BRAID FAMILY REPRESENTATIVES *

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After defining reduced minimum braid word and criteria for a braid family representative, different braid family representatives are derived, and a correspondence between them and families of knots and links given in Conway notation is established.

1. Introduction

In the present article Conway notation [1,2,3,4] will be used without any additional explanation. A *braid-modified Conway notation* is introduced in Section 1, for a better understanding of the correspondence between braid family representatives (BFRs) and families of knots and links (KLs) given in Conway notation.

Minimum braids are defined, described, generated and presented in tables for knots up to ten crossings and oriented links up to nine crossings by T. Gittings [5]. T. Gittings used them for studying graph trees, amphicheirality, unknotting numbers and periodic tables of KLs.

Since knots are 1-component links, the term KL will be used for both knots and links.

* *MSC 2000:* 57M25.

Keywords: knot, link, braid, minimum braid.

† Work supported by the Ministry of Science under contract No *1646.*

‡ Work supported by the Ministry of Science under contract No *1646.*

In Section 2 we define a *reduced braid word*, describe general form for all reduced braid words with $s = 2$ strands, generate all braid family representatives of two-strand braids, and establish a correspondence between them and families of KLs given in Conway notation. In Section 3 we consider the same problem for $s \geq 3$. In Section 4 some applications of minimum braids [5] and braid family representatives are discussed. All computations are made using the knot-theory program *LinKnot* written by the authors [6], the extension of the program *Knot2000* by M. Ochiai and N. Imafuji [7].

2. Reduced Braid Words and Minimum Families of Braids with $s = 2$

We use the standard definition of a braid and description of minimum braids given by T. Gittings [5]. Instead of $a \dots a$, where a capital or lower case letter a appears p times, we write a^p; p is the degree of a ($p \in N$). It is also possible to work with negative powers, satisfying the relationships: $A^{-p} = a^p$, $a^{-p} = A^p$. A number of strands is denoted by s, and a length of a braid word by l.

The operation $a^2 \rightarrow a$ applied on any capital or lower case letter a is called *idempotency*. To every braid word we can apply the operation of idempotency until a reduced braid word is obtained.

Definition 2.1 A *reduced braid word* is a braid word with degree of every capital or lower case letter equal to 1.

By an opposite procedure, *braid word extension*, from every reduced braid word we obtain all braid words that can be derived from it by assigning a degree (that can be greater then 1) to every letter. In this case, a reduced braid word plays a role of a *generating braid word*.

A braid word with one or more parameters denoting degrees greater then one represents a *family of braid words*. If values of all parameters are equal 2, it will be called a *source braid*.

For the *minimality of reduced braids* we are using the following criteria:

(1) minimum number of braid crossings;
(2) minimum number of braid strands;
(3) minimum binary code for *alternating* braid crossings.

According to the first and second criterion minimal reduced braids are the shortest reduced braids with a smallest as possible number of different letters among all equivalent reduced braids representing certain KL. A binary code for any braid crossing can be generated by assigning a zero for an alternating, and a one for a non-alternating crossing. Hence, a priority will be given to alternating braids, and then to braids that differ from them as low as possible. Analogous minimality criteria can be applied to source braids.

Definition 2.2 Among the set of all braid families representing the same KL family, the *braid family representative* (MFB) is the one that has the following properties:

(1) minimum number of braid crossings;
(2) minimum *reduced braid*;
(3) minimum *source braid*.

These criteria are listed in descending order of importance for determining BFRs.

Our definition of BFRs results in some fundamental differences with regard to minimum braids, defined by T. Gittings [5]. Some members of BFRs will be minimum braids, but not necessarily.

For example, the minimum braid of the link .2 1 : 2 (9^3_{11} in Rolfsen [4]) is 9:03-05a $AAbACbACb$ [5,Table 2]. According to the second BFR criterion it will be derived from the generating minimum braid $AbAbACbC$ corresponding to the link .2 1 (8^2_{13}), and not from the non-minimum generating braid $AbACbACb$ corresponding to the same link. Hence, to the three-component link .2 1 : 2 (9^3_{11}) obtained as the first member of BFR AbA^pbACbC for $p = 2$ will correspond the braid $AbAAbACbC$, that is not a minimum braid according to the minimum braid criteria [5].

