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GENERALIZED ORTHOGONAL PROJECTIONS AND SHORTED OPERATORS

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Dedicated to the memory of our friend Chicho Guadalupe

Abstract. Let $\mathcal{H}$ be a Hilbert space, $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and $\langle\ ,\ \rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ the bounded sesquilinear form induced by a selfadjoint $A \in L(\mathcal{H})$, $\langle \xi, \eta\rangle_A = \langle A\xi, \eta\rangle$, $\xi, \eta \in \mathcal{H}$. Given $T \in L(\mathcal{H})$, T is A-selfadjoint if $AT = T^*A$. If $\mathcal{S} \subseteq \mathcal{H}$ is a closed subspace, we study the set of A-selfadjoint projections onto $\mathcal{S}$,

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q\ ,\ R(Q) = \mathcal{S}\ ,\ AQ = Q^*A\}$$

for different choices of A, mainly under the hypothesis that $A \geq 0$. In this paper we study the close relationship between the existence and properties of A-selfadjoint projections onto $\mathcal{S}$ and the shorted operator (also called Schur complement) $A_{/\mathcal{S}}$ of A to $\mathcal{S}$ and the $\mathcal{S}$-compression $A_{\mathcal{S}} = A - A_{/\mathcal{S}}$.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and A a bounded linear positive (semidefinite) operator on $\mathcal{H}$. The pair $(A, \mathcal{S})$ is said to be *compatible* if there exists a bounded linear (not necessarily selfadjoint) projection Q which maps $\mathcal{H}$ onto $\mathcal{S}$ such that AQ is selfadjoint. Thus, if

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q,\ R(Q) = \mathcal{S},\ AQ = Q^*A\},$$

then $(A, \mathcal{S})$ is compatible if and only if $\mathcal{P}(A, \mathcal{S})$ is not empty. In a recent paper [7] the authors introduced and studied this notion (see also Hassi and Nordström [13]). In particular it was shown that there exists a strong relationship between compatibility, the projections of $\mathcal{P}(A, \mathcal{S})$ and the shorted operator $A_{/\mathcal{S}}$ of Krein [15] and Anderson-Trapp [2].

This paper is devoted to refine several results of [7], providing new formulae and properties of the so called *minimal projection* $P_{A,\mathcal{S}}$ of $\mathcal{P}(A, \mathcal{S})$, and new characterization of compatible pairs, in order to apply them to shorted operators and compressions.

Observe that the elements of $\mathcal{P}(A, \mathcal{S})$ are selfadjoint for the sesquilinear form defined by A. Therefore, the usual best approximation properties of selfadjoint

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projections can be extended to the elements of $\mathcal{P}(A, \mathcal{S})$. Let us mention the following application of the notion of compatibility and A-selfadjoint projections to approximation theory.

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_1$, $T \in L(\mathcal{H}, \mathcal{H}_1)$, $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and $\xi \in \mathcal{H}$, an *abstract spline* or a $(T, \mathcal{S})$-*spline interpolant to* ξ is any element of the set

$$\operatorname{sp}(T, \mathcal{S}, \xi) = \{\eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma \in \mathcal{S}} \|T(\xi + \sigma)\|\}.$$

It turns out that, if $A = T^*T$, then $(A, \mathcal{S})$ is compatible if and only if $\operatorname{sp}(T, \mathcal{S}, \xi)$ is not empty for any $\xi \in \mathcal{H}$ and, in that case, $\operatorname{sp}(T, \mathcal{S}, \xi) = \{(1 - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$ for any $\xi \in \mathcal{H} \setminus \mathcal{S}$. Moreover, the vector of $\operatorname{sp}(T, \mathcal{S}, \xi)$ with minimal norm is exactly $(1 - P_{A,\mathcal{S}})\xi$, where $P_{A,\mathcal{S}}$ is a distinguished element of $\mathcal{P}(A, \mathcal{S})$ defined in section 4 which is called the *minimal projection.* See [8] for proofs of these and related facts.

The notion of *shorted operator* of A to $\mathcal{S}$, introduced by M. G. Krein [15] as part of the theory of extensions of Hermitian operators, was later rediscovered by W. N. Anderson and G. E. Trapp [1], [2], who applied it in electrical network theory.

In finite dimensional spaces, the shorted operator is one of the various manifestations of the Schur complement of a matrix. Given a block matrix

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

with B invertible, then $E - DB^{-1}C$ is the Schur complement of B in A. This definition is due to E. Haynsworth [14], but it has appeared in several disguised forms since the beginning of the theory of matrices. The reader is referred to the nice surveys by R. W. Cottle [6] and D. Carlson [5] for many properties and applications. The notion was generalized in several directions. In particular, T. Ando [3] introduced, simultaneously with a generalization of the Schur complement, the concept of $\mathcal{S}$-compression $A_{\mathcal{S}}$ of an operator A in the case of a finite dimensional space. In Ando's definition, if $\mathcal{S}$ is a subspace of $\mathcal{H}$ and A is an operator on $\mathcal{H}$ of the form $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, with B invertible on $\mathcal{S}$, then

$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E - DB^{-1}C \end{pmatrix} \quad \text{and} \quad A_{\mathcal{S}} = \begin{pmatrix} B & C \\ D & DB^{-1}C \end{pmatrix}.$$

W. N. Anderson [1] showed that if $A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}$ is a $n \times n$ positive semidefinite matrix and B is a square $k \times k$ submatrix, then the operator

$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E - DB^{\dagger}C \end{pmatrix},$$

where $B^{\dagger}$ is the Moore-Penrose pseudoinverse of B and $\mathcal{S}$ the subspace of $\mathbb{C}^n$ generated by the fist k canonical vectors, has the following interpretation in electrical network theory: if A is the impedance matrix of a resistive n-port network, then $A_{/\mathcal{S}}$ is the impedance matrix of the network obtained by shorting the first k ports. He proved that

$$A_{/\mathcal{S}} = \max\{X \in \mathbb{C}^{n \times n} : 0 \le X \le A \quad \text{and} \quad R(X) \subseteq \mathcal{S}^{\perp}\}$$

and used this property to extend the notion to Hilbert space positive operators:

Definition 1.1. *Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. Then*

1. *The shorted operator of A by $\mathcal{S}$ is defined by*

$$A_{/\mathcal{S}} = \max\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}^\perp\}$$

   where the maximum is taken for the natural order relation in $L(\mathcal{H})^+$ (see [2]).
2. *The $\mathcal{S}$-compression $A_\mathcal{S}$ of A is defined as $A_\mathcal{S} = A - A_{/\mathcal{S}}$.*

The following general properties about the range and kernel of $A_{/\mathcal{S}}$ and $A_\mathcal{S}$ are proved in section 2:

1. $\overline{\ker A + \mathcal{S}} \subseteq \ker A_{/\mathcal{S}} \subseteq A^{-1/2}(\ \overline{A^{1/2}(\mathcal{S})}\ )$.
2. $\ker A_{/\mathcal{S}} = \ker A + \mathcal{S}$ if and only if $A^{1/2}(\mathcal{S})$ is closed in $R(A)$.
3. $A(\mathcal{S}) \subseteq R(A_\mathcal{S}) \subseteq \overline{A(\mathcal{S})}$ and both inclusions may be strict.
4. $\ker A_\mathcal{S} = A^{-1}(\mathcal{S}^\perp) = A(\mathcal{S})^\perp$.

The following list contains some of the results of the paper relating the compatibility of the pair $(A, \mathcal{S})$ with the properties of $A_{/\mathcal{S}}$ and $A_\mathcal{S}$:

1. If $(A, \mathcal{S})$ is compatible, and $E \in \mathcal{P}(A, \mathcal{S})$, then

$$A_\mathcal{S} = AE \quad \text{and} \quad A_{/\mathcal{S}} = A(1 - E).$$

2. $(A, \mathcal{S})$ is compatible if and only if $A_{/\mathcal{S}} = \min\{R^*AR : R^2 = R, \ker R = \mathcal{S}\}$ (see 5.1).
3. $(A, \mathcal{S})$ is compatible if and only if

$$\ker A_{/\mathcal{S}} = \mathcal{S} + \ker A \quad \text{and} \quad R(A_{/\mathcal{S}}) \subseteq R(A).$$

   In this case, $R(A_{/\mathcal{S}}) = R(A) \cap \mathcal{S}^\perp$ (see 5.4).
4. $(A, \mathcal{S})$ is compatible if and only if $R(A_\mathcal{S}) = A(\mathcal{S})$ (see 5.5).
5. $R(A_{/\mathcal{S}}) \subseteq R(A)$ if and only if the pair $(A, \ker A_{/\mathcal{S}})$ is compatible (see 5.2).

Section 2 contains some properties of shorted operators and compressions we shall use later. In section 3 we present several results about A-selfadjoint operators and compatibility, for A a positive (semidefinite) operator. In section 4 we define and show formulas and properties of the minimal projection $P_{A,\mathcal{S}}$ of $\mathcal{P}(A, \mathcal{S})$. In section 5 we get the mentioned characterizations of compatibility for a pair $(A, \mathcal{S})$, in terms of the properties of shorted operators and compressions. Section 6 contains some examples.

