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Contact Theory for Spacelike Hypersurfaces with Constant Mean Curvature in the Minkowski Space[1]

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Abstract. In this paper we introduce a variational problem for spacelike hypersurfaces in the Minkowski space whose critical points turn out to be constant mean curvature spacelike hypersurfaces which intersect a given hypersurface in a non-transverse way. In the case where the given hypersurface is totally umbilical we prove the non-existence of non-trivial "contact" hypersurfaces.

1. Introduction

In this paper we study spacelike hypersurfaces with constant mean curvature in the Minkowski space. The importance of this kind of hypersurfaces in Lorentzian spaces is well-known, not only from a mathematical point of view, but also from a physical one. For instance, Lichnerowicz proved that maximal hypersurfaces are convenient as initial data for the Cauchy problem of the Einstein equations (Lichnerowicz 1944), while non-zero constant mean curvature spacelike hypersurfaces are used in the study of the propagation of gravitational waves. There are other reasons justifying their importance in relativity and cosmology (see Marsden and Tipler 1980, Stumbles 1980 and references therein).

From a strictly mathematical point of view, constant mean curvature spacelike hypersurfaces are solutions to a variational problem. In fact, they are critical points of the area functional for variations that leave constant a certain volume function (see, for example, Barbosa and Oliker 1993 a,b).

[1]This is an original research article and no version has been submitted for publication elsewhere.

In this paper we introduce a variational problem for spacelike hypersurfaces in the Minkowski space $\mathbf{L}^{n+1}$ (see Section 3). Let us consider $\overline{\Sigma} \subset \mathbf{L}^{n+1}$ an embedded (connected) spacelike hypersurface. Then our objective consists of studying the critical points of a certain energy functional, among all compact spacelike hypersurfaces Σ immersed in $\mathbf{L}^{n+1}$ whose boundary is contained in $\overline{\Sigma}$. The energy functional is given by $E = A - \overline{A}$, where A is the n-dimensional area of Σ and $\overline{A}$ is the n-dimensional area of the compact domain in $\overline{\Sigma}$ bounded by the boundary of Σ. Under these constraints, a hypersurface is called stationary if it has critical energy. The corresponding Euler-Lagrange equation shows that a stationary hypersurface is a constant mean curvature hypersurface and has a contact of order one with $\overline{\Sigma}$ along the common boundary (see the precise definitions in Section 2).

When we consider particular cases for the hypersurface $\overline{\Sigma}$ we are able to get some uniqueness results. For instance, if $\overline{\Sigma}$ is a spacelike hyperplane we prove that:

There exists no non-trivial constant mean curvature spacelike hypersurface that has a contact of order one with a spacelike hyperplane.

On the other hand, if we suppose that $\overline{\Sigma}$ is a hyperbolic space we prove again that:

There exists no non-trivial constant mean curvature spacelike hypersurface that has a contact of order one with a hyperbolic space.

Our approach is based on two integral formulas which allow us to detect when a given hypersurface is totally umbilical. One of these formulas was already obtained in (Alías and Pastor 1998) in order to get a similar uniqueness result for the case of prescribed boundary.

2. Preliminaries

Let us denote by $\mathbf{L}^{n+1}$ the Minkowski space, that is, the vector space $\mathbf{R}^{n+1}$ endowed with the indefinite metric

$$\langle , \rangle = \sum_{i=1}^{n} \mathrm{d}x_i^2 - \mathrm{d}x_{n+1}^2$$

where $(x_1, x_2, ..., x_{n+1})$ are the canonical coordinates in $\mathbf{R}^{n+1}$. A smooth immersion $\psi : \Sigma \longrightarrow \mathbf{L}^{n+1}$ of an n-dimensional connected manifold Σ is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ, which, as usual, is also denoted by $\langle , \rangle$. Now, observe that $(0, ..., 0, 1)$ is a unit timelike vector field globally defined on $\mathbf{L}^{n+1}$ which determines a time orientation on $\mathbf{L}^{n+1}$. Therefore, we can choose a unique unit normal vector field N on Σ which is a future-directed timelike vector in $\mathbf{L}^{n+1}$ and, hence, we may assume that Σ is oriented by N.

