CORRECTED ENERGY OF DISTRIBUTIONS FOR 3-SASAKIAN AND NORMAL COMPLEX CONTACT MANIFOLDS, AND A CRITICAL POINT OF THE ENERGY ON THE IWASAWA MANIFOLD *

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In this paper, recent developments in the energy and the corrected energy of the corresponding distributions on a compact oriented Riemannian manifold are given.

1. Introduction

In [8], Chacón, Naveira and Weston introduced the energy $\mathcal{E}(V)$ of a distribution. They studied the first and second variation of the energy and as an application showed that the Hopf fibration $S^3 \hookrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$ is an unstable critical point. The corresponding result in the case of the energy of a vector field for the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$ is due to C. M. Wood [19]. Wood showed that for $n > 1$, the critical point is unstable; for $n = 1$ Brito [6] showed that this Hopf fibration is a minima.

Subsequently in [7], Chacón and Naveira introduced a corrected energy $\mathcal{D}(V)$ for a q-dimensional distribution on a Riemannian manifold (M, g) and proved that $\mathcal{D}(V)$ is $\geq$ the integral of the sum of the mixed sectional curvatures associated to a compatible basis. As a single application they showed that the Hopf fibration $S^3 \hookrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$ is a minimum of $\mathcal{D}(V)$.

In [3], we showed that this application can be greatly generalized to the natural fibrations on 3-Sasakian manifolds and on normal complex contact

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metric manifolds.In [18], we considered as a further application of the results of [8], the Boothby-Wang fibration of the Iwasawa manifold $S^1 \times S^1 \hookrightarrow H_{\mathbb{C}}/\Gamma \longrightarrow \mathbb{C}^3/\Gamma$. Making use of the complex contact structure on $H_{\mathbb{C}}/\Gamma$ we showed that this fibration is also unstable for the energy.

In [11], Gil-Medrano, González-Dávila and Vanhecke studied conditions under which the energy of a distribution, viewed as a map into the Grassmann bundle, is a harmonic map or minimal immersion.

2. Geometry of Distributions

Let (M^n, g) be a compact oriented Riemannian manifold with a q-dimensional distribution or subbundle $\mathcal{V}$ and let $\mathcal{H}$ denote the orthogonal complementary distribution of dimension $p = n - q$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis on M^n such that $\{e_1, \ldots, e_p\}$ span $\mathcal{H}$ and $\{e_{p+1}, \ldots, e_n\}$ span $\mathcal{V}$ and adopt the index conventions: $1 \leq a, b \leq n$, $1 \leq i, j \leq p$, $p + 1 \leq \alpha, \beta \leq n$. The second fundamental form of the horizontal distribution $\mathcal{H}$ in the direction e_α and that of the vertical distribution $\mathcal{V}$ in the direction e_i are given respectively by $h^\alpha_{ij} = -g(\nabla_{e_i} e_\alpha, e_j)$, $h^i_{\alpha\beta} = -g(\nabla_{e_\alpha} e_i, e_\beta)$. The mean curvature vectors of the horizontal and vertical distributions are given respectively by

$$\vec{H}_{\mathcal{H}} = \sum_{\alpha=p+1}^{n} \Big(\frac{1}{p} \sum_{i=1}^{p} h^\alpha_{ii}\Big) e_\alpha, \quad \vec{H}_{\mathcal{V}} = \sum_{i=1}^{p} \Big(\frac{1}{q} \sum_{\alpha=p+1}^{n} h^i_{\alpha\alpha}\Big) e_i.$$

One can regard a distribution, such as $\mathcal{V}$, as a section of the Grassmann bundle, $G(q, M^n)$, of oriented q-planes in the tangent spaces of M^n. The geometry of this bundle was developed in [8]. We also view $\mathcal{V}$ as a map $\xi : M^n \longrightarrow G(q, M^n)$ where $\xi(x)$ is a unit q-vector with respect to the induced metric on $\bigwedge^q(M^n)$, in particular $\xi(x) = e_{p+1}(x) \wedge \cdots \wedge e_n(x)$. Note that we have chosen a local orthonormal basis; in [19], the variation of unit vector fields is through unit vector fields and the variations of distributions in [8] are through unit q-vectors. The norm of the covariant derivative of ξ is given in terms of the second fundamental forms of $\mathcal{H}$ and $\mathcal{V}$ by

$$\sum_a ||\nabla_{e_a} \xi||^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2 + \sum_{i,\alpha,\beta} (h^i_{\alpha\beta})^2. \tag{1}$$

