A Sociocultural Account of Students' Collective Mathematical Understanding of Polynomial Inequalities in Instrumented Activity

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In this report, we give a sociocultural account of the mediating functions handheld graphing calculators and social interaction play in students' mathematical understanding. We discuss the evolution of students' abilities to symbolize, model, and develop collective mathematical practices about polynomial inequalities in instrumented activity. In our sociocultural model we foreground the social nature of technological tools and the social transactions that take place in a classroom context which assist students as a collective to establish viable shared practices and collective representations. Graphing calculators as psychological tools are characterized as providing intentional, transcendental, and social mediation.

BACKGROUND AND RESEARCH PROBLEM

While a number of psychologically-driven research investigations consider competence in the use of mathematical tools such as notations and representations as a primary goal in the development of students' mathematical understanding (see, for e.g., Thompson 1992 and Kaput 1991), in this report we give a qualitative account of the mediating functions handheld graphing technologies and social activity play in this development. In particular, we discuss the evolution of students' abilities to symbolize, model, and develop collective mathematical practices about polynomial inequalities based on their actions with the TI-89, a handheld graphic calculator, as well as their interaction with other learners. Cobb and colleagues (Cobb 1999; Cobb & Yackel 1996; Cobb, Wood, & Yackel 1992) have clearly demonstrated the independent, but mutually reflexive and determining, roles played by social and psychological factors in mathematical learning. Since their research investigations attempt to account for both collective and individual mathematical development, they focus on social (i.e., classroom norms, sociomathematical norms, and classroom practices) and individual (i.e., personal and mathematical beliefs, nature of mathematical activity, interpretations, and reasoning) mechanisms that contribute to learning acquisition and appropriation. Our overall concern in this investigation is to provide a sociocultural basis for the meaning objectification of mathematical concepts and processes - that is, by surfacing both the social nature of technological tools and the social transactions that take place in classroom activity which assist students as a collective to establish viable shared practices and collective representations (Durkheim 1915).

One basic task in secondary school algebra in the US involves solving inequalities. The algebra and precalculus components of high school mathematics target the solutions of linear, quadratic, polynomial, rational, absolute value, and radical inequalities. Noncalculator-based mathematical analysis texts oftentimes favor a table-of-signs method to solving polynomial inequalities (see Table 1) which is a novel improvement compared to the very cumbersome manner in which they were solved in the 1970s (see Table 2). In this research report, we provide a sociocultural account of how students collectively developed a graphical approach to solving polynomial inequalities. Such an account considers how learners produce truths based on taken-as-shared practices of symbolizing and modeling.

Solve: $(x - 1)(x + 1) < 0$.

x - 1	-		-	0	+
x + 1	-	0	+		+
(x-1)(x+1)	+	-1	-	1	+

Therefore, $(x - 1)(x + 1) < 0$ provided $-1 < x < 1$.

Table 1. Example of a Quadratic Inequality Solution by the Table-of-Signs Method

Solve: $(x - 1)(x + 1) < 0$.

Case 1. $x - 1 < 0$ and $x + 1 > 0$	<----------------------------)
	(------------------------------>
$x < 1$ and $x > -1$	-1 1
Therefore, $-1 < x < 1$.	
OR:	
Case 2. $x - 1 > 0$ and $x + 1 < 0$	<-------------) (-------------->
$x > 1$ and $x < -1$	-1 1
There is no solution.	

Final Solution Set: $(-1, 1) \cup \varnothing = (-1, 1)$ (i.e., $-1 < x < 1$).

Table 2. Example of a Solution of a Quadratic Inequality by the Case Method

THEORETICAL FRAMEWORK

Recent sociocultural investigations in school mathematics concerning ways in which individuals acquire concepts provide strong evidence that learning takes place through experiences that are oftentimes mediated by physical or material and symbolic tools and with assistance drawn from other (competent) individuals (Gravemeijer, Lehrer, van Oers, & Verschaffel 2002; Cobb, Yackel, & McClain 2000). Such tools are capable of influencing learners' thinking about concepts and processes, and they also assist learners to exercise "control" over mental functions that affect their thinking. Following Vygotsky (1978), those devices such as graphing technologies mediate as "psychological tools" between the mind and the required sociocultural acts of mathematizing (Kozulin 1998). Graphing calculators, in particular, provide a convenient virtual environment that enable learners to acquire mathematical processes and concepts. Further, as learners become competent in using them, they begin to implicitly reconstruct and appropriate conventional practices.

