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Nonoverlapping Domain Decomposition Methods for Inverse Problems

Karl Kunisch and Xue-Cheng Tai

1 Introduction

Inverse problems related to the estimation of coefficients of partial differential equations are ill-posed. Practical applications often use the fit-to-data output-least-squares method to recover the coefficients. In this work, we develop parallel nonoverlapping domain decomposition algorithms to estimate the diffusion coefficient associated with elliptic differential equations. In order to realize the domain decomposition methods, we combine the function decomposition approach of [Tai95a] and the augmented Lagrangian techniques of [IK90, KT97b]. The output-least-squares method minimizes the output error over the whole domain. When decomposing the domain into nonoverlapping subdomains the output error over the whole domain equals the sum of the output errors in the subdomains. Thus, by borrowing ideas from [Tai95a], parallel methods can be used to find the minimizer. In this approach the partial differential equation arises as a constraint in the optimization problem whose proper treatment is essential. We incorporate it by an augmented Lagrangian technique.

To present the approach we consider the model problem

$$\left\{ \begin{array}{rcll} -\nabla \cdot (q \nabla u) & = & f & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{1.1}$$

where Ω is a domain with boundary $\partial\Omega$, $q \in L^\infty(\Omega)$ and $q \geq \alpha > 0$ and $f \in L^2(\Omega)$ is a given function. The problem consists in estimating the functional parameter q from an observation u_d of the state variable u. The idea that will be described for this model problem can be extended to other parameter estimation problems of partial differential equations.

The fit-to-data formulation for the above estimation problem is given by

$$(\mathcal{P}) \qquad \left\{ \begin{array}{rl} \min & \frac{\beta}{2}|q - q_d|_N^2 + \frac{1}{2}|u - u_d|_{H^1}^2 \\ \text{subject to} & (q, u) \text{ satisfying (1.1)} . \end{array} \right.$$

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Here $|\cdot|_N$ denotes a norm or seminorm on $H^2(\Omega), q_d$ is an initial guess for the parameter and $\beta \geq 0$ stands for the regularization parameter. Thus $(\mathcal{P})$ represents a regularized least-squares formulation with the partial differential equation as a constraint. When decomposing the domain Ω into nonoverlapping subdomains the output error in the whole domain Ω equals the sum of the output errors in the subdomains. Thus, by borrowing ideas from [Tai95a, Tai95b, Tai94], parallel methods can be used to find the minimizer. However, the partial differential equation which arises as a constraint in $(\mathcal{P})$ represents an essential difficulty. This constraint will be incorporated by an augmented Lagrangian technique of [IK90],[KT97b]. Domain decomposition methods for solving the state equation (1.1) for given q and f have been extensively studied. There is a vast literature of which we only mention some relevant classical papers [BPS86], [BW86] and [MQ89].

2 The Domain Decomposition Approach

Through out this work Ω is assumed to be a two-dimensional, bounded, simply connected convex domain with piecewise smooth boundary. We decompose Ω into finitely many nonoverlapping subdomains. The decomposition is carried out in such a way that all subdomains are marked with two colours, say white and black. Subdomains do not intersect each other and the union of their closures equals the closure of Ω. Moreover subdomains with the same colour do not meet each other along edges but rather only at most at one corner. We denote by Ω_1 and Ω_2 the union of the white and black subdomains respectively. Let $\Gamma_1 = \partial\Omega_1$ and $\Gamma_2 = \partial\Omega_2$. Then the interfaces between the subdomains are $\Gamma = \Gamma_1 \cap \Gamma_2 = \Gamma_1 \setminus \partial\Omega = \Gamma_2 \setminus \partial\Omega$. We shall utilise the following notation:

$$
\begin{aligned}
V_i &= \{v | v \in H^1(\Omega_i), \quad v = 0 \text{ on } \Gamma_i \cap \partial\Omega\},\\
W_i &= \{v | v \in H^2(\Omega_i)\},\\
K_i &= \{v | v \in H^2(\Omega_i), \quad v \geq \alpha > 0 \text{ a.e. in } \Omega_i\}, \text{ for } i = 1, 2,
\end{aligned}
$$

$$
\begin{aligned}
X_1 &= W_1 \times W_2 = \{v | v_{|\Omega_i} \in H^2(\Omega_i), \quad i = 1, 2\},\\
X_2 &= V_1 \times V_2 = \{v | v_{|\Omega_i} \in H^1(\Omega_i), \quad i = 1, 2, v = 0 \text{ on } \partial\Omega\},\\
K &= K_1 \times K_2 = \{v | v_{|\Omega_i} \in H^2(\Omega_i), \quad i = 1, 2, v \geq \alpha > 0 \text{ a.e. in } \Omega\},\\
X &= X_1 \times X_2.
\end{aligned}
$$