The *corrected energy* of [7] is defined by

$$\mathcal{D}(V) = \int_M \Big(\sum_a ||\nabla_{e_a} \xi||^2 + p(p-2)||\vec{H}_{\mathcal{H}}||^2 + q^2||\vec{H}_{\mathcal{V}}||^2\Big) d\mathrm{vol}.$$

We now define the *energy* of a distribution $\mathcal{V}$ as in [8] by

$$\mathcal{E}(V) = \frac{1}{2}\int_M \sum_{a=1}^{n} ||\nabla_{e_a}\xi||^2 d\mathrm{vol} + \frac{n}{2}\mathrm{vol}(M).$$

Denote by $\nabla^*\nabla\xi$ the rough Laplacian

$$\nabla^*\nabla\xi = \sum_{a=1}^{n}(-\nabla_{e_a}\nabla_{e_a}\xi + \nabla_{\nabla_{e_a}e_a}\xi).$$

The main results of [7] and [8] are summarized as follows

Theorem 2.1 *If $\mathcal{V}$ is integrable, then*

$$\mathcal{D}(V) \geq \int_M \sum_{i,\alpha} c_{i\alpha} d\mathrm{vol}$$

where $c_{i\alpha}$ is the sectional curvature of the plane section spanned by $e_i \in \mathcal{H}$ and $e_\alpha \in \mathcal{V}$.

Theorem 2.2 *A distribution $\mathcal{V}$ is a critical point of the energy if and only if $\nabla^*\nabla\xi$ is orthogonal to all tangent vectors of ξ in $\bigwedge^q(M^n)$, i.e.,*

$$\nabla^*\nabla\xi = ||\nabla\xi||^2\xi + \sum \text{terms of type}\, \mathcal{H}\wedge\mathcal{H}\wedge\mathcal{V}\wedge\cdots\wedge\mathcal{V}.$$

If ξ_{st} is a variation of a critical point $\mathcal{V}$ through oriented distributions with tangent fields

$$V = \frac{\partial\xi_{st}}{\partial s}\Big|_{(s,t)=(0,0)} \text{and} \quad W = \frac{\partial\xi_{st}}{\partial t}\Big|_{(s,t)=(0,0)},$$

then

$$\frac{\partial^2\mathcal{E}(\xi_{st})}{\partial s\,\partial t}\Big|_{(0,0)} = \int_M \big(g(\nabla_{\partial s}\nabla_{\partial t}\xi_{st}|_{(0,0)}, \nabla^*\nabla\xi) + g(W, \nabla^*\nabla V)\big) d\mathrm{vol}.$$

3. 3-Sasakian manifolds

By a *contact manifold* we mean a differentiable manifold M^{2n+1} together with a 1-form η such that $\eta\wedge(d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ, such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$; ξ is called the *characteristic vector field* or *Reeb vector field* of the contact form η.

A Riemannian metric g is an *associated metric* for a contact form η if, first of all, $\eta(X) = g(X,\xi)$ and secondly, there exists a field of endomorphisms ϕ

such that $\phi^2 = -I + \eta \otimes \xi$ and $d\eta(X,Y) = g(X,\phi Y)$. We refer to (ϕ,ξ,η,g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold.*

An *almost contact structure*, (ϕ,ξ,η), consists of a field of endomorphisms ϕ, a vector field ξ and a 1-form η such that $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$ and an *almost contact metric structure* includes a Riemannian metric satisfying the compatibility condition $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$.