The third criterion: *minimum source braid* enables us to obtain KLs of a certain family from a single BFR, and not from several different BFRs. For example, applying this criterion, KLs .3.2.2 0, .2.3.2 0 and .2.2.3 0 belonging to the same KL family $.r.p.q\,0$ will be obtained from the single BFR $A^pbA^qbAb^r$. Otherwise, using the minimum braid criteria [5], the knot .3.2.2 0 will be obtained from the family $A^pbAb^qAb^r$, three-component link .2.3.2 0 will be obtained from $A^pbA^qbAb^r$, and the knot .2.2.3 0 will be obtained from $A^pbA^qb^rAb$ for $p = 3$, $q = 2$, $r = 2$. Source braids corresponding to the families $A^pbAb^qAb^r$, $A^pbA^qbAb^r$ and $A^pbA^qb^rAb$ are $A^2bAb^2Ab^2$, $A^2bA^2bAb^2$ and $A^2bA^2b^2Ab$, respectively, and the second source braid is minimal. Hence, the representative of the KL family $.r.p.q\,0$ is BFR $A^pbA^qbAb^r$.

According to this, to every BFR can be associated a single corresponding family of KLs given in Conway notation and *vice versa*.

An overlapping of KL families obtained from BFRs can occur only at their beginnings. For example, distinct BFRs AbA^pbACbC and $A^pbCbAbCb$, giving KL families $.2\,1 : p$ and $.p\,1 : 2$, respectively, for $p = 2$ will have as a joint member aforementioned three-component link $.2\,1 : 2$ (9^3_{11}). According to the second BFR criterion, it will be derived from the minimum generating braid $AbAbACbC$, and not from $AbACbACb$. Hence, BFR AbA^pbACbC giving KLs of the form $.2\,1 : p$ begins for $p = 2$, and $A^pbCbAbCb$ giving KLs of the form for $.p\,1 : 2$ begins for $p = 3$. In this way, all ambiguous cases can be solved.

Every KL is algebraic (if its basic polyhedron is 1^*) or polyhedral, so according to this criterion, all KLs are divided into two main categories: algebraic and polyhedral. Since to every member of a BFR corresponds a single KL, we can introduce the following definition:

Definition 2.3 An alternating BFR is *polyhedral iff* its corresponding KLs are polyhedral. Otherwise, it is *algebraic*. A non-alternating BFR is *polyhedral iff* its corresponding alternating BFR is polyhedral. Otherwise, it will be called *algebraic*.

The division of non-alternating BFRs into algebraic and polyhedral does not coincide with the division of the corresponding KLs [1,2,3], because minimum number of braid crossings is used as the first criterion for the BFRs. Accepting *minimum reduced braid universe* [5] as the first criterion, all KLs derived from the basic polyhedron .1 will be algebraic, because they can be represented by non-alternating minimal (but not minimum [5]) algebraic braids. E.g., the alternating knot $.2.2\,0$ (8_{16}) with the polyhedral braid A^2bA^2bAb can be represented as the algebraic knot $(-3,2)\,(3,-2)$ with the corresponding algebraic braid $A^3b^2a^2B^3$. In this case, to the knot 8_{16} corresponds algebraic braid $A^3b^2a^2B^3$ that reduces to $AbaB$, and not A^2bA^2bAb that reduces to $AbAbAb$.

Another solution of this discrepancy is changing the definition of an algebraic KL into the following:

Definition 2.4 KL is algebraic if it has an algebraic *minimum crossing number* representation.

In this case, all KLs derived from the basic polyhedron .1 (with Conway symbols beginning with a dot) will be polyhedral KLs, because their minimum crossing number representations are polyhedral.

We will consider only BFRs corresponding to prime KLs.

It is easy to conclude that every 1-strand BFR is of the form A^p, with the corresponding KL family p in Conway notation.

Theorem 2.1 *Every reduced BFR with $s = 2$ is of the form $(Ab)^n$, $n \geq 2$.*

This BFR corresponds to the knot 2 2 and to the family of basic polyhedra $.1 = 6^*$, 8^*, 10^*, 12^* (or 12A according to A. Caudron [3]), *etc.* For $n \geq 3$ all of them are n-antiprisms. Let us notice that the first member of this family, the knot 2 2, is not an exception: it is an antiprism with two diagonal bases.

Theorem 2.2 *All algebraic alternating KLs with $s = 2$ are the members of the following families:*

$p\,1\,2$ with the BFR A^pbAb ($p \geq 1$);
$p\,1\,1\,q$ with the BFR A^pbAb^q ($p \geq q \geq 2$);
$p, q, 2$ with the BFR A^pbA^qb ($p \geq q \geq 2$);
$p, q, r\,1$ with the BFR $A^pbA^qb^r$ ($r \geq 2, p \geq q \geq 2$);
$(p, r)\,(q, s)$ with the BFR $A^pb^qA^rb^s$
($p, q, r, s \geq 2, p \geq r, p \geq s, s \geq q$ and if $p = s$, then $r \geq q$).