2. Preliminaries

In this paper $\mathcal{H}$ denotes a Hilbert space, $L(\mathcal{H})$ is the algebra of all linear bounded operators on $\mathcal{H}$, $L(\mathcal{H})^+$ is the subset of $L(\mathcal{H})$ of all (selfadjoint) positive operators, $GL(\mathcal{H})$ is the group of all invertible operators in $L(\mathcal{H})$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$ (positive invertible operators). For every $C \in L(\mathcal{H})$ its range is denoted by $R(C)$ and its nullspace by $\ker C$. Denote by $\mathcal{Q}$ (resp., $\mathcal{P}$) the set of all projections (resp., selfadjoint projections) in $L(\mathcal{H})$:

$$\mathcal{Q} = \mathcal{Q}(L(\mathcal{H})) = \{Q \in L(\mathcal{H}) : Q^2 = Q\}, \quad \mathcal{P} = \mathcal{P}(L(\mathcal{H})) = \{P \in \mathcal{Q} : P = P^*\}.$$

The nonselfadjoint elements of $\mathcal{Q}$ will be called *oblique projections*.

Along this note we use the fact that every $P \in \mathcal{P}$ induces a representation of elements of $L(\mathcal{H})$ by 2×2 matrices: if $T \in L(\mathcal{H})$ decomposes as

$$T = PTP + PT(1-P) + (1-P)TP + (1-P)T(1-P),$$

then T is represented by the matrix $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$, where for example $T_1 = PTP$, which is alternatively viewed as an element of $L(\mathcal{H})$ or $L(P(\mathcal{H}))$. Under this representation P can be identified with

$$\begin{pmatrix} I_{P(\mathcal{H})} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and all idempotents Q with the same range as P have the form

$$Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

for some $x \in L(\ker P, R(P))$.

Now we state the well known criterium due to Douglas [11] about ranges and factorizations of operators:

Theorem 2.1. *Let $A, B \in L(\mathcal{H})$. Then the following conditions are equivalent:*

1. *$R(B) \subseteq R(A)$.*
2. *There exists a positive number λ such that $BB^* \leq \lambda AA^*$.*
3. *There exists $D \in L(\mathcal{H})$ such that $B = AD$.*

Moreover, the operator D is unique if it satisfies the conditions

$$B = AD\,, \quad \ker D = \ker B \qquad \textit{and} \qquad R(D) \subseteq \overline{R(A^*)}.$$

In this case $\|D\|^2 = \inf\{\lambda : BB^* \leq \lambda AA^*\}$ and A is called the ***reduced*** *solution of the equation $AX = B$.*

We state the following elementary result because we shall use it several times in this paper.

Lemma 2.2. *Ler $A \in L(\mathcal{H})^+$. Then*

1. $\ker A = \ker A^{1/2}$.
2. $R(A) \subseteq R(A^{1/2}) \subseteq \overline{R(A)}$.
3. *If $R(A)$ is not closed then $R(A)$ is properly included in $R(A^{1/2})$.*

Proof. Item 1 and 2 are easy to see. If $R(A) = R(A^{1/2})$ and $\xi \in (\ker A)^\perp$, then there exists $\rho \in (\ker A)^\perp$ such that $A^{1/2}\xi = A\rho$. Therefore $A^{1/2}\rho = \xi$ and $R(A^{1/2})$ is closed. Clearly this implies that $R(A)$ is also closed. □

Shorted operator and compressions.

2.3. As before, let $P \in \mathcal{P}$ be the orthogonal projection onto the closed subspace $\mathcal{S} \subseteq \mathcal{H}$. The classical notion of Schur complement of a matrix (see [6] and [5] for concise surveys on the subject) has been extended to positive Hilbert space operators by M. G. Krein [15] and, later and independently, by W. N. Anderson and G. E.

Trapp [2] defining what is called the *shorted operator*: if $A \in L(\mathcal{H})^+$ then there exists

$$A_{/\mathcal{S}} = \max\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}^\perp\}$$

where the maximum is taken for the natural order relation in $L(\mathcal{H})^+$ (see [2]). $A_{/\mathcal{S}}$ is called the *shorted operator* of A to $\mathcal{S}^\perp$. $\Sigma : \mathcal{P} \times L(\mathcal{H})^+ \to L(\mathcal{H})^+$, $(P, A) \mapsto A_{/\mathcal{S}}$. Next we collect some results of Anderson-Trapp and E. L. Pekarev [19] which are relevant in this paper.

Theorem 2.4. *Let $A \in L(\mathcal{H})^+$ with matrix representation* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$.

1. $R(b) \subseteq R(a^{1/2})$ *and if $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the RS of the equation $a^{1/2}\, x = b$ then*
$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}$$
2. *If $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ and $P_\mathcal{M}$ is the orthogonal projection onto $\mathcal{M}$ then*
$$A_{/\mathcal{S}} = A^{1/2}(1 - P_\mathcal{M})A^{1/2}.$$
3. *$A_{/\mathcal{S}}$ is the infimum of the set $\{R^*AR \ : \ R \in \mathcal{Q},\ \ker R = \mathcal{S}\ \}$; in general, the infimum is not attained.*
4. *$R(A) \cap \mathcal{S}^\perp \subseteq R(A_{/\mathcal{S}}) \subseteq R(A_{/\mathcal{S}}^{1/2}) = R(A^{1/2}) \cap \mathcal{S}^\perp$; in general, the inclusions are strict.*

The reader is referred to [2] and [19] for proofs of these facts.

Corollary 2.5. *Let $A \in L(\mathcal{H})^+$. Then*

1. $\overline{\ker A + \mathcal{S}} \subseteq \ker(A_{/\mathcal{S}}) = A^{-1/2}(\,\overline{A^{1/2}(\mathcal{S})}\,)$.
2. $\ker A_{/\mathcal{S}} = \ker A + \mathcal{S}$ *if and only if* $A^{1/2}(\mathcal{S})$ *is closed in* $R(A^{1/2})$.

Proof.

1. By Theorem 2.4, if $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$, then $A_{/\mathcal{S}} = A^{1/2}(1 - P_\mathcal{M})A^{1/2}$. Hence both $\ker A$ and $\mathcal{S}$ are included in $\ker A_{/\mathcal{S}}$. On the other hand,
$$\ker A_{/\mathcal{S}} = \ker A^{1/2}(1 - P_\mathcal{M})A^{1/2} = \ker(1 - P_\mathcal{M})A^{1/2} = A^{-1/2}(\mathcal{M}).$$
2. It is clear that $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ if and only if $\mathcal{M} \cap R(A^{1/2}) = A^{1/2}(\mathcal{S})$ if and only if $A^{-1/2}(\mathcal{M}) = A^{-1/2}(A^{1/2}(\mathcal{S})) = \ker A + \mathcal{S}$.

□

Definition 2.6. *Let $A \in L(\mathcal{H})^+$, $P \in \mathcal{P}$ and $\mathcal{S} = R(P)$. The positive operator*

$$A_\mathcal{S} := A - A_{/\mathcal{S}}$$

will be called the $\mathcal{S}$-compression of A.

Remark 2.7. Let $A \in L(\mathcal{H})^+$, $P \in \mathcal{P}$ and $\mathcal{S} = R(P)$. Using Theorem 2.4 and Proposition 5.1, one can easily deduce the following properties of $A_\mathcal{S}$:

1. $(A_\mathcal{S})_{/\mathcal{S}} = 0$.

2. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and d is the reduced solution of the equation $a^{1/2}x = b$, then

$$A_{\mathcal{S}} = \begin{pmatrix} a & b \\ b^* & d^*d \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ d^* & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & d \\ 0 & 0 \end{pmatrix}.$$

3. $A_{\mathcal{S}} = A^{1/2} P_{\mathcal{M}} A^{1/2}$, where $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.
4. $\ker A_{\mathcal{S}} = A^{-1}(\mathcal{S}^\perp)$. Indeed, since $\mathcal{M}^\perp = A^{-1/2}(\mathcal{S}^\perp)$, then

$$\ker A_{\mathcal{S}} = \ker P_{\mathcal{M}} A^{1/2} = A^{-1/2}(\mathcal{M}^\perp) = A^{-1/2}(A^{-1/2}(\mathcal{S}^\perp)) = A^{-1}(\mathcal{S}^\perp).$$

5. $A(\mathcal{S}) \subseteq R(A_{\mathcal{S}}) \subseteq \overline{A(\mathcal{S})}$ and the inclusions may be strict. Indeed,

$$A(\mathcal{S}) = A_{\mathcal{S}}(\mathcal{S}) \subseteq R(A_{\mathcal{S}}) \subseteq (\ker A_{\mathcal{S}})^\perp = (A^{-1}(\mathcal{S}^\perp))^\perp = \overline{A(\mathcal{S})}.$$

See Example 6.9 in order to see an example of strict inclusions.

