RF03: INTERNATIONAL PERSPECTIVES ON THE NATURE OF MATHEMATICAL KNOWLEDGE FOR SECONDARY TEACHING: PROGRESS AND DILEMMAS

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This research forum addresses the question: what is the nature of the mathematical knowledge that is needed for secondary teaching? Six international contributors respond by making two claims (one related to an area where progress in research has been made and the other related to dilemmas facing researchers): preparing teachers, teaching practice, and research designs and methodologies. This structure provides a way of focusing the discussion among forum participants and a means to develop international points of view on the nature of the mathematical knowledge that is needed for secondary teaching.

GENERAL FRAMEWORK

Over the past two decades, international perspectives on research about the teaching of mathematics have received considerable and increasing attention at PME and by the research community in mathematics education (Ellerton, 1998; Jaworski, Wood & Dawson, 1999). Yet, progress towards changes in teaching practices remains slow and large gaps exist between the highest achieving schools and countries and the lowest achieving schools and countries. Substantial progress has been made in many areas of research related to students' learning along with the emergence of curricular materials and standards documents that reflect findings of this research (e.g., the early numeracy projects in the United Kingdom, New Zealand and Australia). Nevertheless, translating research on mathematical learning into forms that are useful for teaching practice continues to be a difficult problem that varies substantially across schools and countries and progress has been elusive. Difficulties in preparing new teachers are compounded by the disconnection that pre-service teachers can experience between their teacher preparation programs and their experiences in practice. Furthermore, the complexity that characterizes teaching and learning seems to have yielded a multiplicity of research designs and methodologies with insufficient coherence across these research designs to support the development of a shared knowledge base for teaching.

KEY QUESTIONS AND THEMES

There is substantial agreement among mathematics educators that the quality of teachers' subject matter knowledge is necessary but not sufficient for effective teaching. Subject matter knowledge is just one category among many that attempt to capture the complexity of the nature of the mathematical knowledge base that is needed for teaching (Hiebert, Gallimore & Stigler, 2002; Shulman, 1986). Hence, the central focus of this research forum is the nature of the mathematical knowledge that is needed for teaching in secondary schools.

A recent National Research Council report in the United States (NRC, 2003) described mathematical proficiency for students as the simultaneous and integrated acquisition of five strands: (1) conceptual understanding, (2) procedural fluency, (3) strategic competence, (4) adaptive reasoning, and (5) productive disposition. These proficiencies provide one possible framework for considering the mathematical knowledge that is needed by secondary teachers. However, in addition to teachers having this kind of mathematical proficiency, they must also understand (1) how such mathematical proficiencies are developed in curricular materials, (2) the ways in which students' thinking might reveal students' mathematical proficiencies, and (3) how students from diverse cultural and ethnic backgrounds develop these mathematical proficiencies.

Another possible framework comes from the KOM project (Niss, 2003) which provides eight competencies for students, such as 'think mathematically and make use of different representations and translate between them,' that describe the main components for mastering mathematics derived from the work of mathematicians. In addition to having these competencies, teachers must also have competencies in curriculum, teaching, student learning and assessment. We will use these proficiencies and competencies as background for considering the nature of teachers' mathematical knowledge, and then we will describe the challenges and difficulties in designing and implementing research in this area.

This research forum will focus on the following central question: *What is the nature of the mathematical knowledge that is needed for secondary teaching?*

Our goal in this forum is to stimulate discussion on this question through a reporting of research findings that identify areas in which significant progress has been made and where difficulties and persistent obstacles to progress continue to exist. To initiate the discussion, contributors from six different countries that represent international differences in contexts and perspectives report their findings. Each contributor addresses the question from three views: (a) preparing teachers; (b) supporting teachers in practice; and (c) research design and methodologies. Each contributor makes two key claims related to each of the three views of the above question. The first claim reflects an area where substantial research progress has been made in the contributor's country with respect to the nature of the mathematical knowledge that is needed for teaching secondary mathematics. These claims reflect findings that are of significance to the field and are based on a substantial body of research. The second claim reflects a significant dilemma in research or an area where progress has remained elusive. This structure provides both a broad view of the field (as it is seen internationally) and a way of focusing the discussion among forum participants. The contributors' claims are presented in the next section. Following those contributions, we provide a tentative synthesis of the claims and pose some cross-cutting questions that will provide a beginning point for work of the participants in this forum.

PREPARING TEACHERS—PROGRESS AND DILEMMAS

Australia (Kaye Stacey)

Claim 1 Progress: The corpus of research on students' conceptions, thinking and learning in mathematical content areas provides foundation knowledge for a greatly improved teacher education.

Claim 2 Dilemma: This corpus of knowledge needs to undergo substantial didactic transposition before it is maximally useful.

Claim 1 is about creating the scientific basis of a discipline of mathematics didactics (pedagogy) for teacher education. The established sciences and humanities have an accumulated set of well-tested research results, which have been codified and simplified to create learnable disciplines. In mathematics education, we are now reaching a point where we too have a sufficiently strong scientific foundation to undertake this task. We know enough about students' thinking patterns, conceptions and their development to begin the "didactic transposition" from raw research results to learnable and organised material which could form the basis of a new teacher education. I expect these outcomes to be very much more effective than teacher education based around general theories of mathematical development (as was tried, for example, with Piagetian research in times past).

What is the evidence for Claim 1? The extent of the research knowledge is evident from the accumulated proceedings of PME, the handbooks of reviews of research and so forth. The need for this material to undergo a didactic transposition is evident in the lack of textbooks on student's thinking and learning for secondary mathematics teacher education (indeed no textbook is widely used in Australia) and the consequent practice of referring teacher education students directly to research reports rather than to scholarly accounts written for them.

My claim also requires evidence that this new content of teacher education would "make a difference." Two large scale elementary teacher development projects provide some confirmation. *Count Me In Too* (Bobis, 1999) is a New South Wales government professional development initiative where mathematics education researchers turned international research on children's early number development into support material for professional development. Teachers learned about how children's knowledge progressed, assessed children's learning carefully and selected teaching materials to move them along the framework. The *Early Numeracy Research Project* in Victoria had a similar mission and adopted a similar approach, although differing in detail. Both projects, although focused on elementary schooling, demonstrate improved outcomes for students across large numbers of schools, some of them sustained. A difficulty with using a program evaluation as evidence for my claim is that improved learning outcomes are a result of the whole program, rather than one component, such as improved teacher knowledge.

I envisage the process of didactic transposition running some decades behind the research. There is clear evidence that teacher education students find such material interesting and relevant to their future work. We have found that presenting case studies of children's thinking about decimal numbers using simple multimedia products has engaged our pre-service teachers very deeply (Chambers, Stacey, & Steinle, 2003). It has been a powerful way to expose and remediate their own misunderstandings (e.g., Stacey et al., 2001). Several years after working with this material, some of our pre-service teachers have spontaneously recalled the case studies by name and by misconception.