In order to set up the notation to be used later, let us denote by ∇^0 and ∇ the Levi-Civita connections of $\mathbf{L}^{n+1}$ and Σ, respectively. Then the Gauss and Weingarten formulas for Σ in $\mathbf{L}^{n+1}$ are written as

$$\nabla^0_X Y = \nabla_X Y - \langle AX, Y \rangle N,$$

and

$$A(X) = -\nabla^0_X N,$$

where X, Y are tangent vector fields and A stands for the shape operator associated to N. The mean curvature function of the hypersurface Σ is defined by $H = -\frac{1}{n}\mathrm{tr}(A)$.

In this paper we are going to deal with *compact* spacelike hypersurfaces immersed in $\mathbf{L}^{n+1}$. Let us remark that there exists no closed spacelike hypersurface in $\mathbf{L}^{n+1}$. To prove this, let us consider $a \in \mathbf{L}^{n+1}$ a fixed vector and define the height function $\langle a, x\rangle$ on Σ. The gradient of $\langle a, x\rangle$ is given by

$$\nabla\langle a, x\rangle = a^{\top} = a + \langle a, N\rangle N,$$

where $a^{\top}$ is tangent to the hypersurface Σ. Computing now the squared norm we get

$$|\nabla\langle a, x\rangle|^2 = \langle a, a\rangle + \langle a, N\rangle^2 \geq \langle a, a\rangle.$$

In particular, if we choose a a spacelike vector the height function has no critical points in Σ, hence Σ cannot be closed and, therefore, every compact spacelike hypersurface Σ necessarily has non-empty boundary $\partial\Sigma$. As usual, if Γ is a $(n-1)$-dimensional closed submanifold in $\mathbf{L}^{n+1}$, a spacelike hypersurface $\psi : \Sigma \longrightarrow \mathbf{L}^{n+1}$ is said to be a hypersurface *with boundary* Γ if $\psi_{|\partial\Sigma} : \partial\Sigma \longrightarrow \Gamma$ is a diffeomorphism.

Now, let us consider $\overline{\Sigma}$ an embedded connected spacelike hypersurface in $\mathbf{L}^{n+1}$ and we will assume that $\overline{\Sigma}$ is oriented by $\overline{N}$, the unique future-directed unit timelike vector field normal to $\overline{\Sigma}$. Moreover, let us suppose that the boundary Γ is contained in $\overline{\Sigma}$ and bounds a compact n-dimensional domain $\overline{\Omega}$ in $\overline{\Sigma}$. We will denote by ν (resp. $\overline{\nu}$) the *outward pointing* unit conormal vector field to Σ (resp. $\overline{\Omega}$).

In this context, the two hypersurfaces Σ and $\overline{\Sigma}$ are said to have a *contact of order one* if the normal vector fields N and $\overline{N}$ coincide along Γ or, equivalently, if $\nu = \overline{\nu}$ along this common submanifold.

3. The variational problem

Given $\psi : \Sigma \longrightarrow \mathbf{L}^{n+1}$ a smooth spacelike immersion satisfying $\psi(\partial\Sigma) \subset \overline{\Sigma}$, an *admissible variation* of ψ will be a smooth map $\Phi : (-\varepsilon, \varepsilon) \times \Sigma \longrightarrow \mathbf{L}^{n+1}$ such that for each $t \in (-\varepsilon, \varepsilon)$ the map $\Phi_t : \Sigma \longrightarrow \mathbf{L}^{n+1}$ defined by $\Phi_t(p) = \Phi(t, p)$ is a spacelike immersion with

$$\Phi_t(\partial\Sigma) \subset \overline{\Sigma}$$

and

$$\Phi_0 = \psi.$$

For each admissible variation Φ, we may define its *energy function* $E : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}$ as

$$E(t) = A(t) - \overline{A}(t),$$

where

$$A(t) = \text{area}\,(\Sigma, \Phi_t) = \int_{\Sigma} \mathrm{d}A_t$$

is the n-dimensional area of Σ in the metric induced by the immersion Φ_t and

$$\overline{A}(t) = \text{area}(\overline{\Omega}_t) = \int_{\Omega_t} \mathrm{d}\overline{A}$$

