ABSTRACTION IN MATHEMATICS AND MATHEMATICS LEARNING

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It is claimed that, since mathematics is essentially a self-contained system, mathematical objects may best be described as abstract-apart*. On the other hand, fundamental mathematical ideas are closely related to the real world and their learning involves empirical concepts. These concepts may be called* abstract-general *because they embody general properties of the real world. A discussion of the relationship between abstract-apart objects and abstract-general concepts leads to the conclusion that a key component in learning about fundamental mathematical objects is the* formalisation *of empirical concepts. A model of the relationship between mathematics and mathematics learning is presented which also includes more advanced mathematical objects.*

This paper was largely stimulated by the Research Forum on abstraction held at the 26th international conference of PME. In the following, a notation like [F105] will indicate page 105 of the Forum report (Boero et al., 2002).

At the Forum, Gray and Tall; Schwarz, Hershkowitz, and Dreyfus; and Gravemeier presented three theories of abstraction, and Sierpinska and Boero reacted. Our analysis indicates that two different contexts for abstraction were discussed at the Forum: abstraction in mathematics and abstraction in mathematics learning. However, the Forum did not include a further meaning of abstraction which we believe is important in the learning of mathematics: The formation of concepts by empirical abstraction from physical and social experience. We shall argue that fundamental mathematical ideas are formalisations of such concepts.

The aim of this paper is to contrast abstraction in mathematics with empirical abstraction in mathematics learning. In particular, we want to clarify "the relation between mathematical objects [and] thinking processes" (Boero, [F138]).

ABSTRACTION IN MATHEMATICS

What does it mean to say that mathematics is "abstract"?

Mathematics is a self-contained system separated from the physical and social world:

- Mathematics uses everyday words, but their meaning is defined precisely in relation to other mathematical terms and not by their everyday meaning. Even the syntax of mathematical argument is different from the syntax of everyday language and is again quite precisely defined.

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- Mathematics contains objects which are unique to itself. For example, although everyday language occasionally uses symbols like x and P, objects like x^0 and $\sqrt{(-1)}$ are unknown outside mathematics.
- A large part of mathematics consists of rules for operating on mathematical objects and relationships. Sierpinska calls these "the rules of the game" [F132]. It is important that students learn to manipulate symbols using these rules and no others.

We claim that the essence of abstraction in mathematics is that mathematics is self-contained: An abstract mathematical object takes its meaning only from the system within which it is defined. Certainly abstraction in mathematics—at all levels—includes ignoring certain features and highlighting others, as Sierpinska [F130] emphasises. But it is crucial that the new objects be related to each other in a consistent system which can be operated on without reference to their previous meaning. Thus, self-containment is paramount.

Historically, mathematics has seen an increasing use of axiomatics, especially over the last two centuries. For example, numbers were initially mathematical objects based on the empirical idea of quantity. Then mathematicians such as Dedekind and Peano reconceptualised numbers in axiom systems which were independent of the idea of quantity. Euclid, Hilbert, and others performed a similar task for geometry. But, as Kleiner (1991) states, "whereas Euclid's axioms are idealizations of a concrete physical reality … in the modern view axioms are … simply assumptions about the relations among the undefined terms of the axiomatic system" (p. 303). In other words, mathematics has become increasingly independent of experience, therefore more self-contained and hence more abstract.

To emphasise the special meaning of abstraction in mathematics, we shall say that mathematical objects are *abstract-apart.* Their meanings are defined within the world of mathematics, and they exist quite apart from any external reference.

So why is mathematics so useful?

Mathematics is used in predicting and controlling real objects and events, from calculating a shopping bill to sending rockets to Mars. How can an abstract-apart science be so practically useful?

One aspect of the usefulness of mathematics is the facility with which calculations can be made: You do not need to exchange coins to calculate your shopping bill, and you can simulate a rocket journey without ever firing one. Increasingly powerful mathematical theories (not to mention the computer) have led to steady gains in efficiency and reliability.

But calculational facility would be useless if the results did not predict reality. Predictions are successful to the extent that mathematics models appropriate aspects of reality, and whether they are appropriate can be validated by experience. In fact, one can go further and claim that the mathematics we know today has been developed (in preference to any other that might be imaginable) because it does model significant aspects of reality faithfully. As Devlin (1994) puts it:

> How is it that the axiomatic method has been so successful in this way? The answer is, in large part, because the axioms do indeed capture meaningful and correct patterns. … There is nothing to prevent anyone from writing down some arbitrary list of postulates and proceeding to prove theorems from them. But the chance of those theorems having any practical application [is] slim indeed. (pp. 54-55)

Many fundamental mathematical objects (especially the more elementary ones, such as numbers and their operations) clearly model reality. Later developments (such as combinatorics and differential equations) are built on these fundamental ideas and so also reflect reality—even if indirectly. Hence all mathematics has some link back to reality.

EMPIRICAL ABSTRACTION IN MATHEMATICS LEARNING

Learning fundamental mathematical ideas

Students learn about many fundamental, abstract mathematical objects in school. In this section, we discuss the meaning of abstraction in this learning context. We begin by looking at some examples.

