number by means of the relationship of quantities. In L. P. Steffe (Ed.) *Children's capacity for learning mathematics. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. VII* (pp. 207–261). Chicago: University of Chicago.

Rachlin, S., Matsumoto, A., & Wada, L. (1987). The teaching experiment: A model for the marriage of teachers and researchers. *Focus on Learning Problems in Mathematics*, (*9*)3, pp. 21–29.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, *14*, 15–39.

Shavelson, R. J., Phillips, D. C., Towne, L., & Feuer, M. (2003). On the science of education design studies. *Educational Researcher*, *32*(1), pp. 25–28.

Vergnaud, G. (1985). Understanding mathematics at the secondary-school level. In A. Bell, B. Low, & J. Kilpatrick, (Eds.), *Theory, Research & Practice in Mathematical Education* (pp. 27-45). University of Nottingham, UK: Shell Center for Mathematical Education.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Harvard Press: Cambridge.