3. A-selfadjoint projections and compatibility

Throughout, $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and P is the orthogonal projection onto $\mathcal{S}$. As we said in the introduction, we consider a bounded sesquilinear form $\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ determined by a positive operator $A \in L(\mathcal{H})$: $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$, $\xi, \eta \in \mathcal{H}$. This form induces the notion of A-orthogonality. For example, easy computations show that the A-orthogonal of $\mathcal{S}$ is

$$\mathcal{S}^{\perp_A} := \{\xi :\ \langle A\xi, \eta \rangle\ = 0\ \ \forall \eta \in \mathcal{S}\ \} = A^{-1}(\mathcal{S}^\perp) = A(\mathcal{S})^\perp.$$

Given $T \in L(\mathcal{H})$, an operator $W \in L(\mathcal{H})$ is called an A-*adjoint* of T if

$$\langle T\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A, \quad \xi,\ \eta \in \mathcal{H},$$

or, which is the same, if $T^*A = AW$. Therefore, the existence of an A-adjoint W of T is equivalent to $R(T^*A) \subseteq R(A)$. In particular, if $Q \in \mathcal{Q}$, then the existence of an A-adjoint of Q is also equivalent to

(1) $R(A) = R(A) \cap \ker Q^* \oplus R(A) \cap R(Q^*) = R(A) \cap (\ker Q)^\perp \oplus R(A) \cap R(Q)^\perp.$

Observe that T may have no A-adjoint, only one or many of them. We shall not deal in this paper with the general problem of existence and uniqueness of A-adjoint operators. Instead, we shall study the existence and uniqueness of A-selfadjoint projections, i.e., $Q \in \mathcal{Q}$ such that $AQ = Q^*A$. Among them, we are interested in those whose range is exactly $\mathcal{S}$. Thus, the main goal of the paper is the study of the set

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, AQ = Q^*A\}$$

for different choices of A.

We shall state all the results for positive operators, though some of them are still true in a more general case. For general results on A-selfadjoint operators the reader is referred to the papers by Lax [16] and Dieudonné [10]; a recent paper by Hassi and Nordström [13] contains many interesting results on A-selfadjoint projections.

The following lemma gives equivalent conditions for a projection to be A- selfadjoint. Observe that they are similar to those for a selfadjoint projection.

Lemma 3.1. *Let $A \in L(\mathcal{H})^+$ and $Q \in \mathcal{Q}$. Then the following conditions are equivalent:*

1. *Q is A-selfadjoint.*
2. $\ker Q \subseteq R(Q)^{\perp_A}$.
3. *Q is an A-contraction, i.e. $\langle Q\xi, Q\xi\rangle_A \leq \langle \xi, \xi\rangle_A \quad \xi \in \mathcal{H}$.*

Proof. **1 ↔ 2:** If $Q \in \mathcal{P}(A, \mathcal{S})$ and $\xi, \eta \in \mathcal{H}$, then

$$\langle A\eta, Q\xi\rangle = \langle Q^* A\eta, \xi\rangle = \langle AQ\eta, \xi\rangle = \langle Q\eta, A\xi\rangle, \tag{2}$$

so $\ker Q \subseteq A^{-1}(S^{\perp})$. The converse can be proved in a similar way.

1 ↔ 3: First observe that condition 3 is equivalent to $Q^*AQ \leq A$. Now suppose that $Q^*AQ \leq A$. Then, by Theorem 2.1, the reduced solution D of the equation $A^{1/2}X = Q^*A^{1/2}$ satisfies $\|D\| \leq 1$. We shall see that $D^2 = D$. Indeed, note that $AD^2 = Q^*A^{1/2}D = (Q^*)^2A^{1/2} = Q^*A^{1/2}$. Also

$$\ker Q^*A^{1/2} = \ker D \subseteq \ker D^2 \subseteq \ker AD^2 = \ker Q^*A^{1/2}$$

and $R(D^2) \subseteq R(D) \subseteq \overline{R(A^*)}$. Thus, D^2 is a reduced solution of $AX = Q^*A^{1/2}$ and, by uniqueness, $D^2 = D$. Since $\|D\| = 1$, it must be $D^* = D$. Since $Q^*A = A^{1/2}DA^{1/2}$, we conclude that $Q^*A = AQ$. Conversely, note that $AQ = Q^*AQ \geq 0$ and, if $E = 1 - Q$, then also $AE = E^*AE$. Therefore, $A = A(Q + E) = Q^*AQ + E^*AE \geq Q^*AQ$. □

Throughout, we use the matrix representation determined by P. Given $A \in L(\mathcal{H})^+$, $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, where $a = PAP$, $b = PA(I-P)$ and $c = (I-P)A(I-P)$.

Definition 3.2. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. The pair $(A, \mathcal{S})$ is said to be compatible if there exists an A-selfadjoint projection with range $\mathcal{S}$, i.e. if $\mathcal{P}(A, \mathcal{S})$ is not empty.*

Now, we state equivalent conditions to compatibility, in terms of the matrix representation given by P. Let $A \in L(\mathcal{H})^+$ with matrix representation $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$.

Proposition 3.3. *Given $A \in L(\mathcal{H})^+$, the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *$R(PA) = R(PAP)$ or equivalently $R(b) \subseteq R(a)$.*
3. *The equation $ax = b$ admits a solution.*

Proof. **2 ↔ 3:** Apply Theorem 2.1.

1 ↔ 3: Recall that $a = PAP$ and $b = PA(1 - P)$. If Y is a solution to $(PAP)X = PA(1 - P)$, consider $y = PY(1 - P)$ and $Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$. Easy computations shows that $Q \in \mathcal{P}(A, \mathcal{S})$. Conversely if $Q \in \mathcal{P}(A, \mathcal{S})$, $Q = \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}$ then writing the equality $AQ = Q^*A$ in matrix form, we get that q is a solution to $ax = b$. □

Remark 3.4. Let $A \in L(\mathcal{H})^+$, $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$. Then,

1. If $R(PAP)$ is closed, the pair $(A, \mathcal{S})$ is compatible. Indeed, if $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ then, by Theorem 2.4, $R(b) \subseteq R(a^{1/2})$. But if $R(PAP)$ is closed, $R(a^{1/2}) = R(a)$. Then, by Proposition 3.3, the pair $(A, \mathcal{S})$ is compatible. In particular:
2. If $\dim \mathcal{H} < \infty$ then every pair $(A, \mathcal{S})$ is compatible.
3. If $\dim \mathcal{S} < \infty$ then $(A, \mathcal{S})$ is compatible.
4. If $A \in GL(\mathcal{H})^+$, then $R(PAP) = \mathcal{S}$, so that $(A, \mathcal{S})$ is compatible. In this case, the unique projection $P_{A,\mathcal{S}}$ onto $\mathcal{S}$ which is A-selfadjoint, is determined (see [4]) by the formulae

$$P_{A,\mathcal{S}} = P(1 + P - A^{-1}PA)^{-1} = \Big(PAP + (1-P)A(1-P)\Big)^{-1} PA. \tag{3}$$

Example 3.5. Let $A \in L(\mathcal{H})^+$ and consider

$$M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+.$$

If $\mathcal{S} = \mathcal{H} \oplus \{0\}$, then, by Lemma 2.2, the pair $(M, \mathcal{S})$ is compatible if and only if $R(A)$ is closed.

Now we give equivalent conditions to compatibility, in this case in terms of subspaces.

Proposition 3.6. *Given $A \in L(\mathcal{H})^+$, the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. $\mathcal{S} + \mathcal{S}^{\perp_A} = \mathcal{H}$.
3. $R(A^{1/2}) = A^{1/2}(\mathcal{S}) \oplus (A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2}))$.
4. *If $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$, then $R(P_\mathcal{M} A^{1/2}) \subseteq R(A^{1/2}P)$.*

Proof. **1** $\leftrightarrow$ **2:** follows from Lemma 3.1 with $R(Q) = \mathcal{S}$.

2 $\leftrightarrow$ **3:** If $\mathcal{H} = \mathcal{S} + \mathcal{S}^{\perp_A}$ then applying $A^{1/2}$ to both sides of the equality we get that $A^{1/2}(\mathcal{H}) = A^{1/2}(\mathcal{S}) + A^{1/2}(A^{-1}(\mathcal{S}^\perp))$ or $R(A^{1/2}) = A^{1/2}(\mathcal{S}) + A^{-1/2}(\mathcal{S}^\perp) \cap R(A^{1/2}) = A^{1/2}(\mathcal{S}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$. Conversely, from $R(A^{1/2}) = A^{1/2}(\mathcal{S}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$ we get that $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp) + \ker A^{1/2} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$.