Except for K_i and K the above sets represent Hilbert spaces with their conventional inner products and norms. Each of the subdomains Ω_1 and Ω_2 may consist of some disconnected components. Thus W_1, for example, can equivalently be expressed as $\prod_{j=1}^m H^2(\Omega_{1j})$, where m denotes the number of disconnected components Ω_{1j} of Ω_1. The functions from X_1 and X_2 have jumps along the interfaces of Ω_1 and Ω_2.

For $x \in X$ we shall use the notation $x = (q_1, q_2, u_1, u_2) = (q, u)$. The function-space formulation of $(\mathcal{P})$ is given by

$$
\text{(P)} \qquad \left\{ \begin{array}{rl} \min & \frac{\beta}{2}|q - q_d|_N^2 + \frac{1}{2}|u - u_d|_{H^1}^2 \\ \text{subject to} & (q, u) \in K \times H_0^1(\Omega) \text{ satisfying (1.1).} \end{array} \right.
$$

Here (q,u) is called solution to (1.1) if

$$(q\nabla u, \nabla v)_\Omega = (f,v) \text{ for all } v \in H_0^1(\Omega).$$

Using the fact that $q \in L^\infty(\Omega)$ provided that $q \in K$, it is simple to argue the existence of a solution $x^* = (q^*, u^*)$ to (P). Next we define the mappings

$$e_1 : X \to V_1, \qquad e_2 : X \to V_2,$$

such that for $x \in X, e_1(x) \in V_1$ and $e_2(x) \in V_2$ are the solutions to the following problems (2.2) and (2.4):

$$\begin{cases} (\nabla e_1, \nabla v_1)_{\Omega_1} + (e_1, v_1)_{\Omega_1} & = (q_1 \nabla u_1, \nabla v_1)_{\Omega_1} - (f, v_1)_{\Omega_1} \\ & \quad \text{for all } v_1 \in H_0^1(\Omega_1) \ , \\ & e_1 = u_1 - u_2 \text{ on } \Gamma \ , \\ & e_1 = 0 \text{ on } \partial\Omega \cap \Gamma_1 \ . \end{cases} \tag{2.2}$$

For any $v \in V_2$, let Rv be an extension to Ω_1 satisfying $Rv = 0$ on $\partial\Omega \cap \Gamma_1, Rv = v$ on Γ, and

$$\|Rv\|_{H^1(\Omega_1)} \le C_1 \|v\|_{H^1(\Omega_2)}, \tag{2.3}$$

with constant C_1 independent of $v \in V_2$. Note that the harmonic extension operator R_H defined by $R_H v = 0$ on $\partial\Omega \cap \Gamma_1$, $R_H v = v$ on Γ and

$$(\nabla R_H v, \nabla \phi)_{\Omega_1} = 0 \ , \quad \forall \phi \in H_0^1(\Omega_1)$$

satisfies the required properties. For numerical purposes we prefer to use a different extension which will be introduced in section 5. With R thus defined, let $e_2 \in V_2$ be the solution to

$$\begin{cases} (\nabla e_2, \nabla v_2)_{\Omega_2} + (e_2, v_2)_{\Omega_2} = (q_2 \nabla u_2, \nabla v_2)_{\Omega_2} - (f, v_2)_{\Omega_2} \\ \quad + \langle u_2 - u_1, v \rangle_{\partial\Omega_2} + (q_1 \nabla u_1, \nabla R v_2)_{\Omega_1} - (f, R v_2)_{\Omega_1} \text{ for all } v \in V_2, \\ e_2 = 0 \text{ on } \partial\Omega \cap \Gamma_2 \ . \end{cases} \tag{2.4}$$