Minimum braids include one additional braid ($A^pb^qAb^r$) in the case of algebraic alternating KLs with $s = 2$.

Alternating polyhedral KLs with $s = 2$ are given in the following table, each with its BFR. KLs in this table are given in "standard" Conway notation (that is "standardized" for knots with $n \leq 10$ and links with $n \leq 9$ crossings according to Rolfsen's book [4]). This table can be extended to an infinite list of antiprismatic basic polyhedra $(2n)^*$ described by the BFRs $(Ab)^n$, $n \geq 3$ and BFRs with $s = 2$ obtained as their extensions.

Table 1

Basic polyhedron $.1 = 6^*$

A^pbAbAb	$.p$	(1)	$A^pbAbA^qb^r$	$r : p\,0 : q\,0$	(7)
A^pbAbAb^q	$.p.q$	(2)	$A^pbAb^qA^rb^s$	$p.s.r.q$	(8)
A^pbA^qbAb	$.p.q\,0$	(3)	$A^pbA^qbA^rb^s$	$q\,0.p.r\,0.s\,0$	(9)
A^pbAb^qAb	$.p : q\,0$	(4)	$A^pbA^qb^rAb^s$	$.p.s.r\,0.q\,0$	(10)
$A^pbA^qbAb^r$	$.r.p.q\,0$	(5)	$A^pbA^qb^rA^sb^t$	$p.t.s.r.q$	(11)
$A^pbA^qbA^rb$	$p : q : r$	(6)	$A^pb^qA^rb^sA^tb^u$	$p.q.r.s.t.u$	(12)

If we apply minimum braid criteria [5], we need to add ten braids for the basic polyhedron $.1 = 6^*$: (1') $A^pbAb^qAb^r$, (2') $A^pbA^qb^rAb$, (3') $A^pb^qAbAb^r$, (4') $A^pbA^qb^rA^sb$, (5') $A^pb^qAbA^rb^s$, (6') $A^pb^qAb^rAb^s$, (7') $A^pb^qA^rbAb^s$, (8') $A^pb^qAb^rA^sb^t$, (9') $A^pb^qA^rbA^sb^t$, (10') $A^pb^qA^rb^sAb^t$. Applying BFR criteria, according to the minimum source braid criterion all KLs obtained from the braids (1') and (2') will be obtained from BFR (5), KLs obtained from (3') will be obtained from (7), KLs obtained from (4') and (6') will be obtained from (9), KLs obtained from (5') and (7') will be obtained from (8), and KLs obtained from (8'), (9') and (10') will be obtained from (11). Using minimum braid criteria [5], we need to make analogous additions to all classes of BFRs considered in this paper.

For the basic polyhedron 8^* we have:

Basic polyhedron 8^*

$A^pbAbAbAb$	8^*p	$A^pbA^qbAb^rAb^s$	$8^*p : q : .r : s$
$A^pbAbAbAb^q$	$8^*p.q$	$A^pbAb^qA^rbAb^s$	$8^*p.s : .r.q$
A^pbA^qbAbAb	$8^*p : q$	$A^pbA^qbA^rbA^sb$	$8^*p : s : r : q$
A^pbAbAb^qAb	$8^*p : .q$	$A^pbAbA^qb^rA^sb^t$	$8^*p.t.s.r.q$
A^pbAbA^qbAb	$8^*p :: q$	$A^pbA^qbAb^rA^sb^t$	$8^*p.t.s.r : .q$
$A^pbA^qbAbAb^r$	$8^*p.r :: .q$	$A^pbA^qb^rA^sbAb^t$	$8^*p : q.r.s : .t$
$A^pbAbA^qbAb^r$	$8^*p.r : .q$	$A^pbA^qbA^rbA^sb^t$	$8^*p.t.s : r : q$
$A^pbA^qbA^rbAb$	$8^*p : q : r$	$A^pbA^qbA^rb^sAb^t$	$8^*p.t : s.r : q$
$A^pbA^qbAb^rAb$	$8^*p : .r : .q$	$A^pbAb^qA^rb^sA^tb^u$	$8^*p.u.t.s.r.q$
$A^pbAbAbA^qb^r$	$8^*p.r.q$	$A^pbA^qbA^rb^sA^tb^u$	$8^*p.u.t.s.r : q$
$A^pbAbAb^qA^rb^s$	$8^*p.s.r.q$	$A^pbA^qb^rA^sbA^tb^u$	$8^*p.u.t : s.r.q$
$A^pbA^qbAbA^rb^s$	$8^*p.s.r :: q$	$A^pbA^qb^rA^sb^tAb^u$	$8^*p : q.r.s.t : u$
$A^pbAb^qA^rb^sAb$	$8^*p : .s.r.q$	$A^pbA^qb^rA^sb^tA^ub^v$	$8^*p.v.u.t.s.r.q$
$A^pbA^qb^rAbAb^s$	$8^*p.s :: r.q$	$A^pb^qA^rb^sA^tb^uA^vb^w$	$8^*p.q.r.s.t.u.v.w$
$A^pbA^qbA^rbAb^s$	$8^*p.s : .r : q$		