3 $\leftrightarrow$ **4:** If $y \in R(A^{1/2})$ then $y = y_1 + y_2$ for unique $y_1 \in A^{1/2}(\mathcal{S})$ and $y_2 \in A^{1/2}(\mathcal{S})^\perp$, but then $P_\mathcal{M}(y) = y_1 \in R(A^{1/2}P)$. The converse is similar. □

Remark 3.7. If the pair $(A, \mathcal{S})$ is compatible it follows from item 3 of Proposition 3.6 that $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$. Observe that in this case if $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ then

$$R(A^{1/2}) = \mathcal{M} \cap R(A^{1/2}) \oplus \mathcal{M}^\perp \cap R(A^{1/2}).$$

Conversely if $R(A^{1/2}) = \mathcal{M} \cap R(A^{1/2}) \oplus \mathcal{M}^\perp \cap R(A^{1/2})$ and $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ then $(A, \mathcal{S})$ is compatible.

Proposition 3.8. *Let $A \in L(\mathcal{H})^+$, $P \in \mathcal{P}$ and $\mathcal{S} = R(P)$. Then*

1. $({A^2}_{/\mathcal{S}})^{1/2} \leq A_{/\mathcal{S}}$.
2. *If $A(\mathcal{S})$ is closed in $R(A)$, then $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$.*

3. *If $(A, \mathcal{S})$ is compatible, then $A(\mathcal{S})$ is closed in $R(A)$.*

Proof.

1. $A^2{}_{/\mathcal{S}} \leq A^2$ implies that $(A^2{}_{/\mathcal{S}})^{1/2} \leq A$. But $R((A^2{}_{/\mathcal{S}})^{1/2}) \subseteq \mathcal{S}^{\perp}$.
2. Using Corollary 2.5, the fact that $A(\mathcal{S})$ is closed in $R(A)$ implies that
$$\ker A^2{}_{/\mathcal{S}} = \ker A^2 + \mathcal{S} = \ker A + \mathcal{S}.$$
Using item 1, we can deduce that $\ker A_{/\mathcal{S}} \subseteq \ker A + \mathcal{S}$, so that $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$, again by Corollary 2.5.
3. Assume that $(A, \mathcal{S})$ is compatible. By equation (1), if $Q \in \mathcal{P}(A, \mathcal{S})$, then
$$R(A) = R(A) \cap R(Q^*) \oplus R(A) \cap \ker Q^*.$$
Therefore $A(\mathcal{S}) = R(AQ) = R(Q^*A) = R(Q^*) \cap R(A)$ is closed in $R(A)$.

□

Lemma 3.9. *If $A \in L(\mathcal{H})^+$ then*

1. *The following conditions are equivalent:*
   (a) *$R(PAP)$ is closed.*
   (b) *$A^{1/2}(\mathcal{S})$ is closed.*
   (c) *$A(\mathcal{S})$ is closed.*
2. *If $R(PAP)$ is closed, then the pair $(A, \mathcal{S})$ is compatible.*
3. *If the pair $(A, \mathcal{S})$ is compatible, then $\mathcal{S} + \ker A$ is closed.*

Proof.

1. Since $A^{1/2}(\mathcal{S}) = R(A^{1/2}P)$ and $PAP = (A^{1/2}P)^*A^{1/2}P$, we get that (a) is equivalent to (b). Suppose that $R(PAP)$ is closed. Note that $A(S) = R(AP)$ and $R(AP)$ is closed if and only if $R(PA)$ is closed if and only if $R(PA^2P)$ is closed. Note that $(PAP)^2 \leq PA^2P$ and
$$\ker(PAP)^2 = \ker PA^2P = \mathcal{S}^{\perp} \oplus (\mathcal{S} \cap \ker A).$$
Since $PA^2P \geq (PAP)^2 > 0$ in $(\ker(PAP)^2)^{\perp}$ we get that $R(PA^2P)$ is closed. The reverse implication is easy to see.
2. See Remark 3.4.
3. If $(A, \mathcal{S})$ is compatible, then, by item 3 of Proposition 3.6, $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ and then $\mathcal{S} + \ker A = A^{-1/2}(A^{1/2}(\mathcal{S}))$ is closed.

□

The condition "$A(\mathcal{S})$ closed in $R(A)$" (or equivalently "$A(\mathcal{S})$ closed" when A has closed range), which is necessary for the pair $(A, \mathcal{S})$ to be compatible (by Proposition 3.8), turns out to be sufficient when A has closed range, as we will see in the following proposition.

Proposition 3.10. *If $A \in L(\mathcal{H})^+$ has closed range then the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *$R(PAP)$ is closed.*
3. *$\mathcal{S} + \ker A$ is closed.*

Proof. By Lemma 3.9, we know that $2 \to 1 \to 3$. If $\mathcal{S}+\ker A$ is closed then $P_{R(A)}(\mathcal{S})$ is closed. Therefore $A(\mathcal{S}) = A(P_{R(A)}(\mathcal{S}))$ which is closed because $P_{R(A)}(\mathcal{S}) \subseteq R(A)$ is closed. □

Remark 3.11. If $A, B \in L(\mathcal{H})^+$ have both the same closed range, then $\ker A = \ker B$ and, by Proposition 3.10, $(A, \mathcal{S})$ is compatible if and only if $(B, \mathcal{S})$ is compatible. Moreover, $\mathcal{P}(A, \mathcal{S})$ and $\mathcal{P}(B, \mathcal{S})$ are *parallel* affine manifolds by Remark 4.2 above.

For positive injective operators the following equivalences hold:

Proposition 3.12. *If $A \in L(\mathcal{H})^+$ is injective then the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *$\mathcal{S} \oplus \mathcal{S}^{\perp_A} = \mathcal{H}$.*
3. *$\mathcal{S}^\perp \oplus \overline{A(\mathcal{S})}$ is closed.*

Proof. **1 ↔ 2:** follows from Proposition 3.6 and the fact that $\mathcal{S} \cap \mathcal{S}^{\perp_A} = \{0\}$ when A is injective.

2 ↔ 3: First observe that, if $\mathcal{W} = \overline{A(\mathcal{S})}$, then $\mathcal{S}^\perp + \mathcal{W}$ is always a dense set when A is injective because $\overline{\mathcal{S}^\perp + \mathcal{W}} = (\mathcal{S} \cap A(\mathcal{S})^\perp)^\perp = \mathcal{H}$. Then $\mathcal{S}^\perp + \mathcal{W} = \mathcal{H}$ if and only if $\mathcal{S}^\perp + \mathcal{W}$ is closed. The equivalence follows by using the general fact that given closed subspaces $\mathcal{M}$ and $\mathcal{N}$ then $\mathcal{M} \oplus \mathcal{N} = \mathcal{H}$ if and only if $\mathcal{M}^\perp \oplus \mathcal{N}^\perp = \mathcal{H}$. □

Remark 3.13. Given two subspaces $\mathcal{S}$, $\mathcal{T}$, the cosine of the Friedrichs angle between them is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta \rangle| : \xi \in \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp,\ \|\xi\| \le 1,\ \eta \in \mathcal{T} \cap (\mathcal{S} \cap \mathcal{T})^\perp,\ \|\eta\| \le 1\}.$$

It is well known that $c(\mathcal{S}, \mathcal{T}) < 1$ if and only if $\mathcal{S} + \mathcal{T}$ is closed. Then compatibility in the case of a closed range operator or in the injective case is related to an angle condition between two subspaces:

1. If $A \in L(\mathcal{H})^+$ has closed range, then $(A, \mathcal{S})$ is compatible if and only if $c(\mathcal{S}, \ker A) < 1$ (see Proposition 3.10).
2. If $A \in L(\mathcal{H})^+$ is injective, then, by Proposition 3.12, $(A, \mathcal{S})$ is compatible if and only if $c(\mathcal{S}^\perp, \overline{A(\mathcal{S})}\,) < 1$.

4. The minimal projection

Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace such that the pair $(A, \mathcal{S})$ is compatible. Using Lemma 3.1 or Proposition 3.6, it is clear that $\mathcal{P}(A, \mathcal{S})$ is a singleton if and only if $\ker A \cap \mathcal{S} = \{0\}$. If this is no the case, there exists a projection in $\mathcal{P}(A, \mathcal{S})$ with optimal properties:

Definition 4.1. *Let $A \in L(\mathcal{H})^+$ and suppose that the pair $(A, \mathcal{S})$ is compatible. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the reduced solution of the equation $ax = b$, we define the following oblique projection onto $\mathcal{S}$:*

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}.$$

Remark 4.2. Let $A \in L(\mathcal{H})^+$ and suppose that $(A, \mathcal{S})$ is compatible. Denote by $\mathcal{N} = A^{-1}(S^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S}$. Then $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$, $\ker P_{A,\mathcal{S}} = A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$ and $\mathcal{P}(A, \mathcal{S})$ is an affine manifold that can be parametrized as

$$\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} \ + \ L(\mathcal{S}^\perp, \mathcal{N}),$$

where $L(\mathcal{S}^\perp, \mathcal{N})$ is viewed as a subspace of $L(\mathcal{H})$. Observe that $\mathcal{P}(A, \mathcal{S})$ has a unique element $(P_{A,\mathcal{S}})$ if and only if $\mathcal{N} = \{0\}$, i.e. if $\mathcal{S} \oplus A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$.