With e_1 and e_2 defined we introduce $e : X \to X_2$ by

$$e(x) = (e_1(x), e_2(x)).$$

We choose $|\cdot|_N$ in $(\mathcal{P})$ to be the piecewise $H^2(\Omega)$ norm on Ω_1 and Ω_2 and define

$$\begin{aligned} J_1(x) &= \frac{\beta}{2}\|q_1 - q_d\|^2_{H^2(\Omega_1)} + \frac{1}{2}\|u_1 - u_d\|^2_{H^1(\Omega_1)} \\ J_2(x) &= \frac{\beta}{2}\|q_2 - q_d\|^2_{H^2(\Omega_2)} + \frac{1}{2}\|u_2 - u_d\|^2_{H^1(\Omega_2)}, \\ J(x) &= J_1(x) + J_2(x). \end{aligned}$$

We shall focus on the minimization problem

$$(PP) \qquad \min_{(q,u)\in K\times X_2,\ (q,u)=0.} J(x)$$

Its relation to (P) is established in the following lemma: (see [KT97a])

Lemma 2.1 *$x^* = (q^*, u^*)$ is a minimizer of (PP) if and only if it is a minimizer of (P).*

3 The Augmented Lagrangian Method

In this section we develop parallel nonoverlapping domain decomposition algorithms for (PP). In [Tai95a, Tai95b] the following problem was considered:

$$\min_{x \in K} \sum_{i=1}^{m} F_i(x), \qquad K \subset V.$$

Under the assumption that K is convex and closed in the Hilbert space V and that the functions F_i are convex or uniformly convex several parallel algorithms based on the augmented Lagrangian method were obtained. Problem (PP) does not fit into this class of problems since the constraints of (PP) are not convex. We shall therefore combine the ideas from [Tai95a, Tai95b] and the techniques from [IK90], [KT97b] to overcome this difficulty.

Let us review some of the results of [KT97b] and consider

$$\min_{e(q,u)=0,\ (q,u) \in K \times X_2} \frac{\beta}{2}\|q-q_d\|^2_{X_1} + \frac{1}{2}\|u-u_d\|^2_{X_2} \ . \tag{3.5}$$

In [KT97b], the constraint e is assumed to have the special structure

$$e(q,u) = b(q,u) + l_1(q) + l_2(u) + \tilde{f},$$

with $b : X \to Y$, $\quad l_i \in \mathcal{L}(X_i, Y)$, $\quad \tilde{f} \in Y$, $\quad (q_d, u_d) \in X$, $\quad X = X_1 \times X_2$, and b a bounded bilinear form satisfying

$$\|b(q,u)\|_Y \leq \|b\| \ \|q\|_{X_1} \ \|u\|_{X_2} \text{ for all } (q,u) \in X.$$

Here X_1, X_2 and Y are real Hilbert spaces. In the context of Section 2 the spaces X_i assume the specific meaning explained there and $Y = X_2$. For any $q \in X_1$ let $A_q \in \mathcal{L}(X_2, Y)$ denote the operator defined by

$$A_q v = b(q,v) + l_2(v).$$

Under the conditions

(H1) e is continuous from the weak topology on X to the weak topology on Y, (which guarantees the existence of a solution $x^* = (q^*, u^*)$ to (3.5)),
(H2) A_{q^*} is a homeomorphism from X_2 to Y,
(H3) $\|u^* - u_d\|_{X_2} \|(A^*_{q^*})^{-1}\|_{\mathcal{L}(Y,X_2)} \ \|b\| \leq \sqrt{\beta}$,

the solution x^* to (3.5) is unique and the following

Algorithm 1
Step 1. *Choose* $\lambda_0, c > 0, \sigma \in (0, c]$. *For* $n = 1, 2, \ldots$ *do:*
Step 2. *Determine* x^n *as the solution to*

$$(P_{aux}) \qquad \min L_c(x, \lambda^{n-1}) \quad \textit{over } x \in K \times X_2.$$

Step 3. *set* $\lambda^n = \lambda^{n-1} + \sigma e(x^n)$, produces a sequence $\{x^n\}$ such that $\lim_{n\to\infty} x^n = x^*$. If $q^* \in \text{int } K$ and σ is chosen appropriately then x^n converges linearly in X to x^*. Moreover λ^n converges weekly in Y to λ^*, where λ^* is the Lagrange multiplier associated to the constraint $e(x) = 0$. In the above algorithm $L_c(x, \lambda)$ is the augmented Lagrangian functional

$$L_c(x,\lambda) = \frac{\beta}{2}\|q - q_d\|_{X_1}^2 + \frac{1}{2}\|u - u_d\|_{X_2}^2 + (\lambda, e(x))_Y + \frac{c}{2}\|e(x)\|_Y^2.$$

In [KT97a], it was shown that (PP) fits into the general framework of problem (3.5) and that (H1) - (H3) are satisfied.