is the n-dimensional area of $\overline{\Omega}_t \subset \overline{\Sigma}$, the domain in $\overline{\Sigma}$ bounded by $\Gamma_t = \Phi_t(\partial\Sigma)$. Here $\mathrm{d}\overline{A}$ denotes the area element of $\overline{\Sigma}$ with respect to the induced metric and the chosen orientation whereas $\mathrm{d}A_t$ denotes the area element of Σ with respect to the metric induced by Φ_t and the orientation given by the future-directed unit timelike vector field normal to Φ_t.

The *volume function* of the variation $V : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}$ is given by

$$V(t) = \int_{[0,t]\times\Sigma} \Phi^*(\mathrm{d}V)$$

where $\mathrm{d}V$ is the canonical volume element of $\mathbf{L}^{n+1}$. Notice that $V(t)$ represents the volume enclosed between the hypersurfaces $\Phi_0 = \psi$ and Φ_t. The variation is said to be *volume-preserving* if $V(t) = V(0) = 0$ for all t.

Let Y denote the variation vector field of Φ, which is given by

$$Y(p) = \frac{\partial\Phi}{\partial t}(0, p),$$

and along the immersion $\psi : \Sigma \longrightarrow \mathbf{L}^{n+1}$ write

$$Y = Y^\top - \langle Y, N\rangle N$$

where $Y^\top$ is tangent to Σ. It then follows from the first variation formula for the area (see, for instance, Brill and Flaherty 1976, Frankel 1975) that

$$\delta_Y A = A'(0) = -n \int_\Sigma H\langle Y, N\rangle \mathrm{d}A + \oint_{\partial\Sigma} \langle Y, \nu\rangle \mathrm{d}s$$

and

$$\delta_Y \overline{A} = \overline{A}'(0) = \oint_{\partial\Sigma} \langle Y, \overline{\nu}\rangle \mathrm{d}s,$$

where $\mathrm{d}s$ is the induced line element on the boundary $\partial\Sigma$. Hence, the first variation formula for the energy function is given by

$$\begin{aligned}\delta_Y E = E'(0) &= -n \int_\Sigma H\langle Y, N\rangle \mathrm{d}A + \oint_{\partial\Sigma} \left(\langle Y, \nu\rangle - \langle Y, \overline{\nu}\rangle\right) \mathrm{d}s = \\ &= -n \int_\Sigma H\langle Y, N\rangle \mathrm{d}A - \oint_{\partial\Sigma} \langle Y, \overline{\nu}\rangle \left(1 + \langle N, \overline{N}\rangle\right) \mathrm{d}s.\end{aligned}$$

On the other hand, the well-known first variation formula for the volume is given by

$$\delta_Y V = V'(0) = -\int_\Sigma \langle Y, N\rangle dA.$$

We will say that the immersion x is *stationary* if $\delta_Y E = 0$ for every admissible volume-preserving variation of x. The following characterization of stationary hypersurfaces can be easily deduced from the previous formulae.

Proposition 1. *Let $\overline{\Sigma}$ be an embedded connected spacelike hypersurface and let $\psi : \Sigma \longrightarrow \mathbf{L}^{n+1}$ be a compact spacelike immersion such that $\psi(\partial\Sigma) = \Gamma \subset \overline{\Sigma}$ is a closed submanifold contained in $\overline{\Sigma}$ which bounds a compact domain. Then ψ is stationary if and only if its mean curvature H is constant and $N = \overline{N}$ along Γ, that is, the two hypersurfaces Σ and $\overline{\Sigma}$ have a contact of order one along Γ.*

4. Some uniqueness results

In this section we are going to consider some particular cases for the hypersurface $\overline{\Sigma}$. Then, if $\overline{\Sigma}$ is a spacelike hyperplane we are able to give the following non-existence result:

Theorem 1. *There exists no non-trivial constant mean curvature spacelike hypersurface in* $\mathbf{L}^{n+1}$ *that has a contact of order one with a spacelike hyperplane along a common boundary.*