Moreover $P_{A,\mathcal{S}}$ has minimal norm in $\mathcal{P}(A, \mathcal{S})$. Nevertheless, $P_{A,\mathcal{S}}$ is not in general the unique $Q \in \mathcal{P}(A, \mathcal{S})$ that realizes the minimum. For a proof of these facts see 3.6 of [7].

Proposition 3.3 shows that the pair $(A, \mathcal{S})$ is compatible if and only if $R(PA) \subseteq R(PAP)$. Therefore, if $(A, \mathcal{S})$ is compatible, it is natural to look at the reduced solution Q of the equation

$$(PAP)X = PA \tag{4}$$

and its relation with $P_{A,\mathcal{S}}$. Observe that $R(Q) \subseteq \overline{R(PAP)}$ which can be strictly included in $\mathcal{S}$, so that, in general, $Q \neq P_{A,\mathcal{S}}$. Nevertheless:

Proposition 4.3. *Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. Let Q be the reduced solution of the equation (4). Let $\mathcal{N} = \ker A \cap \mathcal{S}$. Then $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$ and*

$$P_{A,\mathcal{S}} = P_{\mathcal{N}} + Q.$$

Proof. Let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$. In $L(\mathcal{S})$, $\ker a = \mathcal{N}$ and $\overline{R(a)} = \overline{R(a^{1/2})} = \mathcal{S} \ominus \mathcal{N}$. Note that $R(Q) \subseteq \overline{R(a)}$. Also $\ker Q = \ker(PA) = A^{-1}(\mathcal{S}^\perp)$. If $\xi \in \mathcal{S} \ominus \mathcal{N}$, then

$$a(Q\xi) = (PAP)Q\xi = PA\xi = PAP\xi = a(\xi).$$

Since a is injective in $\mathcal{S} \ominus \mathcal{N}$, we can deduce that $Q\xi = \xi$ for all $\xi \in \mathcal{S} \ominus \mathcal{N}$. Now, the compatibility of $(A, \mathcal{S})$ implies that $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$. Also $A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S} = \mathcal{N}$. Therefore $A^{-1}(\mathcal{S}^\perp) \oplus (\mathcal{S} \ominus \mathcal{N}) = \mathcal{H}$. Then $Q^2 = Q$ and $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Note that

$$\ker Q = A^{-1}(\mathcal{S}^\perp) \subseteq A^{-1}((\mathcal{S} \ominus \mathcal{N})^\perp) = R(Q)^{\perp_A},$$

so that Q is A-selfadjoint by Lemma 3.1. On the other hand, $(\mathcal{S} \ominus \mathcal{N}) \cap \ker A = \{0\}$, so that Q is the unique element of $P(A, \mathcal{S} \ominus \mathcal{N})$, by Remark 4.2. Observe that $\mathcal{N} \subseteq \ker A \subseteq A^{-1}(\mathcal{S}^\perp)$. Therefore

$$(P_{\mathcal{N}} + Q)^2 = P_{\mathcal{N}} + Q\,, \quad R(P_{\mathcal{N}} + Q) = \mathcal{S} \quad \text{and}$$

$$\ker(P_{\mathcal{N}} + Q) = (A^{-1}(\mathcal{S}^\perp)) \ominus \mathcal{N}.$$

These formulae clearly implies that $P_{\mathcal{N}} + Q = P_{A,\mathcal{S}}$ (see Remark 4.2). □

By Proposition 3.6, the pair $(A, \mathcal{S})$ is compatible if and only if $R(P_{\mathcal{M}} A^{1/2}) \subseteq R(A^{1/2}P)$ or equivalently if equation $A^{1/2}PX = P_{\mathcal{M}}A^{1/2}$ admits a solution. Moreover, equation (4) and equation $A^{1/2}PX = P_{\mathcal{M}}A^{1/2}$ have the same reduced solution as we will see in the following proposition.

Proposition 4.4. *Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. Let $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ and $\mathcal{N} = \ker A \cap \mathcal{S}$. Consider Q the reduced solution of the equation*

$$(A^{1/2}P)X = P_{\mathcal{M}}A^{1/2}. \tag{5}$$

Then $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$ and $P_{A,\mathcal{S}} = P_{\mathcal{N}} + Q$. In particular, if $A^{1/2}(\mathcal{S})$ is closed and $\ker A \cap \mathcal{S} = \{0\}$, *then*

$$P_{A,\mathcal{S}} = (A^{1/2}P)^{\dagger}P_{\mathcal{M}}A^{1/2} = (A^{1/2}P)^{\dagger}A^{1/2} \tag{6}$$

where $(A^{1/2}P)^{\dagger}$ denotes the Moore-Penrose pseudoinverse of $(A^{1/2}P)$.

Proof. We will prove that equations (4) and (5) have the same RS. Denote $B = A^{1/2}$. Recall that $\mathcal{M} = \overline{B(\mathcal{S})} = B^{-1}(\mathcal{S}^{\perp})^{\perp}$. Observe that

$$BP_{\mathcal{M}}B = AP_{A,\mathcal{S}} = APP_{A,\mathcal{S}}. \tag{7}$$

In fact, for $\xi \in \mathcal{H}$, let $\eta = P_{A,\mathcal{S}}\xi$ and $\rho = \xi - \eta \in A^{-1}(\mathcal{S}^{\perp})$; then $B\eta \in \mathcal{M}$ and $B\rho \in B^{-1}(\mathcal{S}^{\perp}) = \mathcal{M}^{\perp}$. Hence $BP_{\mathcal{M}}B\xi = A\eta = AP_{A,\mathcal{S}}\xi$. By Proposition 4.3, the projection $Q = P_{A,\mathcal{S}} - P_{\mathcal{N}}$ is the reduced solution of the equation $PAPX = PA$. We shall see that Q is the reduced solution of the equation (5). First note that, by equation (7), $BP_{\mathcal{M}}B = (AP)P_{A,\mathcal{S}} = (AP)Q$, so $B(P_{\mathcal{M}}B - BPQ) = 0$. But $R(P_{\mathcal{M}}B - BPQ) \subseteq \overline{R(B)} = (\ker B)^{\perp}$. Hence Q is a solution of (5). Note that $\ker P_{\mathcal{M}}B = B^{-1}(B^{-1}(\mathcal{S}^{\perp})) = A^{-1}(\mathcal{S}^{\perp}) = \ker Q$ by Proposition 4.3. Finally,

$$\overline{R((BP)^*)} = \overline{R(PB)} = \overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N} = R(Q).$$

The first equality of equation (6) follows directly. The second, from the fact that

$$(A^{1/2}P)^{\dagger}P_{\mathcal{M}} = (A^{1/2}P)^{\dagger}.$$

□

Formula (6), for operators with closed range, is due to Golomb [12].

Corollary 4.5. *Consider $A \in L(\mathcal{H})^+$ injective such that the pair $(A, \mathcal{S})$ is compatible. Then, with the same notations as in Proposition 4.4,*

$$P_{A,\mathcal{S}} = A^{-1/2}P_{\mathcal{M}}A^{1/2}.$$

5. The relationship with shorted operators

As before, let $P \in \mathcal{P}$ be the orthogonal projection onto the closed subspace $\mathcal{S} \subseteq \mathcal{H}$. The following proposition relates, when $(A, \mathcal{S})$ is compatible, the shorted operator $A_{/\mathcal{S}}$ defined in section 2.3 with the elements of $\mathcal{P}(A, \mathcal{S})$.

Proposition 5.1. *Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. Let $E \in \mathcal{P}(A, \mathcal{S})$ and $Q = 1 - E$. Then*

1. $A_{/\mathcal{S}} = AQ = Q^*AQ$.
2. $A_{/\mathcal{S}} = \min\{R^*AR : R \in \mathcal{Q}, \ker R = \mathcal{S}\}$. *Actually, this property is equivalent to the compatibility of the pair $(A, \mathcal{S})$.*
3. $R(A_{/\mathcal{S}}) = R(A) \cap S^{\perp}$.
4. $\ker A_{/\mathcal{S}} = \ker A + \mathcal{S}$.

Proof.