4 Nonoverlapping Domain Decomposition for (P_{aux})

In this section we describe a Gauss-Seidel iteration to solve (P_{aux}). We utilise the decomposition of Ω into M white and M black subdomains as described at the beginning of Section 2 and use M parallel processors. Each processor takes care of a white and a neighbouring black subdomain.

Let us recall the augmented Lagrangian functional that appears as the cost functional in (P_{aux}):

$$\begin{aligned} L_c(x,\lambda) = {} & J_1(x) + J_2(x) + (\lambda_1, e_1(x))_{V_1} + (\lambda_2, e_2(x))_{V_2} \\ & + \tfrac{c}{2}\|e_1(x)\|_{V_1}^2 + \tfrac{c}{2}\|e_2(x)\|_{V_2}^2, \end{aligned}$$

and $x = (q_1, q_2, u_1, u_2) = (q, u) \in X$. We note that J_i is only a function of $x_i = (q_i, u_i), i = 1, 2$. The coupling between x_1 and x_2 occurs through the boundary constraints described by e_1 and e_2. In the Gauss-Seidel algorithm $L_c(x, \lambda)$ is minimized with respect to x in the following order: $q_1 \to u_1 \to q_2 \to u_2$. The algorithm is:

Algorithm 2

(i) *Choose* $u^{n,0} \in X_2, q_2^{n,0} \in W_2$ *if* $n = 1$, *else set* $u^{n,0} = u^{n-1}, q_2^{n,0} = q_2^{n-1}$.
(ii) *For* $k = 1, 2\ldots$ *do: find* $q_i^{n,k}, u_i^{n,k}$ *such that*
a) $L_c(q_1^{n,k}, u_1^{n,k-1}, q_2^{n,k-1}, u_2^{n,k-1}, \lambda^{n-1}) \le L_c(q_1, u_1^{n,k-1}, q_2^{n,k-1}, u_2^{n,k-1}, \lambda^{n-1})$, *for all* $q_1 \in K_1$.
b) $L_c(q_1^{n,k}, u_1^{n,k}, q_2^{n,k-1}, u_2^{n,k-1}, \lambda^{n-1}) \le L_c(q_1^{n,k}, u_1, q_2^{n,k-1}, u_2^{n,k-1}, \lambda^{n-1})$, *for all* $u_1 \in V_1$.
c) $L_c(q_1^{n,k}, u_1^{n,k}, q_2^{n,k}, u_2^{n,k-1}, \lambda^{n-1}) \le L_c(q_1^{n,k}, u_1^{n,k}, q_2, u_2^{n,k-1}, \lambda^{n-1})$, *for all* $q_2 \in K_2$.
d) $L_c(q_1^{n,k}, u_1^{n,k}, q_2^{n,k}, u_2^{n,k}, \lambda^{n-1}) \le L_c(q_1^{n,k}, u_1^{n,k}, q_2^{n,k}, u_2, \lambda^{n-1})$, *for all* $u_2 \in V_2$.

The sequences $\{u_i^{n,k}\}$ and $\{q_i^{n,k}\}$ converge to the solution of (P_{aux}), see [KT97a].

5 Numerical Tests.

Experiments for one and two dimensional problems were carried out. Uniform triangular mesh and linear finite element functions were used for 2D approximations.

Table 1 The numerical errors of the first 10 steps.