The product $M^{2n+1} \times \mathbb{R}$ carries a natural almost complex structure defined by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big)$$

and the underlying almost contact structure is said to be *normal* if J is integrable. The normality condition can be expressed as $N = 0$ where N is defined by

$$N(X,Y) = [\phi,\phi](X,Y) + 2d\eta(X,Y)\xi,$$

$[\phi,\phi]$ being the Nijenhuis tensor of ϕ.

A *Sasakian manifold* is a normal contact metric manifold. In terms of the covariant derivative of ϕ with respect to the Levi-Civita connection, the Sasakian condition is

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X.$$

As is well known, from this it is easily seen that

$$\nabla_X \xi = -\phi X$$

and in turn that ξ is a Killing vector field, i.e. the contact metric structure is *K-contact.* It is also well known that on a K-contact manifold the sectional curvature of all plane sections containing ξ are equal to $+1$ (see e.g. [2], p. 92).

A manifold admitting three almost contact structures, $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha = 1,2,3$, satisfying

$$\phi_\gamma = \phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta\phi_\alpha + \eta_\alpha \otimes \xi_\beta.$$

$$\xi_\gamma = \phi_\alpha\xi_\beta = -\phi_\beta\xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha$$

is said to have an *almost contact* 3*-structure.* Kuo [17] showed that given such a structure there exists a Riemannian metric g compatible with each of the three almost contact structures giving us an *almost contact metric* 3-*structure* $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. If each of the three structures is Sasakian we have a 3-*Sasakian structure*. A remarkable result of Kashiwada [15] is that if each of the three almost contact metric structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a contact metric structure, then the structure is a 3-Sasakian structure. There are many 3-Sasakian manifolds aside from the sphere S^{4n+3} including several homogeneous spaces; see e.g. [2] pp. 218-220 or the survey of Boyer and Galicki [5].

Using $\nabla_X \xi_\alpha = -\phi_\alpha X$ one readily obtains on a 3-Sasakian manifold that $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$. Thus the distribution $\mathcal{V}$ determined by the tri-vector $\xi = \xi_\alpha \wedge \xi_\beta \wedge \xi_\gamma$ is integrable with totally geodesic leaves. The horizontal distribution $\mathcal{H}$ is defined by $\eta_\alpha = 0$, $\alpha = 1, 2, 3$.

One of the main results of [3] is as follows.

Theorem 3.1 *The vertical distribution $\mathcal{V}$ on a compact* 3*-Sasakian manifold is a minima of the corrected energy $\mathcal{D}(V)$.*

4. Complex contact manifolds

A *complex contact manifold* is a complex manifold of odd complex dimension $2n+1$ together with an open covering $\{\mathcal{U}\}$ by coordinate neighborhoods such that

(1) On each $\mathcal{U}$, there is a holomorphic 1-form θ with $\theta \wedge (d\theta)^n \neq 0$.

(2) On $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$ there is a non-vanishing holomorphic function f such that $\theta' = f\theta$.

The complex contact structure determines a non-integrable distribution $\mathcal{H}$ by the equation $\theta = 0$. A complex contact structure is given by a global 1-form if and only if its first Chern class vanishes [4]. On the other hand let M be a Hermitian manifold with almost complex structure J, Hermitian metric g and an open covering by coordinate neighborhoods $\{\mathcal{U}\}$; M is called a *complex almost contact metric manifold* if it satisfies the following two conditions:

(1) In each $\mathcal{U}$ there exist 1-forms u and $v = u \circ J$, with dual vector fields U and $V = -JU$ and $(1,1)$ tensor fields G and $H = GJ$ such that

$$H^2 = G^2 = -I + u \otimes U + v \otimes V,\ GJ = -JG,\ GU = 0,$$

$$g(X, GY) = -g(GX, Y).$$

(2) On $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, we have

$$u' = au - bv, \quad v' = bu + av, \; G' = aG - bH, \quad H' = bG + aH$$

where a and b are functions on $\mathcal{U} \cap \mathcal{U}'$ with $a^2 + b^2 = 1$.