1. Note that $0 \leq AQ = Q^*AQ \leq A$, by Lemma 3.1. Also $R(AQ) = R(Q^*A) \subseteq R(Q^*) = \mathcal{S}^\perp$. Given $X \leq A$ with $R(X) \subseteq \mathcal{S}^\perp$, then, since $\ker Q = \mathcal{S}$, we have that
$$X = Q^*XQ \leq Q^*AQ = AQ,$$
where the first equality can be easily checked because $X = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$.
2. By item 1, $Q^*AQ = A_{/\mathcal{S}}$ and $\ker Q = \mathcal{S}$. So the minimum is attained at Q by Theorem 2.4. On the other hand, if the minimum is attained at some projection Y, then $Y^*AY = A_{/\mathcal{S}} \leq A$ implies that Y is A-selfadjoint, by Lemma 3.1. Therefore $1 - Y \in \mathcal{P}(A, \mathcal{S})$.
3. Clearly the equation $A_{/\mathcal{S}} = AQ$ implies that $R(A_{/\mathcal{S}}) \subseteq R(A) \cap S^\perp$. The other inclusion always holds by Theorem 2.4.
4. It follows from Remark 3.7 and Corollary 2.5

□

The condition $R(A_{/\mathcal{S}}) \subseteq R(A)$, which is necessary for compatibility, implies that some subspace bigger than $\mathcal{S}$ (actually $\ker A_{/\mathcal{S}}$) is A-compatible:

Proposition 5.2. *Let $A \in L(\mathcal{H})^+$ such that $R(A_{/\mathcal{S}}) \subseteq R(A)$. Denote $\ker A_{/\mathcal{S}} = \mathcal{T}$. Then*

1. $A_{/\mathcal{T}} = A_{/\mathcal{S}}$.
2. *The pair $(A, \mathcal{T})$ is compatible.*
3. *Let Q be the reduced solution of the equation $AX = A_{/\mathcal{S}}$. Then*
$$1 - Q = P_{A,\mathcal{T}}.$$

Proof. Item 1 follows directly from the definition of shorted operator. Condition $R(A_{/\mathcal{S}}) \subseteq R(A)$ implies, by Douglas theorem, that the set
$$\Delta = \left\{ Q \in L(\mathcal{H}) : AQ = A_{/\mathcal{S}} \text{ and } \ker Q = \mathcal{T} \right\}$$
is not empty. Let $Q \in \Delta$. Clearly Q verifies that $\ker Q = \mathcal{T}$ and $Q^*A = AQ$, because $A_{/\mathcal{S}}$ is selfadjoint. In order to prove that $1 - Q \in \mathcal{P}(A, \mathcal{T})$, it just remain to show that $Q^2 = Q$. Let us first prove that, if $\mathcal{Z} = A^{-1/2}(\mathcal{S}^\perp) = A^{1/2}(\mathcal{S})^\perp$, then Q is a solution of the equation $A^{1/2}X = P_\mathcal{Z} A^{1/2}$. Recall that $A_{/\mathcal{S}} = A^{1/2}P_\mathcal{Z}A^{1/2}$, so $A^{1/2}(A^{1/2}Q - P_\mathcal{Z}A^{1/2}) = 0$. Then, if $\xi \in \mathcal{H}$, $P_\mathcal{Z}A^{1/2}\xi = A^{1/2}Q\xi + \eta$ with $\eta \in \ker A^{1/2} = R(A^{1/2})^\perp \subseteq \mathcal{Z}$. So that
$$\|\eta\|^2 = \langle P_\mathcal{Z}A^{1/2}\xi, \eta\rangle - \langle A^{1/2}Q\xi, \eta\rangle = \langle A^{1/2}\xi, P_\mathcal{Z}\eta\rangle = \langle A^{1/2}\xi, \eta\rangle = 0.$$
Therefore $A^{1/2}Q = P_\mathcal{Z}A^{1/2}$. Note that also $A^{1/2}Q^2 = (P_\mathcal{Z})^2A^{1/2} = P_\mathcal{Z}A^{1/2}$, so $A^{1/2}(Q^2 - Q) = 0$. Let $\rho \in R(Q)$. Then $Q\rho - \rho \in \ker A \cap R(Q)$. If $Q\rho - \rho = Q\omega$, for some $\omega \in \mathcal{H}$, then $0 = AQ\omega = A_{/\mathcal{S}}\omega$. So $\omega \in \ker A_{/\mathcal{S}} = \mathcal{T} = \ker Q$. Therefore $Q\rho = \rho$ for every $\rho \in R(Q)$. This clearly implies that $Q^2 = Q$ and $1 - Q \in \mathcal{P}(A, \mathcal{T})$, showing item 2.

Denote by Q_o the reduced solution of $AX = A_{/\mathcal{S}}$. Then $R(Q_o) \subseteq \overline{R(A)} = (\ker A)^\perp$. Also $\ker Q_o = \ker A_{/\mathcal{S}} = \mathcal{T}$ so that $1 - Q_o \in \mathcal{P}(A, \mathcal{T})$ and $R(Q_o) \subseteq A^{-1}(\mathcal{T}^\perp)$. Then $R(1 - Q_o) = \mathcal{T} = R(P_{A,\mathcal{T}})$ and

$$\ker(1 - Q_o) = R(Q_o) \subseteq A^{-1}(\mathcal{T}^\perp) \cap (\ker A)^\perp \subseteq A^{-1}(\mathcal{T}^\perp) \cap (\mathcal{T} \cap \ker A)^\perp = \ker P_{A,\mathcal{T}}$$

by Remark 4.2. Therefore it must be $P_{A,\mathcal{T}} = 1 - Q_o$. □

Remark 5.3.

1. Observe that if A has closed range then $(A, \mathcal{S})$ is compatible if and only if $\ker(A_{/\mathcal{S}}) = \mathcal{S} + \ker A$. Indeed, $(A, \mathcal{S})$ is compatible if and only if $A^{1/2}(\mathcal{S})$ is closed (see Proposition 3.10) if and only if $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ (because $R(A^{1/2}) = R(A)$ is closed) if and only if $R(A_{/\mathcal{S}}) = \mathcal{S} + \ker A$ (see Corollary 2.5). Note that $R(A_{/\mathcal{S}}) = R(A) \cap \mathcal{S}^\perp$ if $R(A)$ closed.
2. If A is injective, using Propositions 5.1 and 5.2, one gets that $(A, \mathcal{S})$ is compatible if and only if $R(A_{/\mathcal{S}}) = R(A) \cap \mathcal{S}^\perp$ and $\ker(A_{/\mathcal{S}}) = \mathcal{S}$ (see also 5.5 of [7]).

Now we state a general result:

Theorem 5.4. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then $(A, \mathcal{S})$ is compatible if and only if $R(A_{/\mathcal{S}}) = R(A) \cap \mathcal{S}^\perp$ and $\ker A_{/\mathcal{S}} = \ker A + \mathcal{S}$.*

Proof. One implication is stated in Proposition 5.1. Conversely, if $R(A_{/\mathcal{S}}) = R(A) \cap \mathcal{S}^\perp$ and $\ker A_{/\mathcal{S}} = \ker A + \mathcal{S} = \mathcal{T}$ then, by Proposition 5.2, pair $(A, \mathcal{T})$ is compatible, or equivalently $\mathcal{T} + A^{-1}(\mathcal{T}^\perp) = \mathcal{H}$. But

$$\ker A \subseteq A^{-1}(\mathcal{S}^\perp) = A(\mathcal{S})^\perp = A(\mathcal{T})^\perp = A^{-1}(\mathcal{T}^\perp),$$

so that $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$. Then $(A, \mathcal{S})$ is compatible. □

Compressions. Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Recall from Definition 2.6, that the *compression* of A by $\mathcal{S}$ is $A_{\mathcal{S}} = A - A_{/\mathcal{S}}$. Using Proposition 5.1, if $(A, \mathcal{S})$ is compatible, then $A_{\mathcal{S}} = AP_{A,\mathcal{S}}$. So that $R(A_{\mathcal{S}}) = A(\mathcal{S})$. In the next Proposition we shall see that this equality actually characterizes compatibility:

Proposition 5.5. *Let $A \in L(\mathcal{H})^+$, $P \in \mathcal{P}$ and $\mathcal{S} = R(P)$. Then*

1. *The pair $(A, \mathcal{S})$ is compatible if and only if $R(A_{\mathcal{S}}) = A(\mathcal{S})$.*
2. *If $(A, \mathcal{S})$ is compatible and Y is the reduced solution of the equation $(AP)X = A_{\mathcal{S}}$ and $\mathcal{N} = \ker A \cap \mathcal{S}$, then $Y = P_{A,\mathcal{S}\ominus\mathcal{N}}$ and*

$$P_{A,\mathcal{S}} = Y + P_{\mathcal{N}}.$$

Proof. If $(A, \mathcal{S})$ is compatible then from the properties of $A_{\mathcal{S}}$ above, $R(A_{\mathcal{S}}) = A(\mathcal{S})$. Conversely, $R(A_{\mathcal{S}}) = A(\mathcal{S})$ implies that the equation $APX = A_{\mathcal{S}}$ admits a solution (apply Douglas' theorem). Denote by Y the reduced solution of the equation $APX = A_{\mathcal{S}}$. Then

$$\ker Y = \ker A_{\mathcal{S}} = A(\mathcal{S})^\perp \quad \text{and} \tag{8}$$

$$R(Y) \subseteq (\ker AP)^\perp = (\mathcal{S}^\perp + \mathcal{N})^\perp = \mathcal{S} \ominus \mathcal{N} \subseteq \mathcal{S}. \tag{9}$$