Addition. Between the ages of 3 and 6, most children learn that a given set of objects contains a fixed number of objects. A little later, they realise that two sets can be combined and that the number of objects in the combined set can be determined from the number of objects in each set—a procedure which later becomes the operation of addition. Students learn these fundamental arithmetical ideas from counting experiences: They find that repeatedly counting a given set of objects always gives the same number, no matter how often it is done and in which order. As they recognise more and more patterns, counting a combined set is gradually replaced by "counting on" and eventually the use of "number facts" (Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Angles. There is good evidence that, at the beginning of elementary school, students have already formed classes of angle situations such as corners, slopes, and turns (Mitchelmore, 1997). To acquire a general concept of angle, students need to see the similarities between them and identify their essential common features (two lines meeting at a point, with some significance to their angular deviation). Even secondary students find it difficult to identify angles in slopes and turns, where one or both arms of the angle have to be imagined or remembered (Mitchelmore & White, 2000).

Rate of change. The most fundamental idea in calculus is *rate of change*, leading to differentiation. A major reform movement over the last decade or so has been concerned with making this idea more meaningful by initially exploring a range of realistic rate of change situations. In this way, students build up an intuitive idea of rate of change before studying the topic abstractly. A leading US college textbook (Hughes-Hallett et al., 1994) devotes a whole introductory chapter to exploring realistic situations, and in Australia similar materials have been published for high school calculus students (Barnes, 1992).

Characteristics of empirical abstraction

The above examples show how fundamental mathematical ideas are based on the investigation of real world situations and the identification of their key common features. Hence, a characteristic of the learning of fundamental mathematical ideas is *similarity recognition.* The similarity is not in terms of superficial appearances but in underlying structure—for example, in counting, space, and relationships. To get below the surface often requires a new viewpoint, as when a student imposes imaginary initial and final lines on a turning object in order to obtain an angle.

There is a leap forward when students recognise such a similarity: As students relate together situations which were previously conceived as disconnected, they become able to do things they were not able to do before. More than that, they form new ideas (such as addition, angle, and rate of change) and are incapable of reverting to their previous state of innocence. In a sense, these new ideas *embody* the similarities recognised. Of course, single ideas rarely evolve in isolation; for example, the idea of angle is inextricable linked to ideas such as point, line, parallel, intersection and measurement which can also be traced to similarities students recognise in their environment.

This process of similarity recognition followed by embodiment of the similarity in a new idea is an *empirical abstraction* process. It is well described by Skemp (1986):

> *Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept.* (p. 21, italics in original)

Thus number, addition, angle and rate of change are all empirical concepts, and they take their place in students' learning alongside other empirical concepts such as colour, friend, and fairness.

Piaget (1977) made a distinction between abstraction on the basis of superficial characteristics of physical objects (*abstraction à partir de l'objet*) and abstraction on the basis of relationships perceived when the learner manipulates these objects (*abstraction à partir de l'action*). But both are based on the child's physical and social experience, and in both similarity recognition is essential. In using the term *empirical abstraction* to cover both cases, we are making the distinction between abstraction on the basis of experience and what we shall call *theoretical abstraction* (see below).

EMPIRICAL ABSTRACTION AND MATHEMATICAL ABSTRACTION

From empirical concept to mathematical object

When students learn a fundamental mathematical idea in the way described above, three things happen: They learn an empirical concept, they learn about a mathematical object, and they learn about the relationship between the empirical concept and the mathematical object. Empirical concepts are often rather fuzzy and difficult to define. For example, the empirical concept of circle is that of a perfectly round object—but "perfect roundness" can only be defined by showing examples. A circle becomes a mathematical object only when it is defined as the locus of points equidistant from a fixed point: It is then clearly defined in terms of other mathematical objects. However, for this definition to be meaningful, an individual must see that the locus of points equidistant from a fixed point gives a perfectly round object and vice versa.

We have already referred to mathematical objects as abstract-apart. To emphasise the distinction between abstraction in mathematics and mathematics learning, we shall call empirical concepts *abstract-general*: Each concept embodies that which is general to the objects from which the similarity is abstracted.

Gravemeier also focuses on how "formal mathematics grows out of the mathematical activity of the students" [F125], calling the process *emergent modelling.* The Realistic Mathematics Education movement, to which Gravemeier belongs, has previously called it *vertical mathematisation* (Treffers, 1987). We prefer to call this process *formalisation*, since its main purpose is to select abstract-apart relationships which capture the *form* of an abstract-general concept. (So "formal mathematics" is the study of mathematical forms.) For example, the locus definition of a mathematical circle precisely expresses the perfect roundness of an empirical circle.

Linking mathematical objects to empirical concepts

There is strong evidence that many student difficulties in learning mathematics can be traced to the fact that, when they learned about an abstract-apart mathematical object, they made no link to the corresponding abstract-general concept (Mitchelmore & White, 1995). Consider again the previous three examples.