Within a sociocultural context, knowledge is seen as arising out of collective representations that are (historically) rooted in a community (Durkheim 1915). For instance, a classroom community consists of learners who participate with other learners in an effort to construct shared knowledge among themselves. In mathematical settings, knowledge evolves as a common representation for all; it is shared by all members in a community and, hence, is a form of social relation. A collective representational view of mathematical knowledge enables individual and groups of learners to construct universal understandings; for any meaningful mathematical knowledge constructed does not inhere entirely on the individual learner but from the communities in which the learner transacts with and which determine the manner in which the knowledge is constructed (Cobb, Stephan, McClain, & Gravemeijer 2002; Gravemeijer, Lehrer, van Oers, & Verschaffel 2002).

Mathematical learners in social activity develop understanding by means of participation, cooperation, co-construction, negotiation or transaction, and, ultimately, intersubjective agreement among learners. These social performances enable them to form a "conscience collective," and this form of solidarity of social consciousness or shared understanding appears to be distinct from the individual consciousness or understanding of members in the community (Durkheim 1915). In other words, within a conscience collective, since mathematical knowledge is an established social fact, it resists individual interpretations of learners whose understanding appears different from what it has taken to be its "true" nature. Individual understandings are, thus, forced to reconcile with understandings that are allowed in the conscience collective. This is not to say that individuals could not negotiate. They could, certainly, but they are not allowed to establish freely on their own because mathematical knowledge as a social fact has the power to sanction or to constrain the manner in which individual learners develop their understanding of it.

Collective mathematical practices emerge from learners in social activity behaving as a conscience collective. Such practices are "perceptions of the acts and artifacts as manifestations of culture (as an analytic construct), and of the social relationships, which exist in the field of collectively held representations" (Bohannan 1960 p. 94). The symbolizing nature of collective mathematical practices is, thus, inherently social – that is, they possess social attributes whereby the meanings and practices associated with them have not been drawn from the objects of knowledge, at least not entirely, but more so in the manner in which the practices have become for the conscience collective their collective representation.

METHODS

Thirteen males and seventeen females of mixed mathematical abilities comprise this class of juniors and seniors (mean age:16.63; 26 Asians and Asian Americans, 4 Hispanic-Americans). The first author taught the class while the resident teacher observed and took down notes throughout the investigation. Twenty-one sessions each lasting 55 minutes were needed to accomplish the goal of the entire teaching experiment. In each session, students would usually work in pairs first and then would later regroup for a whole-classroom discussion. The basic design of the classroom teaching experiment focused on initially providing the participants in the study with an experientially real context for thinking about a way to solve a polynomial inequality, and a TI-89 could facilitate the construction of such a context. Thus, instead of simply telling the students how polynomial inequalities were solved using the standard methods such as those in Tables 1 and 2 above, we wanted them to construct a model that would make much more sense to them based on their collective mathematical experiences and with assistance from a TI-89.

RESULTS

Developing a Model of Solving Inequalities Graphically Using Linear and Quadratic Functions. The class took ten 55-minute sessions exploring ways to solve linear and quadratic inequalities graphically. The sequence of activities implemented at this stage was meant to help students obtain a relatively simple structure for solving inequalities. In solving linear and quadratic inequalities, the students performed the following activities: (1) they investigated the general behavior of the graphs of linear and quadratic functions; (2) they investigated situations in which y = ax+b and $y = ax^2 + bx + c$ <0, >0, $\leq 0, \geq 0$ graphically. Due to limitations in space, we discuss only linear inequalities. From the graphs shown in the TI-89, Pair 1 interpreted the inequality y < ax + b as "getting the values of x where y is negative." They suggested "imagine shading" portions of the graph where y = ax + b was below the x-axis and then determining the range of values of x where the regions applied. This proved to be difficult to accept by other pairs since they could not make a connection between the two variables ("we are graphing the y-values but we are solving for x?"; "isn't the inequality expressed as y? … So why not get all the y-values instead of x?"). Pair 2 then suggested to the class to "think of interval notations as boundaries where the graph lies below the x-axis." The "graphic action" enabled the class to conclude that linear inequalities involved determining intervals for x in which y was below the x-axis. Pair 3 made another suggestion, that is, to examine the table generated in the TI-89 to determine the range of values of x wherein y was negative. But this suggestion proved to be impractical and was eventually abandoned. Later, the students' attempts to write the correct solution had them discussing the significance of knowing the x-intercepts because the solution intervals relied on them.