The didactic transposition is not, however, unproblematic. In a teacher education course with limited time, what is the right "grain size" for knowledge about student's learning so that it can guide teaching actions? Our research catalogue of decimal misconceptions (e.g., Steinle & Stacey, 2003) with 12 major types is probably too large for teachers to act upon in real time in classrooms, without technological assistance (Stacey, et al., 2003). (Even in this paper, we only use the simplest examples.) *Count Me in Too*, for example, presented a considerably finer analysis of early number learning than the Victorian *Early Numeracy Research Project*. There are many other questions to be answered. Are there (or where are there) strong commonalities in learning trajectories that will assist transfer of knowledge between different teaching areas? For reasons such as these, Claim 2 is a call for research on our research.

Brazil (Marcelo Borba)

Claim 1 Progress: The notion that we need to search for the particularities of mathematics that should be taught to teachers with a broad view of mathematical content. Adding more content to teacher preparation programs is not a solution.

There are a significant number of teacher educators who are investigating what specific mathematics should be included in teacher education programs. In Brazil, this discussion has taken on new dimensions, since there is an established tradition of research on ethnomathematics that began over a quarter of a century ago. The idea that different cultural groups produce different mathematics (D'Ambrosio, 2001; Borba, 1987) is very well-accepted. Various researchers seem to have chosen to extend this idea into teacher education, and to consider pre-service teachers as members of the mathematics education community (Lave & Wenger, 1991), even if they do not necessarily address the problem using the specific constructs of ethnomathematics or "community of learners."

Different researchers in Brazil have emphasized that simply adding more content is not the solution to the problem of what should be taught to pre-service mathematics teachers. Instead, such content should be seen as embedded in cultural and social issues regarding the context, philosophical themes related to what should be taught and historical aspects of the change in mathematical knowledge over time. Mathematics teachers, beginning in their pre-service education, should become members of the mathematics education community and not members of the mathematics community.

Claim 2 Dilemma: Although there seems to be a consensus that the education of mathematics teachers should be different from those who will go on to do research in mathematics, there is no consensus regarding whether or not the education of these two types of students should be totally distinct.

On the one hand, there are teacher educators who believe that it is important for future teachers to interact with professional mathematicians and with students who will become mathematicians. On the other hand, there are mathematics educators who believe that it is impossible to do so, and that to keep these two kind of students in the same structure means that pre-service teachers will be educated like mathematicians for two years and only in the final years they will be prepared to become teachers. According to Fiorentini et al (2002) dilemmas such as this, involving the tension between mathematicians and mathematics educators, have been present in Brazilian research about teacher education for a more than a decade.

Israel (Ruhama Even)

Claim 1 Progress: Regular university or college mathematics courses do not support the development of adequate mathematical knowledge for teaching secondary school mathematics.

A traditional approach to equip secondary school mathematics teachers with adequate mathematical knowledge is quantitative in nature: “more is better.” This approach is based on the premise that teachers already learned, and therefore know, school mathematics; and that teachers should know more mathematics than the mathematics their students have to learn, and therefore, advanced mathematics studies are a good indicator of adequate teacher mathematical knowledge. However, several research studies that examined teachers’ mathematical content knowledge (e.g., Even, 1990, 1992, 1998; Knuth, 2002; Lipman, 1994; Shriki & David, 2001) suggest that secondary school mathematics teachers often do not hold a sound understanding of the mathematics they need to use and teach in school. This includes fundamental concepts from the secondary school curriculum, such as functions and proof.

For example, the following problem was presented to 162 American (Even, 1992) and to 45 Israeli (Lipman, 1994) prospective teachers from several universities (U.S.) and teacher colleges (Israel), in the last stage of their formal pre-service preparation.

> A student said that there are 2 different inverse functions for the function $f(x)=10^x$: One is the root function and the other is the log function. Is the student right? Explain.

Many did not answer correctly. Some chose the root function as the inverse function, using a naive conception of “undoing” as their interpretation of inverse function. The xth root of 10 seemed to them to “undoes” what 10^x does in the following manner: In order to get 10^x, one starts with 10 and then raises it to the xth power. By taking the xth root of 10^x, one gets 10 back. Accepting the root function as an inverse function because of its “undoing” appeal created for many of the American prospective teachers a cognitive dissonance; they remembered that log was the appropriate inverse function and that the inverse function for any given function is unique. To solve this uncomfortable situation these students decided that the log function and the root function were both inverse functions of the given function *since they were the same function*. For example, “I believe that there is only one function. The root function and the log function are just two different ways of representing the same function.”

Such findings indicate that relying on advanced mathematical studies at the college or university level to account for adequate teacher mathematical knowledge of secondary school mathematics is problematic. Apparently, even though teachers have already learned as students the mathematics they need to teach, and then studied even more advanced mathematics, they still need to re-learn the mathematics they have to teach.

Claim 2 Dilemma: What would support the development of adequate mathematical knowledge for teaching secondary school mathematics?

Several programs and courses for in-service secondary teachers in Israel include as one of their components the deepening of the participants’ knowledge of the mathematics they need to use and teach at school (e.g., Even & Bar-Zohar, 1997; Zaslavsky & Leikin, 1999). However, this is less common in pre-service teacher education. At any rate, we do not have enough research findings to provide adequate answer to the above question.

Norway (Bodil Kleve and Barbara Jaworski)

Claim 1 Progress: The problematic nature of mathematics teacher education in Norway is at last being recognized and addressed.

Norway is a long and thin country in Northern Europe (Scandinavia) covering 324,000 km^2 of land, approximate in size to Poland. It is covered largely with lakes, fjords, mountains and forests and is only sparsely populated: its population is 4.5 million, of which 0.6 is in the capital, Oslo. Thus, for geographic and demographic reasons, many schools are small and this affects the organisation of education. Multi-grade teaching is common, and teachers need to teach a wide range of subjects.

As a consequence of this geographical spread and an educational philosophy of inclusion, all teachers educated in teacher education colleges in Norway are general teachers. This means that they have formal competence to teach *all* subjects in grades 1 to 10 (age 6 to 15). Mathematics has been a compulsory subject in teacher education only since 1992, which implies that there are many teachers in Norwegian schools (including lower secondary school, grade 7-10) teaching mathematics without any formal competence within the subject.

To start teacher education study, students need what we call a general study competence from upper secondary school (grades 11-13). The first year in upper taught 5 lessons a week out of a total of 30 lessons. To obtain general study competence, students need only this basic course in mathematics from upper secondary school, although it is possible do more. In teacher education before 1992, a short course in mathematical didactics was taught to all pre-service teachers who, at that time, could also choose to study mathematics (per se) for one-fourth, one-half or 1 year of study. From 1992 to 1998 studying mathematics became compulsory in teacher education for all pre-service teachers for one-fourth year of study. Since 1998 it has been compulsory to include one-half year of mathematics, with the option of up to one and one-half years of study. Currently, in upper secondary school, mathematics teachers usually have 1-3 years of education in mathematics from a university, some having a degree in mathematics.