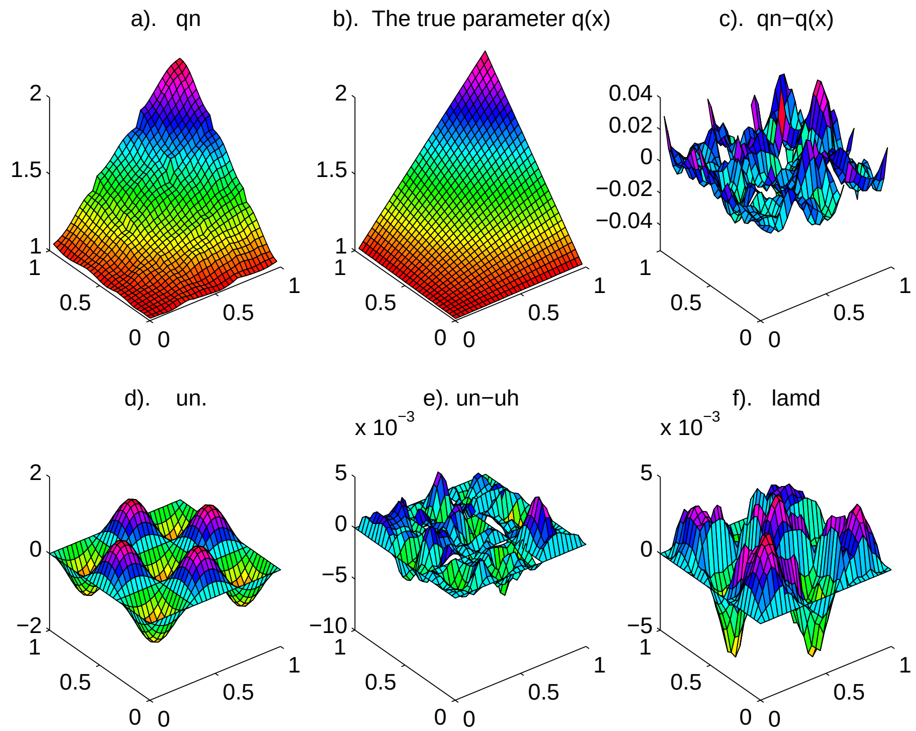
Iteration	err_q^*	err_q
1	0.03938	0.3363
2	0.02473	0.01972
3	0.0191	0.007926
4	0.01607	0.004409
5	0.01418	0.002879
6	0.01292	0.002071
7	0.01204	0.001589
8	0.01142	0.001273
9	0.01096	0.001055
10	0.01064	0.0008957

For a given q^* and f, the observation u_d is obtained by adding uniformly distributed random numbers in $[-\delta, \delta]$ to the finite element solution at the nodal points. In the simulations, Algorithm 2 is applied to find the minimizer of (P_{aux}) and only 3 iterations are performed between the subproblems a)–b)–c)–d) in our computations. A very simple extension operator is used in the computations by choosing $Rv = v$ in Ω_2 and $Rv(x_i) = 0$ if x_i is an inner node of Ω_1. Condition (2.3) is fulfilled with a constant C_1 depending on the mesh size h. Our one dimensional tests show that the convergence rate does not depend on the size of the extension. This may be due to the fact that the most important source for inaccurate reconstruction of q is the ill-posedness of the estimation problem.

From our numerical experiments, we find that there are two issues that require special care in using Algorithm 2. First, it must be guaranteed that the subproblems a) and c) of Algorithm 2 are identifiable when decomposing the domain. Sufficient conditions for the identifiability can be found in [IK94]. Second, identifying q_2 from c) of Algorithm 2 is very sensitive to observation errors and the errors caused by the initial values. One way to overcome such a problem is to choose the boundary conditions according to the flow directions. Dirichlet boundary conditions shall be used in the outflow boundaries and Neumann boundary conditions need to be used on the inflow boundaries. This is beyond the scope of the present work and shall be reported in detail in [KT97a].

Figure 1 depicts a typical numerical result. We identify $q(x,y) = e^{xy}$ from $u(x,y) = sin(3\pi x)sin(3\pi y)$. The domain $\Omega = (0,1)\times(0,1)$ is divided into $3\times 3 = 9$ subdomains. Dirichlet boundary conditions are used for the subdomains at the 4 corners and the one in the middle. Extension operators are used for the other subdomains. Observation error is added with $\delta = 0.01$. In the computations, we use mesh size $h = 1/30$, $c = 100$, $\sigma = 100$, and $\beta = 0.1$. The convergence for the first 10 iterations are shown in Table 1. In the table, $err_q^* = \|q_1^n - q^*\|_{L^2(\Omega_1)} + \|q_2^n - q^*\|_{L^2(\Omega_2)}$ and $err_q = \|q_1^n - q_1^{n-1}\|_{L^2(\Omega_1)} + \|q_2^n - q_2^{n-1}\|_{L^2(\Omega_2)}$.

Figure 1 The identified parameter by domain decomposition.



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