Since u and v are dual to the vector fields U and V, we easily see from the second condition that on $\mathcal{U} \cap \mathcal{U}'$, $U' = aU - bV$ and $V' = bU + aV$. Also since $a^2 + b^2 = 1$, $U' \wedge V' = U \wedge V$. Thus U and V determine a global vertical distribution $\mathcal{V}$ by $\xi = U \wedge V$ which is typically assumed to be integrable.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is $u - iv$ to within a non-vanishing complex-valued function multiple and the local tensor fields G and H are related to du and dv by

$$du(X,Y) = g(X,GY) + (\sigma \wedge v)(X,Y), \;\; dv(X,Y) = g(X,HY) - (\sigma \wedge u)(X,Y)$$

where $\sigma(X) = g(\nabla_X U, V)$, ∇ being the Levi-Civita connection of g (Ishihara and Konishi [14], Foreman [9]). We refer to a complex contact metric manifold with a complex almost contact metric structure satisfying these conditions as a *complex contact metric manifold.*

Ishihara and Konishi [12], [13] introduced a notion of normality for complex contact structures. Their notion is the vanishing of the two tensor fields S and T given by

$$\begin{aligned} S(X,Y) = {} & [G,G](X,Y) + 2g(X,GY)U - 2g(X,HY)V + 2(v(Y)HX \\ & - v(X)HY) + \sigma(GY)HX - \sigma(GX)HY \\ & + \sigma(X)GHY - \sigma(Y)GHX, \\ T(X,Y) = {} & [H,H](X,Y) - 2g(X,GY)U + 2g(X,HY)V + 2(u(Y)GX \\ & - u(X)GY) + \sigma(HX)GY - \sigma(HY)GX \\ & + \sigma(X)GHY - \sigma(Y)GHX. \end{aligned}$$

However this notion is too strong; among its implications is that the underlying Hermitian manifold (M,g) is Kähler. Thus while indeed one of the canonical examples of a complex contact manifold, the odd-dimensional complex projective space, is normal in this sense, the complex Heisenberg group, is not. In [16] B. Korkmaz generalized the notion of normality and we adopt her definition here. A complex contact metric structure is said to be *normal* if

$$S(X,Y) = T(X,Y) = 0, \text{ for every } X, Y \in \mathcal{H}$$

$$S(U,X)=T(V,X)=0, \text{ for every } X.$$

Even though the definition appears to depend on the special nature of U and V, it respects the change in overlaps, $\mathcal{U}\cap\mathcal{U}'$, and is therefore a global notion. With this notion of normality both odd-dimensional complex projective space and the complex Heisenberg group with their standard complex contact metric structures are normal.

One important consequence of normality for us is that the sectional curvature of a plane section spanned by a vector in $\mathcal{V}$ and a vector in $\mathcal{H}$ is equal to $+1$ (cf. Korkmaz [16]). Another consequence of normality is that

$$\nabla_X U = -GX + \sigma(X)V, \qquad \nabla_X V = -HX - \sigma(X)U. \tag{2}$$

Another important result of [3] is the following.

Theorem 4.1 *If M is a compact normal complex contact metric manifold, then the vertical distribution is a minima of the corrected energy, i.e.*

$$\mathcal{D}(V)=\int_M \sum_{i,\alpha} c_{i\alpha} d\text{vol}.$$

5. Complex Heisenberg group and the Iwasawa manifold

The complex Heisenberg group is the closed subgroup $H_{\mathbb{C}}$ of $GL(3,\mathbb{C})$ given by

$$H_{\mathbb{C}}=\left\{\begin{pmatrix}1 & z_2 & z_3\\ 0 & 1 & z_1\\ 0 & 0 & 1\end{pmatrix}\Big| z_1,z_2,z_3\in\mathbb{C}\right\}\cong\mathbb{C}^3.$$