In 1995, a group of experienced mathematics teachers from all levels in the school and college system were asked by the Ministry of Education (KUF) to undertake a survey of the subject of mathematics from primary school to university level. The goal of this 3-year project, MISS–MATEMATIKK I SKOLE OG SAMFUNN (Mathematics in School and Society, Bekken, 1997), was to improve the teaching of mathematics for all students by identifying basic problems and suggesting strategies and initiatives for improvement of teaching competence with reference to teacher education (pre-service and in-service) and to textbooks and teaching material. The background for the project included changes in the need for computational skills in the light of computer technology with increased emphasis on understanding of concepts. Students' and pre-service teachers' attitudes to mathematics were an important focus.

The work of the group relied mainly on three sources: the experience of the members of the group, findings in other (research) documents and some small investigations done by members of the group. Outcomes from the work are to be found in articles in three reports, June 95, 96 and 97 and in a final report from November 97. The articles were written by individuals, discussed and sometimes revised by the whole group before being printed. Thus they vary in reflecting individual or group perspectives.

The final work of MISS concludes with 42 proposals for changes, where 14 are labelled as key proposals. Those significant for mathematics teaching and teacher education include: to give teachers a sabbatical year to study more mathematics; to establish a forum for the didactical development of mathematics teachers; to enable teachers and teacher educators to collaborate in developmental projects in schools; to start research and development projects designed to create an extensive plan for in-service teacher education; to establish a requirement for at least two years of mathematics from upper secondary school in order to start higher education studies involving mathematics (science, economics, and teacher education).

Claim 2 Dilemma: Students entering higher education involving studies in mathematics do not have command of all basic skills in mathematics that one would expect at this level. This is especially dramatic within teacher education. Norway has

Norsk Matematikkråd (NMR), the Norwegian Mathematics Council, has constructed a survey (Halvorsen & Johnsbråten, 2002) which is administered to pre-service teachers at the starting point of teacher education to analyse their performance in mathematics. The test is also given to students entering other studies in mathematics such as engineering or computer science. Ninety percent of the items relate to mathematics that is covered by the syllabus in lower secondary school (grades 8-10). The items test mathematical skills, procedural knowledge and facts rather than students' conceptual knowledge in mathematics. The survey was conducted every other autumn between 1982 and 1991, and every autumn since 1999. Results show that there has been a decrease in performance in recent years. In 2001, 4,737 students participated with an average number of correct answers of 52%. These figures include 732 pre-service teachers. Of all groups pre-service teachers had the lowest average number, 29.5%; 516 of these students had only the basic course in mathematics from upper secondary school. These results reveal that among students starting higher education involving mathematics, pre-service teachers are those who perform lowest with regard to basic skills within the subject. With this background, The Norwegian Mathematics Council has suggested at least two years of mathematics from upper secondary school should be required in order to start higher education studies in teacher education.

Taiwan (Fou-Lai Lin)

Claim 1 Progress: Based on research process and results, several mathematics education courses have been developed in teacher education program.

Both the MUT (Mathematics Understanding of Taiwanese students) program carried out in the eighties (cf., Lin, 1989) and CD–MIT (Concept Development-Mathematics in Taiwan) program conducted recently (cf., Lin & Chen, 2003) studied students' conceptual understanding of most topics in school mathematics. The MUT program has generated a course "Mathematics Learning" for pre-service teachers and the CD–MIT program enhanced its content. The HPM (History and Pedagogy of Mathematics) program in Taiwan has developed more than thirty learning units based on historical text and have published their results in a monthly newsletter, *HPM Forum* (cf., Horng, 2002). Those results shaped the content of "Mathematics History" course towards a pedagogical orientation.

On his website (http://math.ntnu.edu.tw/~cyc/), Chen, Taso and others have demonstrated many learning activities developed with GSP (Geometric Sketch Pad). Those learning activities serve as the foundation for a "Computer and Mathematics" course. Some other mathematics education courses, such as "Mathematics Activity and Thinking", "Mathematics Problem Solving", "Mathematics Teaching and Assessment" have also benefited from the results of varied research projects. By taking those courses, pre-service teachers have experienced multiple didactic views about mathematics, such as mathematics as a model of thinking, school mathematics is about students' thinking and strategies, and mathematics has a cultural and dynamic nature.

Claim 2 Dilemma: Pre-service teachers are still experiencing two contrasting views about learning mathematics, one from university mathematics courses and the other from mathematics education courses they take.

A survey (Huang, 2001) aimed to investigate the learning phenomena of mathematics pre-service teachers has revealed the seriousness of the conflict. To reflect the multiple didactic views about mathematics, instruction in mathematics education courses very often is activity-based, process-oriented and includes multi-media aids. Such a process-oriented view about learning was challenged by pre-service teachers because of their own experiences in learning university mathematics. Within university mathematics classes, how much content should be covered still is the main concern among most mathematicians. To cover the content, their instruction very often keeps a traditional exposition on formal structured content. Pre-service teachers, therefore, have no choice but to experience two contrasting approaches of learning--process-oriented versus content-oriented.

USA (Helen Doerr)

Claim 1 Progress: Pre-service teachers tend to hold beliefs about the nature of mathematics and its teaching and learning that are at odds with views put forth by the Standards documents (NCTM, 1989, 1991, 2000) and by teacher educators.

It is widely accepted in the US that pre-service teachers come to their teacher preparation programs with beliefs about mathematics "as a set of discrete rules best learned through repeated practice. Based on their own experiences as students, prospective teachers think of 'doing math' as a matter of completing a page of forty problems" (Feiman-Nemser & Remillard, 1996, p. 70.) A view of mathematics as doing procedural problems is generally accompanied by an image of teaching as clearly presenting, showing and explaining to students how to follow the rules of mathematics and to do particular problems. In 1992, Thompson provided a detailed review of the beliefs of teachers and later work by Cooney and colleagues (1998) has provided more detailed descriptions of the beliefs structures of pre-service teachers. Work by Frykholm (1996) has documented the difficulties and challenges that pre-service teachers face when attempting to implement standards-based teaching practices that attend to the conceptual development of mathematical ideas through a focus on problem-solving, reasoning, communication, and connections. Frykholm found that the pre-service teachers lacked the tools to implement standards-based lesson and were influenced more by their cooperating teachers who did not make the standards a primary focus of their teaching than by their university-based methods course that did.

The perception of mathematics as centered on the knowledge and application of rules is aptly illustrated by Kinach (2002) who routinely found that pre-service teachers' descriptions of explaining the operations with integers to someone just learning it focused on giving students rules for signs. As Kinach observed, none of the pre-service teachers had any representational notions (other than arrows) to draw on or initially saw the inadequacy of simply telling students the rules rather than providing a good mathematical explanation.