Addition. Many young students experience difficulty learning elementary arithmetic. One explanation is that they do not understand the empirical meaning of the operations: Symbols such as + and × are learned apart from the abstract-general concepts of addition and multiplication on which they are based. Early number research (Steffe et al., 1983; Wright, 1994) has led to projects such as *Count Me In Too* which have closely linked early arithmetic to students' counting experiences, with a measurable improvement in learning (Mitchelmore & White, 2003).

Angle. Many student difficulties with angles arise because the angle diagram is abstract-apart. Williams (2003) gives a particularly extreme example: Her case-study secondary school student successfully made a generalisation about the angle sum of a polygon, but he could not identify the angles of the triangles into which he had divided the polygon. In fact, it is quite possible to teach an abstract-general concept of angle as early as Grade 3, as White & Mitchelmore (2003) have shown.

Calculus. Calculus instruction based on abstract-apart differentiation leads to a *manipulation focus* (White & Mitchelmore, 1996). Students do not see symbols as representing anything, so they cannot use the manipulative techniques they have learned to solve contextual problems. Their concept of differentiation has been truly decontextualised and therefore impoverished, instead of being abstract-general and rich (Van Oers, 2001).

The preceding discussion emphasises the value of making a clear distinction between empirical concepts and mathematical objects.

MORE ADVANCED MATHEMATICS LEARNING

The learning of fundamental mathematical ideas is only one component of learning mathematics: More advanced ideas need to be developed out of the fundamental ideas. Some of these ideas (such as square roots) can be readily linked back to abstract-general concepts; others (such as a zero exponent) seem to have no counterpart in normal experience. In addition, students need to learn to operate within an abstract-apart system—an aspect of mathematics learning which takes on increasing significance in university mathematics as the links to experience become thinner and thinner. But even professional mathematicians use empirical concepts as an aid to intuition (Boero, [F137]).

The formation of new ideas within mathematics is well described by the Schwarz-Hershkowitz-Dreyfus *Nested RBC Model of Abstraction*. They define abstraction as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" [F121]. New mathematical objects are constructed by "the establishment of connections, such as inventing a mathematical generalization, proof, or a new strategy of solving a problem" [F121]. This abstraction process is quite different from empirical abstraction, and is best described as *theoretical abstraction.* Sierpinska's ignoring/highlighting process is another example of theoretical abstraction.

Gray & Tall's idea of a *procept*—"the amalgam of three components: a *process* which produces a mathematical *object,* and a *symbol* which is used to represent either process or object" [F117]—also clarifies the development of ideas within mathematics. The construction of a procept seems to us, however, to be more akin to formalisation than abstraction.

Historically, some more advanced mathematical objects have been constructed by a process similar to empirical abstraction. An example is group theory:

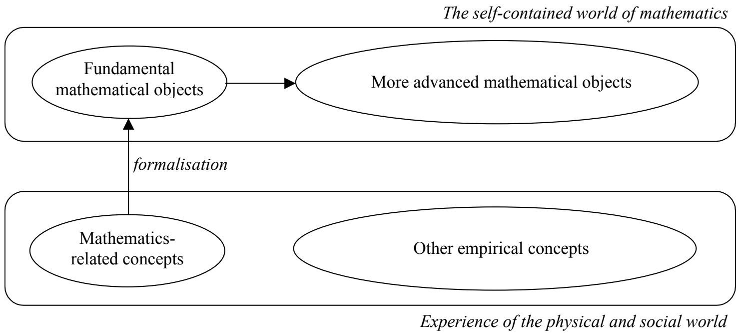
> The abstract concept of a group arose from different sources. Thus polynomial theory gave rise to groups of permutations, number theory to groups of numbers and of "forms" … and geometry and analysis to groups of transformations. Common features of these

concrete examples of groups began to be noted, and this resulted in the emergence of the abstract concept of a group in the last decades of the 19th century. (Kleiner, 1991, p. 302)

Other examples are rings, fields and vector spaces. Our arguments above would suggest that the learning of such mathematics would be most effective if it were based on a process of similarity recognition followed by formalisation.

SUMMARY AND CONCLUSION

The term *abstraction* has different meanings in relation to mathematics and the learning of mathematics. Previous abstraction theorists have tended to focus on the process of developing ideas within mathematics. In this paper, we have tried to redress the balance by exploring the role of empirical abstraction in the formation of fundamental mathematical ideas. This is a crucial process, since many fundamental, abstract-apart mathematical objects need to be linked to abstract-general empirical concepts if their learning is to be meaningful.



Grossly over-simplified, we see the whole picture as follows:

In practice, the formation of mathematics-related empirical concepts and their formalisation into mathematical objects may occur simultaneously—especially in school learning. Also, more advanced mathematical objects may be linked directly to empirical concepts and not only indirectly via fundamental objects.

Like Boero [F138], we believe that "we are still far from a comprehensive theoretical answer to the challenge of mathematical abstraction in mathematics education". A clear response to this challenge would be of great value to researchers and teachers alike. Examining and differentiating the different forms of abstraction involved in learning mathematics constitute one step along the path to this goal.

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