Proof. Let us suppose that $\overline{\Sigma}$ is a spacelike hyperplane given by

$$\overline{\Sigma} = a^{\perp},$$

where a is a timelike unit future-directed vector. Let us consider now the height function $\langle a, x\rangle$ defined on the hypersurface Σ. The Laplacian of $\langle a, x\rangle$ with respect to the metric induced on Σ is

$$\Delta\langle a, x\rangle = -\langle a, N\rangle \mathrm{tr}\,(A) = nH\langle a, N\rangle. \tag{4.1}$$

Integrating now (4.1) we have by the divergence theorem that

$$\oint_{\partial\Sigma} \langle a, \nu\rangle \mathrm{d}s = n\int_{\Sigma} H\langle a, N\rangle \mathrm{d}A. \tag{4.2}$$

If we consider now the function $\langle a, N\rangle$ a direct computation (see, for instance, Alías and Pastor 1998) gives us for the gradient

$$\nabla\langle a, N\rangle = -A\left(a^{\top}\right),$$

whereas for the Laplacian,

$$\Delta\langle a, N\rangle = n\langle a, \nabla H\rangle + \langle a, N\rangle \mathrm{tr}\left(A^2\right). \tag{4.3}$$

When H is constant, from (4.1) and (4.3), we obtain that

$$\Delta\left(H\langle a, x\rangle - \langle a, N\rangle\right) = -\langle a, N\rangle\left(\mathrm{tr}\left(A^2\right) - nH^2\right)$$

and integrating in this expression we get

$$\oint_{\partial\Sigma}\left(\langle A(\nu), \nu\rangle + H\right)\langle \nu, a\rangle \mathrm{d}s = -\int_{\Sigma}\langle a, N\rangle\left(\mathrm{tr}\left(A^2\right) - nH^2\right)\mathrm{d}A.$$

Now, since $\overline{\Sigma} = a^{\perp}$ and we are supposing that both hypersurfaces have a contact of order one we get $\overline{N} = a$, $\nu = \overline{\nu}$ and then $\langle \nu, a\rangle = \langle \overline{\nu}, \overline{N}\rangle = 0$. So,

$$\int_{\Sigma}\langle a, N\rangle\left(\mathrm{tr}\left(A^2\right) - nH^2\right)\mathrm{d}A = 0$$

and Σ is a totally umbilical hypersurface. On the other hand, from (4.2) we get

$$\int_{\Sigma} H\langle a, N\rangle \mathrm{d}A = 0$$

and since $\langle a, N\rangle = \langle\overline{N}, N\rangle \leq -1$ we can conclude that $H = 0$. Therefore, Σ is totally geodesis and $\Sigma = \overline{\Sigma}$. □

Using a similar procedure as in the previous proof, we may obtain another non-existence theorem supposing that $\overline{\Sigma}$ is a hyperbolic space.

Theorem 2. *There exists no non-trivial constant mean curvature spacelike hypersurface in* $\mathbf{L}^{n+1}$ *that has a contact of order one with a hyperbolic space along a common boundary.*

Proof. Now, we are assuming that $\overline{\Sigma}$ is a hyperbolic space of radius $r > 0$ which is given by

$$\overline{\Sigma} = \left\{q \in \mathbf{L}^{n+1} : \langle q, q\rangle = -r^2, q_{n+1} > 0\right\}$$

with constant mean curvature $\overline{H} = 1/r$. Let us consider now the function $\langle x, x\rangle$ defined on the hypersurface Σ. The gradient of $\langle x, x\rangle$ with respect to the metric induced in Σ is

$$\nabla\langle x, x\rangle = 2x^{\top} = 2\left(x + \langle x, N\rangle N\right).$$