While the beliefs of pre-service teachers, as I have just described them, would resonate well with the experiences of teacher educators and researchers, I have found no systematic, large scale study of the beliefs about mathematics that pre-service teachers bring to their preparation programs. However, I am not suggesting that such studies be conducted, but rather that we shift our focus from the nature and structure of pre-service teachers' beliefs systems—which can often appear to be impermeable and not particularly easy to directly address—to an examination of the issue of the mathematical knowledge that is needed to begin learning to teach.

Claim 2 Dilemma: One of the central dilemmas of learning to teach is found in the struggle of moving past the apprenticeship of observation and the years of experience as a learner of the rules and procedures of mathematics.

Unlike their elementary counterparts, who often found frustration and confusion as they encountered difficulty in trying to make sense of mathematics, pre-service secondary teachers were by and large successful (and often very successful) in their experiences as learners in K-12. Hence, pre-service secondary teachers are less likely to find a practice focused on the mastery of procedures to be problematic. Furthermore, there is no clear evidence as to how or to what extent pre-service teachers' undergraduate experiences in mathematics reinforce notions of mathematics as a fixed body of rules to be mastered. Field experiences at the secondary level (as noted above) may reinforce traditional views of learning mathematics; secondary practice in the US has been especially resistive to change. This, of course, situates pre-service teachers in the gap between the realities of classroom practice and the goals of their preparation programs.

This leaves teacher educators and researchers facing two difficult issues: (1) How and what do pre-service teachers learn about the nature of mathematics as a discipline in their undergraduate experiences with mathematics? And how does this influence their beginning ideas about how others might learn mathematics? (2) How do pre-service teachers negotiate the constraints and limitations of field experiences? To what extent do those experiences impede and support their understanding of the mathematics that is needed for teaching?

PRACTICING TEACHERS—PROGRESS AND DILEMMAS

Australia (Kaye Stacey)

Claim 1 Progress: Many examples demonstrate that teachers' deep content knowledge and extensive pedagogical content knowledge improves students'

Claim 2 Dilemma: There is insufficient evidence to convince a skeptic that teachers' deep content knowledge and extensive pedagogical content knowledge improve students' learning.

Evidence for Claim 1 arises in most studies of classroom learning which gather relevant data. For example, for a variety of interesting reasons, about 10-15% of secondary school students are likely to believe that a decimal which looks smaller (e.g., 0.45 looks smaller than 0.4567) is actually larger, and another significantly sized group has a great deal of difficulty with zero, as the number and as a digit in decimal numbers. Teachers who understand these problems can address them in their teaching—others will not and as a result, misconceptions cluster in classes and schools (Steinle & Stacey, 1998). This illustrates an unfortunate but unavoidable feature of research in this area; it is easier to trace the impact of errors and misunderstanding, than of teachers' good understandings. Other research demonstrates that a minimal intervention which demonstrates to teachers how students might be thinking about decimal numbers and provides some targeted teaching tasks can make a long-term difference to children's understanding (Helme & Stacey, 2000). Some of the difficulties that students have, often for years, are not necessarily difficult to fix, but a teacher needs to understand their importance.

I find Claim 1, supported by many small examples, compelling because it gels with my own experiences of teaching mathematics. However, there is little hard data to support it: hence Claim 2. The most internationally influential studies are far from conclusive. Ma (1999) asserts, from only tens of examples, that differences in teachers' ability to make connections among mathematical ideas are largely responsible for the difference in performance between Chinese and U.S. students. The work of Ball (2000) very usefully emphasized how a myriad teaching decisions, such as what questions to ask, what test items to set, what examples to choose, are affected by teachers' knowledge, but again this information is case-based.

There is some large-scale quantitative data to support Claim 1. In considering teachers' characteristics and their association with children's numeracy performance in Britain, Askew et al (1997) identified teachers' recognition of deep connections between mathematical ideas as one of the few predictors of high learning gains by children. On the one hand, effective teachers of numeracy saw mathematics as richly connected and adopted classroom strategies that helped children to make links. On the other hand, the correlations found are surprisingly low. In sum, the data convinces the believers, but not skeptics. Furthermore, the above studies focus on elementary mathematics and elementary mathematics teachers; it is even less clear how to generalize this evidence to the secondary level.

Brazil (Marcelo Borba)

Claim 1 Progress: Online support has been shown to be useful in continuing teacher education projects as means of collaborating with teachers in the implementation information and communication technology in the mathematics classroom.

Online support has been used in continuing teacher education projects, both for research and extension courses. One of these projects (Borba, in press; Gracias, 2003) has been developing 100% on-line courses for teachers in Brazil and in countries like Argentina and Venezuela. This appears to be a solution for continuing education in countries like Brazil—with huge geographical size but with a concentration of research centers in just one small region. For some, the concern with mathematics teacher education is almost synonymous with mathematics education, as pre-service and continuing education can be seen as the "trunk" for all other aspects of mathematics education.

Among the researchers who investigate how teachers deal with the introduction of information and communication technology (ICT), there seems to be a tendency to share one certainty these days: short term courses are positive, but they are far from enough if teachers are to incorporate changes in the classroom. An alternative that goes beyond courses, but without discarding them, is one based on collaborative practices. Penteado and Borba (2000) developed a project that merged short-term courses on basic use of technology and on mathematics education software together with support for teachers to use them in the classroom. Teachers would prepare classes using software with the help of members of the research team who had more experience with given software, and who, at the same time, would help to frame the problems to be investigated (and maybe solved) during this interaction which joined extension courses and research. More recently, researchers such as Penteado (http://ns.rc.unesp.br/igce/matematica/interlk) have been leading a project in which there is collaboration between researchers and teachers in order to provide support for teachers who want to use geometry, function or other types of software in the classroom. Different research projects focusing on the relationship among members of this support network are developed and, at the same time, provide solutions for problems and help to bring teachers into graduate programs.

Studies have shown the transformation of the interaction in these courses, which focus on trends in mathematics education, when we compare it to the regular interaction we had in graduate courses in which teachers and researchers take part. For instance, when we have a synchronous interaction in a chat, multiple dialogues may happen at the same time. Participants may switch from one discussion to another and the teacher may have to deal with several questions and issues at the same time. (See http://www.rc.unesp.br/igce/pgem/gpimem.html for papers on these types of interactions.)

Based on the assessment made at the end of each course (five have been offered so far), this model has had a significant impact in terms of bringing members of different communities into the discussion regarding mathematics education and giving them access to professors from one of the most prestigious mathematics education graduate programs in Brazil with whom they would otherwise not have an opportunity to interact.

Claim 2 Dilemma: Online support also raises problems that are far from being solved

The first problem is related to continuous support, as discussed before. In this sense, we need to have an increasing number of people giving support to teachers who participate in the course if they are to bring change to the mathematics that is taught in the classroom. On the one hand, this can be considered to be more of an extension course problem, but on the other hand, it is a logistical problem for researchers if we want to assess change with teachers who participate in such courses.