So that $PY = Y$ and $A_\mathcal{S} = AY = Y^*A$, which means that Y is A-selfadjoint. On the other hand, because $A\big|_\mathcal{S} = A_\mathcal{S}\big|_\mathcal{S}$ and the fact that $A\big|_{\mathcal{S}\ominus\mathcal{N}}$ is injective, we can deduce that $Y\xi = \xi$ for every $\xi \in \mathcal{S} \ominus \mathcal{N}$, which means that $Y^2 = Y$. Then Y^2 is the reduced solution and $Y = Y^2$. So $\mathcal{H} = R(Y) + \ker Y \subseteq \mathcal{S} + A(\mathcal{S})^\perp$ and the pair $(A, \mathcal{S})$ is compatible. Using formulae (8) and (9), item 2 follows as in the proof of Proposition 4.3. □

6. Some examples

Example 6.1. Given a positive injective operator $A \in L(\mathcal{H})$ with non-closed range. Let $\xi \in R(A^{1/2})$ and let P_ξ be the orthogonal projection onto the subspace $\langle\xi\rangle$ generated by ξ. Then $R(P_\xi) \subseteq R(A^{1/2})$, so that, by Douglas' theorem, $P_\xi \leq \lambda A$ for some positive number λ which we can suppose equal to 1, by changing A by λA. It is well known that this implies that the operator $B \in L(\mathcal{H} \oplus \mathcal{H})$ defined by

$$B = \begin{pmatrix} A & P_\xi \\ P_\xi & A \end{pmatrix}$$

is positive. By Lemma 2.2, $R(A)$ is strictly contained in $R(A^{1/2})$. Suppose that $\xi \in R(A^{1/2})\backslash R(A)$. Let $\mathcal{S} = \mathcal{H}_1 = \mathcal{H}\oplus 0$. Then $\mathcal{S}^\perp = \mathcal{H}_2 = 0\oplus\mathcal{H}$. We shall see that B is injective, $\ker B_{/\mathcal{S}} = \mathcal{S}$, moreover $B(\mathcal{S})$ is closed in $R(B)$ (this condition is necessary for compatibility and it implies that $B^{1/2}(\mathcal{S})$ is closed in $R(B^{1/2})$ i.e. $\ker B_{/\mathcal{S}} = \mathcal{S}$, by Proposition 3.8), but the pair $(B, \mathcal{S})$ is incompatible. Indeed, it is clear that B does not verify condition 3 of Proposition 3.3, so the pair $(B, \mathcal{S})$ is incompatible. Let D be the reduced solution of $P_\xi = A^{1/2}X$. Then $B_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & A - D^*D \end{pmatrix}$. Note that $\ker D = \ker P_x$ implies $DP_\xi = D$. So $D^*D = P_\xi D^*D$. Then, if $0 \oplus \eta \in \ker B_{/\mathcal{S}}$,

$$A\eta = D^*D\eta = P_\xi D^*D\eta = \lambda\xi \text{ for some } \lambda \in \mathbb{R} \quad \Rightarrow \quad \eta = 0$$

because $\xi \notin R(A)$ and A is injective. So $\ker B_{/\mathcal{S}} = \mathcal{S}$. Also

$$B(\omega \oplus \eta) = 0 \oplus 0 \Rightarrow A\omega + P_\xi\ \eta = 0 = A\eta + P_\xi\ \omega \Rightarrow A\omega = A\eta = 0 \Rightarrow \omega = \eta = 0,$$

so that B is injective. Finally, $\mathcal{H} \oplus \langle\xi\rangle \cap R(B) = B(\mathcal{H} \oplus 0)$, because if $\omega \neq 0$, then $A\omega \notin \langle\xi\rangle$ and $B(\eta\oplus\omega) \notin \mathcal{H}\oplus\langle\xi\rangle$ for every $\eta \in \mathcal{H}$. Therefore $B(\mathcal{S})$ is closed in $R(B)$.

Remark 6.2. Let $P \in \mathcal{P}$, $R(P) = \mathcal{S}$ and $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})^+$. It is well known that the positivity of A implies that $R(b) \subseteq R(a^{1/2})$. Therefore it is true, without restrictions on A, that if $\dim\mathcal{S} < \infty$, then the pair $(A, \mathcal{S})$ is compatible, since in this case $R(a) = R(PAP)$ must be closed, so $R(b) \subseteq R(a^{1/2}) = R(a)$ and Proposition 3.3 can be applied. On the other hand, if $\dim\mathcal{S}^\perp < \infty$ and $R(A)$ is closed then, by Proposition 3.10, $(A, \mathcal{S})$ is compatible. However, if $R(A)$ is not closed, then Example 6.3 shows that the result fails in general.

Example 6.3. Let $A \in L(\mathcal{H})^+$ be injective non invertible. Let $\xi \in \mathcal{H} \setminus R(A)$ a unit vector. Denote by $\mathcal{S}^\perp$ the subspace generated by ξ, $P = P_\mathcal{S}$ and $P_\xi = 1 - P$. If

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

in terms of P and $A\xi = \lambda\xi + \eta$ with $\eta \in \mathcal{S}$, then $\lambda = \langle A\xi, \xi\rangle \neq 0$ and $\eta \neq 0$ (otherwise $\xi \in R(A)$). Therefore $c = \lambda P_\xi$ and $b(\mu\xi) = \mu\eta$, $\mu \in \mathbb{C}$.

Suppose that $\eta \in R(a)$, i.e., there exists $\nu \in \mathcal{S}$ which verifies $a\nu = b\xi$. Then $PA(\nu - \xi) = a\nu - b\xi = 0$, so $A(\nu - \xi)$ is a multiple of ξ, which must be 0 ($\xi \notin R(A)$). So $\nu = \xi$, a contradiction. Therefore $R(b) \not\subseteq R(a)$ and the pair $(A, \mathcal{S})$ is incompatible.

Let d be the reduced solution of the equation $a^{1/2}x = b$. The facts that $\eta \notin R(a)$ and that $a^{1/2}$ is injective in $\mathcal{S}$ clearly implies that $R(a^{1/2}) \cap R(d) = \{0\}$. Consider now the operator

$$B = \begin{pmatrix} a & b \\ b^* & dd^* \end{pmatrix} \geq 0.$$

Then the pair $(B, \mathcal{S})$ is also incompatible and $B_{/\mathcal{S}} = 0$. But in this case B is injective. Indeed,

$$B = \begin{pmatrix} a & a^{1/2}d \\ d^*a^{1/2} & dd^* \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ d^* & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & d \\ 0 & 0 \end{pmatrix}$$

and therefore

$$\ker B = \ker \begin{pmatrix} a^{1/2} & d \\ 0 & 0 \end{pmatrix} = \{0\}$$

because $R(a^{1/2}) \cap R(d) = \{0\}$, $a^{1/2}$ is injective in $\mathcal{S}$ and d is injective in $\mathcal{S}^\perp$. This example shows the intrinsic necessarity of the condition $\ker A_{/\mathcal{S}} = \mathcal{S}$ in the equivalence given in Theorem 5.4: if A is injective, the pair $(A, \mathcal{S})$ is compatible $\iff$ $R(A_{/\mathcal{S}}) \subseteq R(A)$ and $\ker A_{/\mathcal{S}} = \mathcal{S}$. In fact the example shows that $R(A_{/\mathcal{S}}) \subseteq R(A)$ does not imply $\ker A_{/\mathcal{S}} = \mathcal{S}$ neither in the injective case. In this sense this example complements Example 6.1.

6.4. Two positive operators $A, B \in L(\mathcal{H})$ are in the same "Thompson component", if

$$A \sim B \iff R(A^{1/2}) = R(B^{1/2}) \iff \lambda A \leq B \leq \mu A$$

for some constants λ, μ in $\mathbb{R}_+$. A natural question is: given $\mathcal{S}$ a closed subspace of $\mathcal{H}$, is it true that $(A, \mathcal{S})$ is compatible if and only if $(B, \mathcal{S})$ is compatible? This is true for closed range operators by Remark 3.11. Unfortunately, in general the answer is no, as we shall see in the following example. We first need a lemma:

Lemma 6.5. *Let $A, B \in L(\mathcal{H})^+$.*

1. *If $R(A) = R(B)$ then $R(A^t) = R(B^t)$ for $0 \leq t \leq 1$. In particular $A \sim B$.*
2. *If $A \in L(\mathcal{H})^+$ and $R(A)$ is not closed, then there exists $B \in L(\mathcal{H})^+$ such that $A \sim B$ but $R(A) \neq R(B)$.*

Proof.