Thus, the students' model of solving linear and quadratic inequalities consists of the following: transforming a given inequality into its standard form in which one side would contain the algebraic expression while the other side is set to 0; obtaining the graph of the inequality in standard form and calculating the zeros of the corresponding function either algebraically or graphically; determining the appropriate domain that satisfies the inequality, and; expressing the final answer in interval notation form.

Developing a Model For Solving Polynomial Inequalities. The class needed eleven 55-minute sessions to accomplish this task. The model they developed for solving polynomial inequalities graphically could be broken down into three stages below using different types of tools and in which the TI-89 served as the primary tool for the progressive evolution of the two later tools.

I <u>Using the TI-89</u> as a tool for investigating the following: (1) graphs of even- and odd-powered polynomial functions in factored form; (2) graphs of polynomial functions in factored form that contained odd and even multiplicities; (3) graphs of polynomial functions in factored form that contained imaginary zeros; (4) solving polynomial inequalities in factored form graphically.

II <u>Using a constructed Cartesian plane on paper</u> as a tool for solving polynomial inequalities in factored form graphically.

III <u>Using a number line</u> as a tool for solving polynomial inequalities in factored form graphically (and the same tool was used later in the case of polynomial inequalities expressed in the general (non-factored) form.

Initially, the students relied on the TI-89 to obtain generalizations about the graphs of polynomial functions subject to certain restrictions (see (I) above). They also used it to solve inequalities and to see the significance of knowing how the x-intercepts played out in the solution process. The TI-89 enabled them to develop their initial ability to describe and to reason perceptually about graphs of polynomial functions and their relationship to solving polynomial inequalities that were all initially expressed in factored form. In establishing a model for solving inequalities graphically, two additional shifts took place, and both shifts were unaided by the graphing tool. When the students were prompted to solve a polynomial inequality independent of a TI-89, Pair 3 suggested for the class to draw a Cartesian plane, plot the real x-intercepts, and use what they initially learned about the graph of the corresponding polynomial function to draw a sketch of its graph, and then to write down the intervals in which the inequality made sense. It took students some time to accomplish this because they had to calculate specific points on the graph. A number of them obtained values for y by beginning with $x = 0, 1, 2, 3$, and so on, which did not make sense in many cases of polynomial functions and, hence, did not gain much support from the class. One collective practice that emerged from a whole-group discussion came from Pair 4 who suggested obtaining points that lie between x-intercepts that the class immediately accepted. A second collective practice came from Pair 5 who suggested that to solve a polynomial inequality graphically, a rough sketch of its corresponding graph together with all the x-intercepts was all that were needed and that none of the other points mattered. See Figure 1 for a sample of two students' work based on these two collective practices.

As a homework assignment, the students were asked to solve a number of polynomial inequalities of varying difficulty without the use of the TI-89. During the next day's whole-class discussion, Tran raised the viability of solving a polynomial inequality by simply using a number line and not the Cartesian plane as a solution tool (see Figure 2). Tran did not see the significance of constructing a y-axis since the solution of any polynomial inequality depended primarily on the x-intercepts.

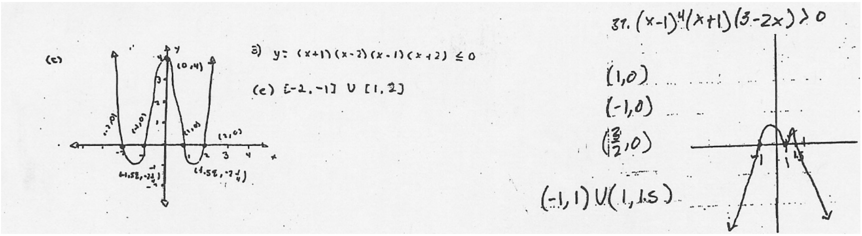

Figure 1. (a) #5: Solution With x-intercepts and Other Points
Figure 1. (b) #37: Solution With x-intercepts Only

Further, consistent with earlier practices, Tran interpreted interval notations as consisting of x-intercepts in which portions of the corresponding polynomial graph satisfied the indicated inequality. What his classmates obtained from his argument became the third collective practice for the entire class. This practice, which was perceptual in origin, is metaphorically equivalent to the table-of-signs method (see Table 1) that was conceptual and algebraic.