The second question is related to the very notion of what mathematics should be taught once a specific function or geometry software is in use. Pre-service teachers should be exposed to changes that software brings to the mathematics in the classroom, as most Brazilian researchers on technology believe. However, there is no such discussion regarding the case of the Internet and distance education. The question, "What kind of change will be brought to mathematical content as Internet use becomes more intense, in face-to-face as well as distance education?" has only recently been posed (Borba, 2004) and as of yet, not even a tentative answer exists. Posing the question in another way, we can think of an example: Does it make sense to spend too much time on techniques of differentiation if we have software that does this rapidly? Is there an equivalent change in content in the case of the Internet?

Another open problem is related to education for teachers who will teach distance education courses. Is it possible to have education for teachers who will teach or participate in distance education courses? In fact, is it possible or desirable to have full distance pre-service education courses? What should be done when participants drop out of courses like this? These are some of the questions which have been addressed in more detail (albeit not answered) by Borba (2004).

Israel (Ruhama Even)

Claim 1 Progress: Conceptual frameworks for mathematical knowledge for teaching are being developed.

A general suggestion for a conceptual framework may be found in Shulman's influential paper (1986) which emphasizes two kinds of understanding of the subject matter that teachers (not necessarily of mathematics) need to have—knowing *that* something is so and knowing *why* it is so. This may seem an almost trivial statement when mathematics knowledge is concerned, although research suggests that quite often teachers know that something in mathematics is so, but not why it is so (Ball, 1990; Even, 1993; Even & Tirosh, 1995.)

The National Council of Teachers of Mathematics (NCTM, 1991) suggests a more detailed mathematical perspective on teacher subject-matter knowledge, stressing that the education of teachers of mathematics should develop their knowledge of the content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality; and, in addition, develop their perspectives on the nature of mathematics, the contributions of different cultures toward the development of mathematics, and the role of mathematics in culture and society; the changes in the nature of mathematics and the way we teach, learn, and do mathematics resulting from the availability of technology; school mathematics within the discipline of mathematics; the changing nature of school mathematics, its relationships to other school subjects, and its applications in society (NCTM, 1991, p. 132).

Focusing on the quality of teacher mathematics knowledge, researchers further emphasize the importance of teacher understanding of the ‘big ideas’ of mathematics, and the connections among and within different ideas, representations and areas of mathematics (Ball, 1991; Even, Tirosh, & Robinson, 1993; Simon, 1993), and of teacher “profound understanding of fundamental mathematics” (Ma, 1999). These qualitative approaches, although some of them are the products of studies that focused on elementary school teachers, are helpful as they acknowledge the complexity of knowing mathematics for teaching and they point at some promising avenues that researchers and teacher educators may explore when designing learning experiences in mathematics for teachers. Still, these approaches do not provide satisfactory answers to questions, such as, what is the meaning of teacher knowledge and understanding about a specific mathematical concept or topic? Is it important that prospective teachers think that the root function is the inverse function of an exponential function, or that they think that the log and the root functions are the same thing? Why is it important, or, Why not? What should a mathematics course for teachers on a specific mathematical topic focus on?

To answer such questions we need a conceptual framework that the mathematics teacher educators could use for the development of mathematics courses for teachers (and the researcher could use to frame studies on teacher subject-matter knowledge of a specific mathematics topic or concept). For this, we draw on a line of research in mathematics education that Dörfler (2003) terms mathematicology–meta-study of mathematics as a human phenomenon and activity. For example, by analyzing mathematical topics from the secondary school curriculum for teaching, using the framework developed by Even (1990). Illustrations of using the framework to analyze the concept of function for teaching (Even, 1990) and the topic of probability for teaching (Kvatinsky & Even, 2002) suggest *what* (but not *how*) needs to be addressed mathematically (e.g., why understanding inverse function is important for secondary school teachers).

Another way to approach the issue of teacher subject-matter knowledge is to adopt a different starting point, as suggested by Ball, Lubienski and Mewborn (2001), and to start with practice in order to uncover knowledge. Ball et al. point out that often teachers do not use what they know, nor does what teachers know fully accommodate the demands of their practice.

Claim 2 Dilemma: What conceptual frameworks for mathematical knowledge for teaching are appropriate?

As of today, there is not enough research in this area. This contributes to the current situation in Israel, where the mathematical preparation of prospective secondary school mathematics teachers is based on traditional advanced mathematics courses with occasionally idiosyncratic innovative courses–the latter dominate enhancement of mathematical knowledge of in-service mathematics teachers.

Norway (Bodil Kleve and Barbara Jaworski)

Claim 1 Progress: The problematic nature of education in mathematics at a variety of levels in Norway is at last being recognized and addressed.

Low competence in mathematics among teachers has been addressed as a possible explanation for low performance among students. However, there have been some positive indications in recent years for practicing teachers as well as pre-service teachers. Now in 2003, teachers who want to take time out to study mathematics can be supported with NOK 100,000 (£10,000). LAMIS, Landslaget for Matematikk i Skolen (the National Society for Mathematics in School) has grown and it receives official support to arrange a conference every summer. There is a nationwide plan for in-service education of teachers (Brekke, et al., 2000) and there have been several collaborating projects in mathematics between teacher educators and teachers in schools. There is an ongoing collaboration project between six colleges of education funded by SOFF, Sentralorganet for Fleksible Læring i Høgre Utdanning (Central Organ for Fexible Learning in Higher Education) where distance-learning or school-based courses in mathematics are offered. In the latter, in-service education takes place as collaboration between teachers and researchers in the classroom.

The project KIM, "Kvalitet i Matematikkundervisningen," (Quality in Mathematics Teaching) was initiated by the Norwegian Ministry of Education in 1993. Like international studies such as TIMSS and PISA, KIM gives us broad information about students' knowledge. Its main focus was to direct teachers' attention to conceptual development in mathematics through materials and guidelines linked to diagnostic testing of students' conceptions.

KIM developed sets of diagnostic test items. The different sets were linked to a specific part of the mathematics curriculum, and thus intended to cover most of the concepts of school mathematics. Choice of items used in the tests was based on research literature, curriculum and textbooks and was made in cooperation with a group of teachers who conducted trials in several rounds. A national standardisation was carried out at two or three grade levels (e.g., 6 and 9) in which written responses were gathered from approximately 2000 students from 100 schools. A survey of students' and teachers' beliefs and attitudes was also conducted. Sadly very few teachers responded to the survey.

Materials produced drew on analyses of the national data obtained from the test items according to identifications of misconceptions and of conceptual obstacles. The associated guidelines for teachers suggest teaching activities designed to create a cognitive conflict for resolution in the classroom. Teachers are encouraged to give students the opportunity to stop and reflect on their actions and experiences in their process of developing a concept. One aim is that students should become aware of their own learning processes. Through use of materials and guidelines, KIM has provided a background for in-service education programs for teachers of mathematics and has become a central focus of pre-service teacher education (Brekke et al, 2000).