1. By Douglas theorem, $R(A) = R(B)$ implies that there exist $\lambda, \mu > 0$ such that $\lambda A^2 \leq B^2 \leq \mu A^2$. Then, by Löwner theorem [17], $\lambda^t A^{2t} \leq B^{2t} \leq \mu^t A^{2t}$ and $R(A^t) = R(B^t)$, for $0 \leq t \leq 1$. Taking $t = 1/2$ one gets that $A \sim B$.
2. Denote $C = A^{1/2}$. If $G \in GL(\mathcal{H})^+$, then $R(C) = R(CG^{1/2}) = R((CGC)^{1/2})$. We claim that G can be chosen in such a way that $R(A) \neq R(CGC)$. Indeed, take $\xi \in R(C) \setminus R(A)$, $\eta \in (\ker C)^\perp$ such that $C\eta = \xi$, and $\rho \in R(C)$ such that $\langle \rho, \eta \rangle > 0$ (recall that $R(C)$ is dense in $(\ker C)^\perp$). Choose $G \in GL(\mathcal{H})^+$ such that $G\rho = \eta$. This can be done working separately in the subspace $\mathcal{Z}$ generated by ρ and η, and in $\mathcal{Z}^\perp$. The condition $\langle \rho, \eta \rangle > 0$ is sufficient by an easy 2×2 argument. Then $\xi = C\eta = CG\rho \in R(CGC) \setminus R(A)$. Take $B = CGC$.

□

Example 6.6. Let $A \in L(\mathcal{H})^+$ injective but not invertible. Suppose that $A \sim B$ and $\lambda A \leq B \leq \mu A$ with $\lambda < 1 < \mu$. By last lemma, we can also suppose that $R(A) \neq R(B)$. So there exists $\xi \in R(A) \setminus R(B) \subseteq R(A^{1/2}) = R(B^{1/2})$. Let P_ξ be the orthogonal projection onto the subspace generated by ξ. Then $R(P_\xi) \subseteq R(A^{1/2}) = R(B^{1/2})$. So that, by Douglas theorem, we can suppose $2P_\xi \leq A$ and $2P_\xi \leq B$. As in Example 6.1, the operators $M_A, M_B \in L(\mathcal{H} \oplus \mathcal{H})$ defined by

$$M_A = \begin{pmatrix} A & P_\xi \\ P_\xi & A \end{pmatrix}, \quad M_B = \begin{pmatrix} B & P_\xi \\ P_\xi & B \end{pmatrix}$$

are positive. Let $\mathcal{S} = \mathcal{H}_1 = \mathcal{H} \oplus 0$. Then $\mathcal{S}^\perp = \mathcal{H}_2 = 0 \oplus \mathcal{H}$. In Example 6.1 it is shown that M_B is injective but the pair $(M_B, \mathcal{S})$ is incompatible. On the other hand, since $\xi \in R(A)$, then the pair $(M_A, \mathcal{S})$ is compatible. We shall see that $M_A \sim M_B$, thus contradicting the previous conjecture. Indeed, note that

$$2P_\xi \leq A \text{ and } \frac{1}{\mu}\, B \leq A \quad \Rightarrow \quad 2A - \frac{1}{\mu}\, B \geq 2P_\xi \geq \Big(2 - \frac{1}{\mu}\Big) P_\xi.$$

Therefore

$$2M_A = 2\begin{pmatrix} A & P_\xi \\ P_\xi & A \end{pmatrix} \geq \frac{1}{\mu}\begin{pmatrix} B & P_\xi \\ P_\xi & B \end{pmatrix} = \frac{1}{\mu}\, M_B.$$

Analogously $2P_\xi \leq B$ and $\lambda A \leq B$ implies that $2B - \lambda A \geq 2P_\xi \geq (2-\lambda)P_\xi$. Therefore

$$2M_B = 2\begin{pmatrix} B & P_\xi \\ P_\xi & B \end{pmatrix} \geq \lambda \begin{pmatrix} A & P_\xi \\ P_\xi & A \end{pmatrix} = \lambda M_A.$$

Example 6.7. Let $\mathcal{A} = \{(A, P) \in L(\mathcal{H})^+ \times \mathcal{P} : \text{ the pair } (A, \mathcal{S}) \text{ is compatible}\}$. If $\dim \mathcal{H} = \infty$, then the space $\mathcal{A}$ is neither open nor closed in $L(\mathcal{H})^+ \times \mathcal{P}$. Indeed, the proper subset $GL(\mathcal{H})^+ \times \mathcal{P} \subseteq \mathcal{A}$ of $\mathcal{A}$ is dense in $L(\mathcal{H})^+ \times \mathcal{P}$, so $\mathcal{A}$ is not closed. On the other hand, let A be a positive injective operator in $L(\mathcal{H})$ with non-closed range and $\xi \in R(A^{1/2})$. Consider the operator $B \in L(\mathcal{H} \oplus \mathcal{H})$ defined in Example 6.1. If $\mathcal{S} = \mathcal{H}_1 = \mathcal{H} \oplus 0$, then $(B, \mathcal{S})$ is compatible if and only if $\xi \in R(A)$. It is easy to see that some $\xi \in R(A)$ can be approached by elements of $R(A^{1/2}) \setminus R(A)$ and so, the compatible pair $(B, \mathcal{S})$ can be approached by non compatible pairs. Since $\mathcal{H} \oplus \mathcal{H}$ is isomorphic to $\mathcal{H}$, this shows that $\mathcal{A}$ is not open in $L(\mathcal{H})^+ \times \mathcal{P}$.

Example 6.8. Consider the map $\alpha : \mathcal{A} \to \mathcal{Q}$ given by $\alpha(A, P) = P_{A,\mathcal{S}}$, where $\mathcal{A}$ is the set defined in Example 6.7. We shall see that α is not continuous. Indeed, fix $\mathcal{S} \subseteq \mathcal{H}$ and consider $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, such that $R(b) = R(a)$ is a closed subspace $\mathcal{M}$ properly included in $\mathcal{S}$. Denote by $\mathcal{N} = \mathcal{S} \ominus \mathcal{M}$ and consider the projection $P_{\mathcal{N}}$ and some element $u \in L(\mathcal{S}^\perp, \mathcal{N}) \subseteq L(\mathcal{H})$, $u \neq 0$. Consider, for every $n \in \mathbb{N}$,

$$\begin{aligned} A_n &= A + \frac{1}{n}\,(P_{\mathcal{N}} + u)^*(P_{\mathcal{N}} + u) \\ &= A + \frac{1}{n}\begin{pmatrix} 1 & 0 & u \\ 0 & 0 & 0 \\ u^* & 0 & u^*u \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix} \\ &= \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}\,u \\ 0 & a & b \\ \frac{1}{n}\,u^* & b^* & c + \frac{1}{n}\,u^*u \end{pmatrix} \geq A \geq 0. \end{aligned}$$

It is clear that $A_n \to A$. Note that a is invertible in $L(\mathcal{M})$. Then, by Remark 4.2,

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix} .$$

Note that $a + \frac{1}{n}P_{\mathcal{N}}$ is invertible in $L(\mathcal{S})$. Then, by equation (3),

$$P_{A_n,\mathcal{S}} = \begin{pmatrix} n & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}\,u \\ 0 & a & b \\ \frac{1}{n}\,u^* & b^* & c + \frac{1}{n}\,u^*u \end{pmatrix} = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix}$$

for all $n \in \mathbb{N}$. Therefore $\alpha(A_n, P) = P_{A_n,\mathcal{S}} \not\to P_{A,\mathcal{S}} = \alpha(A, P)$. Remark that the sequence $\alpha(A_n, P)$ converges (actually, it is constant) to $P_{A,\mathcal{S}} + u$, which belongs to $\mathcal{P}(A, \mathcal{S})$ by Remark 4.2. Also, it is easy to see that, for every $n \in \mathbb{N}$, $(A_n)_{/\mathcal{S}} = A_{/\mathcal{S}}$.

Example 6.9. Let $A \in L(\mathcal{H})^+$ and

$$M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+$$

like in Example 3.5. Denote by $\mathcal{S} = \mathcal{H} \oplus \{0\}$ and by $N = \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix}$. Since $M = N^*N$, then $\ker M = \ker N = \{\xi \oplus -A^{1/2}\xi : \xi \in \mathcal{H}\}$ which is the graph of $-A^{1/2}$. Note that $R(N) = (R(A^{1/2}) + R(I)) \oplus \{0\} = \mathcal{S}$, so that $R(M)$ is also closed. Also $M_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & P_{\ker A} \end{pmatrix}$, because the reduced solution of the equation $A^{1/2}X = A^{1/2}$ is $D = P_{R(A)}$.

If A is injective not inversible, then $(M, \mathcal{S})$ is not compatible (because $R(A)$ is properly included in $R(A^{1/2})$). Also $M = M_{\mathcal{S}}$ and $M(\mathcal{S}) \neq R(M_{\mathcal{S}})$. Hence in this example $R(M_{\mathcal{S}}) = \overline{M(\mathcal{S})}$ while $M(\mathcal{S})$ is not closed (see Proposition 5.5).

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