As we have seen, a complex contact manifold admits a complex contact structure. Here $H_{\mathbb{C}}\cong\mathbb{C}^3$ and $\theta=\frac{1}{2}(dz_3-z_2dz_1)$ is global, so the structure tensors may be taken globally.With J denoting the standard almost complex structure on $\mathbb{C}^3$, we may give a complex almost contact structure to $H_{\mathbb{C}}$ as follows. Since θ is holomorphic, set $\theta=u-iv, v=u\circ J$; also set $4\frac{\partial}{\partial z_3}=U+iV$. Then, with respect to the metric g below,

$$u(X)=g(U,X), \qquad v(X)=g(V,X).$$

Since we will work in real coordinates, G and H are given by

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & y_2 & -x_2 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_2 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \end{bmatrix}.$$

Moreover relative to the coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ the Hermitian metric

$$g = \frac{1}{4}\begin{bmatrix} 1+x_2^2+y_2^2 & 0 & 0 & 0 & -x_2 & -y_2 \\ 0 & 1+x_2^2+y_2^2 & 0 & 0 & y_2 & -x_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -x_2 & y_2 & 0 & 0 & 1 & 0 \\ -y_2 & -x_2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In addition $\{e_1, e_{1^*}, e_2, e_{2^*}, e_3, e_{3^*}\}$ is an orthonormal basis where

$$\begin{aligned} e_1 &= 2\left(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + y_2\frac{\partial}{\partial y_3}\right), \quad e_{1^*} = 2\left(\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial y_3}\right), \\ e_2 &= 2\frac{\partial}{\partial x_2}, \quad e_{2^*} = 2\frac{\partial}{\partial y_2}, \quad e_3 = U = 2\frac{\partial}{\partial x_3}, \quad e_{3^*} = -V = 2\frac{\partial}{\partial y_3}. \end{aligned} \tag{3}$$

For the purpose of computation we give the Levi-Civita connection of g. For the Lie algebra of the Lie group $H_{\mathbb{C}}$ we have

$$[e_1, e_2] = -2e_3,\ [e_1, e_{2^*}] = -2e_{3^*},\ [e_{1^*}, e_2] = -2e_{3^*},\ [e_{1^*}, e_{2^*}] = 2e_3 \tag{4}$$

and the other Lie brackets are zero.The non-zero covariant derivatives of the basis elements are the following

$$\begin{aligned} \nabla_{e_2} e_3 = \nabla_{e_{2^*}} e_{3^*} &= -e_1, & \nabla_{e_{2^*}} e_3 &= -\nabla_{e_2} e_{3*} = e_{1^*}, \\ \nabla_{e_1} e_3 = \nabla_{e_{1^*}} e_{3^*} &= e_2, & \nabla_{e_1} e_{3^*} &= -\nabla_{e_{1^*}} e_3 = e_{2^*}, \\ -\nabla_{e_1} e_2 = \nabla_{e_{1^*}} e_{2^*} &= e_3, & \nabla_{e_1} e_{2^*} &= \nabla_{e_{1^*}} e_2 = -e_{3^*}. \end{aligned}$$

In [16] (see also [2] p.203) B. Korkmaz computed the covariant derivatives of G and H as

$$(\nabla_X G)Y = g(X,Y)U - u(Y)X - g(X,JY)V - v(Y)JX + 2v(X)GHY \tag{5}$$

$$(\nabla_X H)Y = g(X,Y)V - v(Y)X + g(X,JY)U + u(Y)JX - 2u(X)GHY. \tag{6}$$

In [1] and [2] the following are also listed for the complex Heisenberg group

$$g(\nabla_X U, V) = 0,\ \nabla_X U = -GX,\ \nabla_X V = -HX \tag{7}$$

Now let

$$\Gamma = \left\{ \begin{pmatrix} 1 & \gamma_2 & \gamma_3 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{pmatrix} \Big| \gamma_k = m_k + in_k, m_k, n_k \in \mathbb{Z} \right\};$$

Γ is a subgroup of $H_{\mathbb{C}} \cong \mathbb{C}^3$ and the $1-$ form $dz_3 - z_2 dz_1$ is invariant under the action on Γ. Hence the quotient $H_{\mathbb{C}}/\Gamma$ is a compact complex contact manifold with a global complex contact form. $H_{\mathbb{C}}/\Gamma$ is known as the *Iwasawa manifold* and it fibres over a complex torus $\mathbb{C}^2/\Gamma$ with $\xi = U \wedge V$ giving the vertical distribution $\mathcal{V}$. Moreover the integral submanifolds of $\mathcal{V}$ are tori $S^1 \times S^1$; this fibration is known as the Boothby-Wang fibration of $H_{\mathbb{C}}/\Gamma$ [10]. The Iwasawa manifold has no Kählerian structure, but it does have an indefinite Kählerian structure and it has symplectic forms.