Claim 2 Dilemma: Students achievement is lower on national examinations.

Recent research (Alseth et al, 2003) has shown students' performance on national tests to be lower in relation to the L97 curriculum than a similar evaluation of the previous curriculum, M87 (KUF 1987). How should such results be reconciled with materials for teaching development based on students' conceptions and difficulties?

Taiwan (Fou-Lai Lin)

Claim 1 Progress: Multiple didactical views of mathematics are used as content and learning strategies within various teacher professional development programs.

In Taiwan, a generally accepted responsibility of secondary mathematics teachers is helping their students to pass an entrance examination to go on to senior high school at age 14^+ or to the university at age 17^+. The Entrance Examination Center for college organizes workshops to help mathematics teachers develop four types of entrance examination tasks for their students. The four types of tasks developed (conceptual understanding tasks, contextual tasks, argumentation tasks and heuristic tasks) are implemented during workshops. Conceptual understanding tasks assess students' common-sense and intuition of mathematics. Contextual tasks assess mathematics as connections, situational reasoning and modeling. Argumentation tasks assess mathematics as communication and as a deductive system. Heuristic tasks assess comprehension of reading a mathematics text and analogical ability. In addition to the algorithmic nature of the mathematics examinations, these didactic views of mathematics are embedded in the exam tasks and reflect the key nature of mathematics knowledge needed for Taiwan secondary teaching.

Studies on teacher professional development often have designed certain activities as learning strategies for teachers. For instance, analyzing learning cases from practice in which the cases may reveal students' mathematics cognition or affect (Lin, 2003; Leung, 1999; Lee, 2003), developing generic examples, either historical or phenomenological examples (Horng, 2000; Lin, 2000), and communicating the underlying rationale of ones' own teaching to reveal ones' pedagogical values (Chin & Lin, 2000; Leu, 2001). Such activities reflect researchers' didactic views of mathematics.

Claim 2 Dilemma: Didactic views of mathematics besides those of examination mathematics are hardly implemented nationally.

Teaching in Taiwan secondary schools is examination driven. The algorithmic nature of mathematics found in examination mathematics (Lin & Tsao, 2000) is widely adopted by the majority of secondary mathematics teachers in their classrooms.

Cooperating with the Entrance Examination Center to cover different didactic views of mathematics, such as students' conceptions and inappropriate strategy/reasoning and connections, in exam tasks is an effective approach for implementing such views. However, the limitation of examination mathematics, such as a time limit for doing the task, still contradicts some views necessary for teaching well from a teacher's perspective, (e.g., modeling, mathematics with graphic calculators and mathematics investigations). Teachers accept the views of mathematics relevant to the entrance exam very passively. Teachers are waiting for a systematic textbook related to the particular didactic view of mathematics, such as modeling, generic examples, dynamic geometry and so forth. From teachers' professional autonomy point of view, engaging actively in designing learning activities to expand their didactic views of mathematics seems a necessary process for teachers' development.

USA (Helen Doerr)

Claim 1 Progress: The importance of subject matter knowledge is widely agreed upon (CBMS, 2001), despite some claims (Begle, 1979; Darling-Hammond, 2000) that would suggest a ceiling effect beyond which teachers' additional knowledge of mathematics has no added influence on student learning.

Several important areas of secondary teachers' subject matter knowledge have had important beginnings, notably studies on teachers' knowledge of algebra and functions (see Doerr (in press) for an extensive review of this area), but large areas of teacher subject matter knowledge remain relatively unexplored: e.g. statistics, probability, rational numbers, geometry, measurement, and topics in advanced mathematics. For example, several researchers have documented how a limited understanding of the concept of function can restrict the kinds of tasks that teacher choose for students to engage with, the depth of questions that are posed, and the connections that are made within mathematics (Haimes, 1996; Heid, Blume, Zbiek & Edwards, 1998; Wilson, 1994).

Other researchers (Chazan, 1999; Lloyd & Wilson, 1998) have shown how the well-connected content knowledge of the teacher can be used to shift from a procedural approach to a more conceptual approach in the teaching of algebra. Such a conceptual approach emphasized a co-variation as well as a dependence approach to functions, the use of graphs to understand patterns and families of functions, the flexible use of multiple representations and the use of meaningful discussions to support student learning. This line of work suggests that well-connected subject matter knowledge is a necessary condition for expertise in teaching algebra, but such subject matter knowledge is not sufficient for expertise in teaching. In the case of functions, the teachers had transformed their own understanding of the concept into an understanding of the concept *for teaching*, or what Shulman (1986) would call pedagogical content knowledge.

Claim 2 Dilemma: The dilemma facing teacher educators and mathematics education researchers is in understanding how subject matter knowledge becomes transformed into the understanding of the subject that is needed for teaching.

What does such subject matter knowledge look like in practice and how do teachers acquire it? At the secondary level, we are lacking the fine-grained accounts of such mathematical understanding as have been generated around topics in elementary mathematics teaching. See, for example, Ball, Lubienski and Mewborn (2001) for a detailed account of the knowledge needed for teaching multiplication of decimal numbers. Moreover, in elementary mathematics education, the development of teachers' knowledge seems to be enhanced by focusing on their understandings of how students think about various topics and how students' ideas might develop (Fennema et al., 1996). In other words, using students' conceptions is a guiding principle for driving instruction at the elementary school level. Almost no work has been done investigating this same principle at the secondary level. The central question is how would teachers learn to use student thinking in practice?

RESEARCH DESIGN--PROGRESS AND DILEMMAS

Australia (Kaye Stacey)

Claim 1 Progress: We have mastered the art of in-depth case studies and of the careful quantitative analysis of videotapes of randomly selected lessons.

Claim 2 Dilemma: To provide convincing evidence of the nature of knowledge that really makes a difference to secondary mathematics teaching, we need to bridge the gap.

These claims follow from the discussion above about current practice in Australia. I have been impressed by how fully the large-scale TIMSS video studies have been able to describe classroom teaching. Hollingsworth, Lokan and McCrae (2003), for example, give us an unprecedented look at teaching in a random sample of Year 8 (age 13) classrooms, which can be studied from a cross-cultural perspective or as a description against standards. These studies however cannot reveal much about how teachers' knowledge can impact on students' learning, except as noted above occasionally in the negative. Similar studies that look at how the nature of teachers' knowledge impacted on students' learning would need to be designed differently–a topic for discussion.

Brazil (Marcelo Borba)

Claim 1 Progress: Collaborative investigations are the viewed as an effective means to change in schools.

The main consensus related to a research methodology issue is that collaborative research is the way that investigation in this area can lead to change in schools. No one seems to believe that top down models work or that courses for teachers, that take place during vacations or on weekends, are the only way that researchers and teachers should interact. Instead, teachers should collaborate and become researchers in mathematics education research--and this is already happening in Brazil.