We need to make some computations in order to get the Laplacian of this function. Taking X a vector field defined on Σ we have

$$X = \nabla^0_X(x) = \nabla^0_X\left(x^{\top}\right) - \nabla^0_X\left(\langle x, N\rangle N\right) =$$

$$= \nabla_X\left(x^{\top}\right) + \sigma\left(X, x^{\top}\right) - X\left(\langle x, N\rangle\right)N - \langle x, N\rangle\nabla^0_X(N) =$$

$$= \nabla_X\left(x^{\top}\right) - \langle x, N\rangle\nabla^0_X(N)$$

and then

$$\nabla_X\left(x^{\top}\right) = X - \langle x, N\rangle A(X).$$

Therefore, if $\{e_1, e_2, ..., e_n\}$ is an orthonormal local frame in Σ the Laplacian is given by

$$\Delta\langle x, x\rangle = 2\sum_{i=1}^{n}\langle\nabla_{e_i}\left(x^{\top}\right), e_i\rangle =$$

$$= 2\sum_{i=1}^{n}\langle e_i - \langle x, N\rangle A(e_i), e_i\rangle = 2n\left(1 + \langle x, N\rangle H\right). \tag{4.4}$$

On the other hand, considering now the function $\langle x, N\rangle$ we get for the gradient that

$$\nabla\langle x, N\rangle = -A\left(x^{\top}\right)$$

whereas for the Laplacian we need to make, as in the previous case, some easy computations. If we take X a vector field on Σ then we have

$$\nabla_X\left(A\left(x^{\top}\right)\right) = (\nabla A)\left(x^{\top}, X\right) + A\left(\nabla_X\left(x^{\top}\right)\right),$$

hence,

$$\Delta\langle x, N\rangle = -\sum_{i=1}^{n}\langle\nabla_{e_i}\left(A\left(x^{\top}\right)\right), e_i\rangle =$$

$$= -\sum_{i=1}^{n} \left(\langle (\nabla A) \left(x^{\top}, e_i \right), e_i \rangle + \langle A \left(\nabla_{e_i} \left(x^{\top} \right) \right), e_i \rangle \right).$$

Applying the Codazzi equation to the first member of this expression we may write

$$\nabla_X \left(A \left(x^{\top} \right) \right) = -\sum_{i=1}^{n} \left(\langle (\nabla A) \left(e_i, x^{\top} \right), e_i \rangle + \langle A \left(\nabla_{e_i} \left(x^{\top} \right) \right), e_i \rangle \right) =$$

$$= -\mathrm{tr} \left(\nabla_{x^{\top}} A \right) - \sum_{i=1}^{n} \langle \nabla_{e_i} \left(x^{\top} \right), A(e_i) \rangle =$$

$$= -x^{\top} \left(\mathrm{tr} \left(A \right) \right) - \sum_{i=1}^{n} \langle e_i - \langle x, N \rangle A(e_i), A(e_i) \rangle =$$

$$= n \langle \nabla H, x \rangle + nH + \langle x, N \rangle \mathrm{tr} \left(A^2 \right). \tag{4.5}$$

When H is constant we get from (4.4) and (4.5) ,

$$\Delta \left(H \langle x, x \rangle - \langle x, N \rangle \right) = -\langle x, N \rangle \left(\mathrm{tr} \left(A^2 \right) - nH^2 \right)$$

and applying the divergence theorem to this equality we have

$$\oint_{\partial \Sigma} \langle x, \nu \rangle \left(\langle A \left(\nu \right), \nu \rangle + H \right) \mathrm{d}s = -\int_{\Sigma} \langle x, N \rangle \left(\mathrm{tr} \left(A^2 \right) - nH^2 \right) \mathrm{d}A. \tag{4.6}$$

The normal vector field to $\overline{\Sigma}$ is given by $\overline{N} = \frac{1}{r} x$ and, since we are assuming that, along the boundary Γ, exists a contact of order one, we have that $N = \overline{N} = \frac{1}{r} x$ on the boundary and (4.6) becomes

$$\int_{\Sigma} r \left(\mathrm{tr} \left(A^2 \right) - nH^2 \right) \mathrm{d}A = 0,$$

hence, the hypersurface Σ is totally umbilical. Immediately, we can suppose that Σ is a hyperbolic space with radius r' and then,

$$N = \frac{1}{r'} x \quad \text{whereas } H = \frac{1}{r'}.$$

Since on the boundary we have $N = \overline{N}$ then $r = r'$ and the two hypersurfaces coincide. □

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