The main results of [18] are summarized as follows.

Theorem 5.1 *The vertical distribution $\mathcal{V}$ of the complex contact structure on the Iwasawa manifold is a critical point of the energy.*

Theorem 5.2 *The Boothby-Wang fibration $S^1 \times S^1 \hookrightarrow H_{\mathbb{C}}/\Gamma \longrightarrow \mathbb{C}^2/\Gamma$ is an unstable critic point of the energy.*

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References

1. C. Baikoussis, D. E. Blair and F. Gouli-Andreou, *Holomorphic Legendre Curves in the Complex Heisenberg Group*, Bull. of the Inst. of Math. Aced. Sinica, **26**, (1988), 179-194.
2. D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, In: Progress in Mathematics, **203**, Birkhäuser, Boston, 2002.
3. D. E. Blair and A. Turgut Vanli, *Corrected Energy of Distributions for* 3-*Sasakian and Normal Complex Contact Manifolds*, Osaka J. Math. **43** (2006), 193–200.
4. W. M. Boothby, *Homogeneous complex contact manifolds*, Proc. Symp. Pure Math. III, Amer. Math. Soc., (1961), 144-154.

5. C. P. Boyer and K. Galicki, 3-*Sasakian manifolds*, In: Essays on Einstein Manifolds, International Press, Cambridge, MA, (1999), 123-184.
6. F. G. B. Brito, *Total bending of flows with mean curvature correction*, Diff. Geom. Appl. **12** (2000), 157-163.
7. P. M. Chacón and A. M. Naveira, *Corrected energy of distributions on Riemannian manifolds*, Osaka J. Math. **41** (2004), 97-105.
8. P. M. Chacón, A. M. Naveira and J. M. Weston , *On the energy of distributions with application to quaternionic Hopf fibrations*, Monatsh. Math. **133** (2001), 281-294.
9. B. Foreman, *Variational Problems on Complex Contact Manifolds with Applications to Twistor Space Theory*, Thesis, Michigan State University, 1996.
10. B. Foreman, *Boothby-Wang fibration on complex contact manifolds*, Diff. Geo. and its Appl. **13** (2000), 179-196.
11. O. Gil-Medrano, J. C. Gonzalez-Davila and L.Vanhecke, *Harmonicity and minimality of oriented distributions, Israel J. Math.*, **143**, (2004), 253-279.
12. S. Ishihara and M. Konishi, *Real contact* 3-*structure and complex contact structure*, Southeast Asian Bulletin of Math. **3** (1979), 151-161.
13. S. Ishihara and M. Konishi, *Complex almost contact manifolds*, Kōdai Math. J. **3** (1980), 385-396.
14. S. Ishihara and M. Konishi, *Complex almost contact structures in a complex contact manifold*, Kōdai Math. J. **5** (1982), 30-37.
15. T. Kashiwada, *On a contact* 3-*structure*, Math. Z. **238** (2001), 829-832.
16. B. Korkmaz, *Normality of complex contact manifolds*, Rocky Mountain J. Math., **30** (2000), 1343-1380.
17. Y. Y. Kuo, *On almost contact* 3-*structure*, Tôhoku Math. J. **22** (1970), 325-332.
18. A. Turgut Vanli and D. E. Blair, *The Boothby-Wang Fibration of the Iwasawa Manifold as a Critical Point of the Energy* , Monatsh. Math. **147** (2006), 75–84.
19. C. M. Wood, *On the energy of a unit vector field*, Geom. Dedicata **64** (1997), 319-330.