Claim 2 Dilemma: Achieving collaboration is not as simple as it looks.

Although there is a consensus that both research agendas and research practice should be developed in a democratic collaboration, there are issues regarding authorship and ethical issues which can make such collaboration a mere formality. For instance, if the researcher is developing a Ph.D. dissertation, even if there is a genuine collaboration of the teacher in the design and development of the research, the authorship of the report and of the analysis belongs to the researcher. Depending on the school and on the content of the research, the teacher may have to suppress his/her name on papers and reports of the problem under scrutiny. Therefore, collaboration is desirable but hard to achieve within the academic and school culture that exists.

Israel (Ruhama Even)

Claim 1 Progress: There is now more appreciation of, and attention to, the complexity of studying the nature of mathematical knowledge for secondary teaching.

Whereas research on student learning has been part of research in mathematics education for almost three decades, reaching a high level of sophistication by means of focus and research design, this has not been the case with research on teachers and teaching. Early Israeli research on teacher mathematical knowledge was mainly evaluative, aiming to measure teachers' knowledge of mathematics.

Data collection for such studies was based mainly on multiple-choice questionnaires, requiring teachers to solve standard mathematics problems. It took time until the mathematics education community began to employ the same level of complexity and depth used in research on students' mathematical knowledge to research on *teachers'* mathematical knowledge. More recent studies on mathematics knowledge for secondary teaching use varied data sources that provide richer information, mainly, open-ended questionnaires and interviews (e.g., Even, 1990, 1998; Hartman, 1997; Leikin, Chazan, & Yerushalmy, 2001; Lipman, 1994; Tsamir, 1999; Shriki & David, 2001; Zaslavsky & Peled, 1994), aiming at better understanding the nature of teachers' mathematical knowledge instead of measuring it. For example, a study that examined the nature of the cognitive processes involved when prospective secondary school teachers work with different representations of functions (Even, 1998) analyzed data from a questionnaire that included non-standard mathematics problems and from an interview that focused on the prospective teachers' explanations of what they had answered on the questionnaire, and why. The results of this study go beyond the conclusion that the prospective teachers had difficulties when needed to flexibly link different representations of functions. Rather, the study illustrates how prospective secondary teachers' knowledge about different representations of functions is not independent, but rather interconnected with knowledge about different approaches to functions, knowledge about the context of the representation and knowledge of underlying notions.

Claim 2 Dilemma: We do not know much about the nature of the interactions between teachers' mathematical knowledge and the practice of secondary mathematics teaching.

How is teachers' mathematical knowledge enacted in the practice of secondary mathematics teaching? The study of the nature of mathematical knowledge for teaching is still often approached cognitively only, and is usually conducted away from the authentic place where this knowledge is enacted, used and constructed—the actual classroom teaching where socio-cultural aspects interact with cognitive ones and where knowledge interacts with practice.

Mathematics teaching relies on deliberate use of knowledge in context. Similar to the dissatisfaction of the mathematics education community for the limited (although important) information obtained from traditional cognitive studies of students' mathematical knowledge and understanding that are conducted outside the classroom, and the consequent expansion of research on students' mathematical knowledge and understanding to classroom studies that incorporate cognitive and socio-cultural aspects (e.g., Hershkowitz & Schwarz, 1999), there is a need to design research studies that focus on studying the interaction of teachers' mathematics knowledge and the practice of (secondary) mathematics teaching; the enactment of mathematical knowledge for secondary teaching in context. This would mean the use of additional data sources, such as, in-class observations and various artifacts (lesson plans, exams, etc.) to be able to answer these new research questions.

Norway (Bodil Kleve and Barbara Jaworski)

Claim 1 Progress: At governmental level, serious recognition of a need to develop research capacity in Norway is resulting in funding being directed at programmes which simultaneously develop research capacity and include teachers in collaborative developmental practices with a research basis.

Under a general title of "Knowledge, Development and Learning," the Norwegian Research Council has granted substantial funding for a four year project in mathematics education. In this programme, didacticians and teachers will work closely to develop 'communities of inquiry' to design classroom activity involving students in inquiry approaches to learning mathematics. Funding includes provision for doctoral stipends so that new researchers can be trained within this programme. Development of inquiry communities draws teachers into design and research activity through which their thinking and teaching develop. Research will be a fundamental basis for development in three ways: 1) Researching activity in workshops in which teachers and didactical work together to explore mathematics, and processes and practices in the learning and teaching of mathematics; 2) Researching teacher group activity in schools in which teachers, with support from their didactician colleagues, design innovative activity for classrooms; 3) Researching teaching of designed innovative activity in classrooms and the associated learning of students. A parallel longitudinal study will explore the status quo of classroom learning and teaching at the beginning and at two further stages within the project.

A new Doctoral Program in Mathematics Education at Agder University College was started in 2002 and given 4 professorships; 8 doctoral students are now registered in the programme and 5 further stipends are advertised. Most of the research generated within this programme involves studies of mathematics learning, teaching and teaching development. Other current moves to build capacity have also been made. A Quality Committee (Kvalitetsutvalget) set up by The Royal Ministry of Educational Affaires, suggests educating resource-teachers in Norwegian, English and Mathematics, and encourages development of master programmes for teachers in the subjects. Several University Colleges in Norway have already responded by developing masters' programmes in Mathematical Education, and are prepared to offer masters studies beginning in 2005.

Claim 2 Dilemma: Despite gaining knowledge through the KIM study about students' learning, and students' conceptions and misconceptions, a recent study (Alseth et al., 2003) shows that students' performance has not improved. Thus we are more aware of the nature of students' knowledge and understanding, but not yet developing this awareness into practices through which learning can be improved.

Instruments and research approaches for studying students' learning, both instrumentally and conceptually, are now well developed in Norway. Despite progress in research-related understandings of students' learning, and opportunities for teachers to be aware of and to use such findings, it appears that recorded learning outcomes are comparatively poor. Thus, research needs to explore relationships between teachers' learning of teaching (both pre-service and in-service) and students' learning of mathematics.

Taiwan (Fou-Lai Lin)

Claim 1 Progress: (Searching for Simplicity) "Making sense of mathematics" as a fundamental view about mathematics teaching has been tested.

Regarding the complexity of mathematics teaching and learning, a simple slogan "teaching for sense making" has been tested within a teacher education program for six years (Lin, 2002). To enhance student's sense making, teaching is encouraged for: developing students' intuition, both first and second order (Fischbein, 1987); situational connection and analogical connection; and assessing students diagnostically.

Being sensitive to the sense students are making about learning content is addressed as the main focus in the teacher education program. Teaching for sense making has been analyzed as a fundamental view about teaching because the teaching strategies have integrated multiple didactic views of mathematics. A group of 30 pre-service teachers have been educated in this program and eight case studies on their teaching in secondary schools were reported as satisfied (cf., Lee & Lin, 2003; Chang & Lin, 2001; Chen & Lin, 2004; Chiang & Lin, 2002).