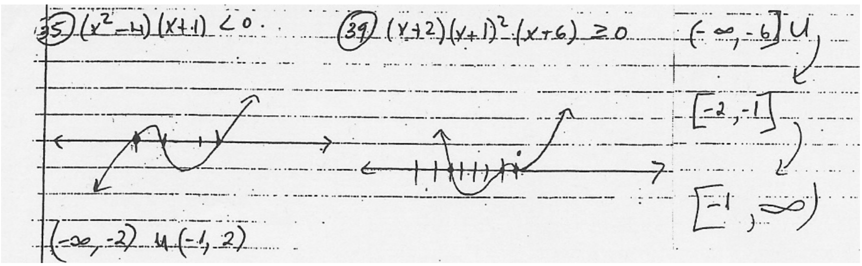

Figure 2. Solution Using the Number Line Method

DISCUSSION

While the initial model for solving polynomial inequalities graphically was mediated by the use of a TI-89, it was the simultaneous negotiation and participation that took place among the students that enabled the development of other models leading to a formal process. Such a process was not something that has been merely foisted upon them as an external reality; it evolved out of their actions with other learners and the TI-89. Formal thought emerging from informal activity in this manner is seen as the "creation of a new mathematical reality" (Gravemeijer & Stephan 2002).

The students capitalized on those shared understandings that they developed amongst themselves in the public space of the classroom. The TI-89 made learning less problematic for them because they used it mainly to draw sketches of several graphs all at once within some limited time. Hence, a shift of focus took place from being able to draw the graphs to being able to collectively make sense of the graphs as they pertained to inequality solving. Both the TI-89 and other learners mediated in ways that made it difficult to analyze the influence of one apart from the other, which is but an effect of instrumented activity (Rivera forthcoming).

Another effect of instrumented activity is seen on students' developing representational fluency. Based on my analysis of the students' written work, about 70% could easily switch from graphic to algebraic and vice-versa. The students could use the trace function to obtain approximate roots, or press one of the math keys to solve for the roots of the corresponding equations, or apply the appropriate algebraic forms to determine exact values. A sociocultural account for finding exact values is worth discussing briefly. When Nestor asked for the best way to obtain correct zeros on the TI-89, he received different approximations for the same root from others (even though they all used the math function "intersection" in the TI-89 to obtain it). Duong suggested that the class agree on how to round numbers, while Salvador insisted on identifying the appropriate upper and lower bounds so as to minimize errors. When the class was nowhere near an agreement, Nestor insisted that they obtain exact answers instead (for e.g., using the quadratic formula for irrational roots). In this situation, their understanding was shaped by the negotiation that occurred as a result of a calculator constraint which nobody was able to resolve. Thus, representational fluency was not forced on the students to achieve; the need to be fluent arose from a social conflict that needed to be settled.

Isolating the role played by the TI-89 in instrumented activity leads to several insights. First, because the various commands and functions in the TI-89 reflect mainstream mathematical processes, students are implicitly provided with an "intentional experience" (Kozulin 1998 p. 65). For example, my students did not have to create the graphs themselves since the calculator did it for them. They did not haphazardly construct their own graphs because the TI-89 has been program-med in such a way that the graphs reflect standard and correct features and characteristics (subject to the appropriate windows, of course). The experience of interpreting is intentionally directed in a way that reflects prevailing conventions. Second, the TI-89 functions as a "transcendent mediation" (*ibid.* p. 66). This was evident among my students when their thinking transitioned from a model of solving inequalities using linear and quadratic functions to a model for solving any polynomial inequality. How they developed the "model of" phase became the basis for constructing the "model for" phase. Students initially obtained solutions of linear and quadratic inequalities by imitating and copying the graphs they saw in the TI-89. In the final stage of their understanding, the essential elements they acquired from the TI-89 were used later as they shifted to more general models for solving inequalities involving the Cartesian plane and the number line. Third, the mathematical functions in the TI-89 will fail to mediate if learners fail to invest them with the correct meanings. While the TI-89 displays the symbols and objects that reflect our cultural inheritance, they are not embodied productions – that is, meaning does not reside in them. However, acting on them individually and in social activity is tantamount to loading them with value and purpose. The students found it significant to use the TI-89 because of the collective meanings that they developed amongst themselves which proved to be especially meaningful in the more formal stages.

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