Claim 2 Dilemma: It is crucial in mathematics teacher education to design a well-tested research program on the development of teachers' multiple didactic views about mathematics that are necessary for teaching well. Such a research program is expected to be able to develop a learning theory for teachers.

Regarding the domination of examination mathematics in a secondary teacher's mind, a well-tested research design that aims to develop teachers' multiple didactic views about mathematics becomes a great challenge. The challenge is not about teachers' understanding but about teachers' constructing of multiple didactic views about mathematics as their beliefs. Taiwan secondary schools might not provide necessary "doubt and evidence," the key elements that changes one's belief, for teachers to change their view with examination mathematics. The expected learning theory derived from such research program might show a strong societal feature.

USA (Helen Doerr)

Claim 1 Progress: A shift in research on teaching over the past 40 years has been from a process-product paradigm towards more naturalistic inquiry into the complexities of teaching practice.

This shift can be described in Schön's (1983) terms as moving from the high ground of technical rationality to the "swampy lowlands" of practice. This has led to a numerous detailed studies on mathematics teaching, especially at the elementary level. This dominance of investigations at the elementary level is reflected in two recent reviews of teacher knowledge by Ball, Lubienski and Mewborn (2001) and Bransford, Brown and Cocking (2000). We do have some studies that are fine-grained analyses of secondary teachers' learning in practice (e.g. Lloyd & Wilson, 1998; Chazan, 1999). We also have a few medium scale studies that give characteristics of the teaching in effective secondary classrooms (e.g., Henningsen, Smith, 1997; Swafford, Jones & Thornton, 1997). However, the methodologies used at the fine-grained level of analysis do not necessarily scale well to medium- or large-scale studies nor are the results of such research easily aggregated across studies. This presents us with several dilemmas.

Claim 2 Dilemma: Understanding the nature of the mathematical knowledge needed for teaching is important at multiple levels of educational practice. However, the design of research studies is plagued by difficult problems of scale, limitations in the usefulness of the forms of results, and challenges in aggregating results across studies.

At the level of policy making and program funding (whether for research, for professional development or for schools), decision makers are confronted with the need to know what is effective and what works in schools under what conditions. Those who design teacher preparation programs and those who certify teachers for jobs in public schools need to know how to make tradeoffs between mathematical coursework and field experiences and how to design courses and experiences that lead to more effective teaching (particularly as measured by student achievement in the current climate). These needs would seem particularly well-served by studies of larger numbers of teachers over a range of conditions (e.g. Schön, Cebulla, Finn & Fi, 2003).

On the other hand, researchers and teacher educators need empirical work based on close observation and embedded in the complexities of practice, attending to the multiple interactions of students, teachers, tasks, curricula, technologies, local school settings, and state policies and mandates. In other words, learning about how teachers learn to teach must be studied in the context of practice. However, such studies are often of the form in which the number of subjects is N=1 or which involve the self-study of teaching, sometimes using a member of the research team. While such studies do provide us with important insights into teacher learning, it remains difficult to scale the methodologies or the results of such studies to larger numbers of teachers.

Another problem of scale can be seen in the dimension of time. It would appear from current research that studies on the development of teachers' knowledge need to be of the order of several years, rather than the several months (or even weeks) that can be sufficient to investigate the conceptual growth of children. The scope of the data collection and analysis are particular problems for research on teacher learning. The potential data sources for understanding teaching are vast, including volumes of student work, reams of observational notes, and boxes of video and audiotape of teaching episodes. Much of this data is not of the form of artifacts or tools that could be used by teachers in the improvement of practice. Much of the resulting analysis is not of the form where findings can be easily aggregated across studies.

SYNTHESIS

During the forum, three matrices will be presented that summarize the claims made above. In the first matrix on the progress and dilemmas in the preparation of teachers, there is consensus among most of the contributors that there has been considerable progress in our understanding that preparation for teaching mathematics is more than knowing advanced mathematics. Although as pointed out by the situation in Norway, knowing mathematics at some level of competence is necessary, but in addition to teach mathematics there is a need for teachers to acquire a 'different' knowledge of mathematics. However, what this mathematical knowledge for teaching is lacks clear definition. In some cases, this knowledge is defined as school mathematics knowledge with specific 'big ideas' such as function, and in others it is seen as distinct from the mathematics of mathematicians. In addition, progress has been made in gaining knowledge of students' conceptions of mathematics, but transposing these conceptions into teaching knowledge is missing. Finally, in preparing teachers mathematically there is still a disconnection between what students experience as mathematics and teaching mathematics in formal mathematics courses and mathematics education courses.

The second matrix addresses claims about progress and dilemmas in terms of mathematical knowledge for practicing teachers and extends the insights drawn from preparing teachers. Here a wider variety of claims and dilemmas exists that extend from acknowledging progress in providing multiple didactic views of mathematics and teachers' understanding in some areas of mathematics to the use of technology as a tool to support practicing teachers. Questions for discussion might include: What is the 'mathematics' that is needed for secondary teaching? How is this 'mathematics' for secondary teaching fundamentally different from the mathematics of advanced courses or mathematicians? What are the elements that define the critical aspects of the mathematical knowledge needed for teaching? How is the question of this forum, what is the nature of the mathematical knowledge that is needed for secondary teaching, connected to the conference theme of diversity and inclusion?

Other discussion questions might be: How can the knowledge of students' mathematical conceptions be didactically transposed in ways that are most useful for teaching? How is teacher's mathematical knowledge transformed when understanding mathematics for secondary teaching?

The third matrix addresses claims about progress and dilemmas in terms of research design. It is clear that qualitative research design and methodology provides valuable insight into mathematics teaching and collaborative research among practitioners, teacher educators and researchers are a means by which to develop not only teaching but research capacity. Yet there is a need for longitudinal studies of mathematical teaching practices, a need to provide evidence for claims of the impact of teacher mathematical knowledge on student learning and a need to define the mathematical knowledge for teaching that can be understood and influence policy at many levels.

Questions for discussion might include: What kinds of research studies might be conducted collaboratively internationally that would address teacher mathematical knowledge in relation to student learning? How can research on teacher knowledge be designed so as to promote the sharing of results in ways that will lead to the development of a knowledge base for teaching? What research designs directly address how changes in teachers' knowledge are generated and sustained beyond the intervention of the research? In other words, what designs enable us to investigate teachers' learning as it occurs and is sustained over time in practice? What kinds of research studies might also be conducted in the same manner that would influence policy on the mathematical knowledge needed for teaching? What research studies might be conducted to address diversity and inclusion?

Participants in this forum are invited to engage in a discussion of these claims, perhaps providing additional supporting or contradictory evidence and additional insights from their particular perspective.

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