Trying to better understand the correspondence between BFRs and Conway symbols of KLs, we can introduce *modified Conway notation*. Most of KLs can be given in Conway notation by several different symbols (and this is the main disadvantage of Conway notation). In a similar way as with the classical notation, where every KL is given by its place in knot tables, we need to use some "standard" code, according to the notation introduced in the original Conway's paper [2] and in the papers or books following it [1,3,4]. For example, the same polyhedral knot $.p$ can be given by $..p$, $: p$, $: .p$, …, or even as 6^*p, $6^*.p$, $6^* : .p$, …

Working with BFRs we introduce a *braid-modified Conway notation* that will be more suitable for denoting KLs obtained from BFRs. We are trying to have a same degree p at the first position of a braid, and as the first element of Conway symbol corresponding to it. Whenever possible, the order of degrees will be preserved in the corresponding Conway symbol. By using this notation, we can recognize a very simple pattern for BFRs derived from the generating minimum braids of the form $(Ab)^n$: by denoting in a Conway symbol corresponding to a given braid every sequence of single letters of a length k by $k+1$ dots, we obtain the Conway symbol of a given braid. In order to recognize this pattern for KLs derived from basic polyhedra, first we need to use only one basic polyhedron 6^* with $n=6$ crossings, and not two of them (.1 and 6^*). In this case, the Table 1 will look as follows:

Basic polyhedron 6^*

A^pbAbAb	6^*p	$A^pbAbA^qb^r$	$6^*p\ ::\ q.r$
A^pbAbAb^q	$6^*p\ ::\ .q$	$A^pbAb^qA^rb^s$	$6^*p\ :\ .q.r.s$
A^pbA^qbAb	$6^*p\ :\ q$	$A^pbA^qbA^rb^s$	$6^*p : q : r.s$
A^pbAb^qAb	$6^*p\ :\ .q$	$A^pbA^qb^rAb^s$	$6^*p : q.r : s$
$A^pbA^qbAb^r$	$6^*p : q : .r$	$A^pbA^qb^rA^sb^t$	$6^*p : q.r.s.t$
$A^pbA^qbA^rb$	$6^*p : q : r$	$A^pb^qA^rb^sA^tb^u$	$6^*p.q.r.s.t.u$

and for the basic polyhedron 8^* we have:

Basic polyhedron 8^*

$A^pbAbAbAb$	8^*p	$A^pbA^qbAb^rAb^s$	$8^*p\ :\ q\ :\ .r\ :\ s$
$A^pbAbAbAb^q$	$8^*p\ :::\ .q$	$A^pbAb^qA^rbAb^s$	$8^*p\ :\ .q.r\ :\ .s$
A^pbA^qbAbAb	$8^*p\ :\ q$	$A^pbA^qbA^rbA^sb$	$8^*p\ :\ q\ :\ r\ :\ s$
A^pbAbAb^qAb	$8^*p\ ::\ .q$	$A^pbAbA^qb^rA^sb^t$	$8^*p\ ::\ q.r.s.t$
A^pbAbA^qbAb	$8^*p\ ::\ q$	$A^pbA^qbAb^rA^sb^t$	$8^*p\ :\ q\ :\ .r.s.t$
$A^pbA^qbAbAb^r$	$8^*p\ :\ q\ ::\ .r$	$A^pbA^qb^rA^sbAb^t$	$8^*p\ :\ q.r.s\ :\ .t$
$A^pbAbA^qbAb^r$	$8^*p\ ::\ q\ :\ .r$	$A^pbA^qbA^rbA^sb^t$	$8^*p\ :\ q\ :\ r\ :\ s.t$
$A^pbA^qbA^rbAb$	$8^*p\ :\ q\ :\ r$	$A^pbA^qbA^rb^sAb^t$	$8^*p\ :\ q\ :\ r.s\ :\ t$
$A^pbA^qbAb^rAb$	$8^*p\ :\ q\ :\ .r$	$A^pbAb^qA^rb^sA^tb^u$	$8^*p\ :\ .q.r.s.t.u$
$A^pbAbAbA^qb^r$	$8^*p\ :::\ q.r$	$A^pbA^qbA^rb^sA^tb^u$	$8^*p\ :\ q\ :\ r.s.t.u$
$A^pbAbAb^qA^rb^s$	$8^*p\ ::\ .q.r.s$	$A^pbA^qb^rA^sbA^tb^u$	$8^*p\ :\ q.r.s\ :\ t.u$
$A^pbA^qbAbA^rb^s$	$8^*p\ :\ q\ ::\ r.s$	$A^pbA^qb^rA^sb^tAb^u$	$8^*p\ :\ q.r.s.t\ :\ u$
$A^pbAb^qA^rb^sAb$	$8^*p\ :\ .q.r.s$	$A^pbA^qb^rA^sb^tA^ub^v$	$8^*p\ :\ q.r.s.t.u.v$
$A^pbA^qb^rAbAb^s$	$8^*p\ :\ q.r\ ::\ s$	$A^pb^qA^rb^sA^tb^uA^vb^w$	$8^*p.q.r.s.t.u.v.w$

$A^pbA^qbA^rbAb^s$ $8^*p:q:r:.s$

Unfortunately, it is not possible to express every family of KLs in the braid-modified Conway notation. Another problem is that it strongly differs from the standard Conway notation. Therefore, the braid-modified Conway notation is used only when after some slight modification standard Conway symbols remained completely understandable to a reader familiar with them.

In the same way, it is possible to continue with the derivation of BFRs from basic polyhedra with a higher number of crossings.

Hence, we conclude that:

Corollary All alternating KLs with $s=2$ are described by Theorem 2 and by an infinite extension of Table 1.

From alternating BFRs we obtain non-alternating BFRs by crossing changes. This way, from BFRs derived from the generating minimum braid $(Ab)^2$ we obtain the following families of non-alternating BFRs and corresponding new KL families:

A^pBaB	$(p\ -\ 1)\,3$	$A^pba^qb^r$	$p,(q-1)\,1,-(r+1)$
A^pba^qb	$p,(q-1)\,1,-2$	$A^pBA^qB^r$	$p,q,-r\,1$
A^pBA^qB	$p,q,-2$	$A^pB^qa^rB^s$	$(-p,r)\,(q,s)$
A^pBaB^q	$(p\ -\ 1)\,2\,q$	$A^pB^qA^rB^s$	$(p,r)\ -\ (q,s)$
$A^pB^qaB^r$	$p\ -\ 1,q,r+$		

In the same way, we can derive non-alternating BFRs with $s=2$ from the generating BFR $(Ab)^n$, $n\geq 3$.

3. Braid Family Representatives with $s\geq 3$

In order to continue derivation of BFRs and corresponding KLs for $s\geq 3$ first we derive all different reduced minimum braid words. It is possible to establish general construction rules for generating minimum braid words.

Definition 3.1 For a given generating minimum braid word $W=wL$ that ends with a capital or lower case letter L, a replacement of L by a word w_1 in W will be called *extending by replacement*. An addition of the word w_1 to W is *extending by addition*. The both operations are *extending operations*.

Definition 3.2 Let $W=wL_s$ and $w_1=L_{s+1}L_sL_{s+1}$ be generating minimum braids with s and $s+1$ strings, where L_s denotes sth letter and

L_{s+1} denotes $(s+1)$th letter. The word extending operations obtained this way will be called, respectively, $(s+1)$-*extending by replacement*, and $(s+1)$-*extending by addition*. The both operations are $(s+1)$-*extending operations*.

For example, the first operation applied on $AbAb$ gives $AbACbC$, and the other $AbAbCbC$.

The $(s+1)$-extending by replacement is sufficient for construction of generating minimum braids for a given s, with $l = 2s$, corresponding to KLs of the form $2\,\ldots\,2 = 2^s$, where 2 occurs s times. For $2 \leq s \leq 6$ as the result we obtain: $AbAb$, $AbACbC$, $AbACbdCd$, $AbACbdCEdE$, $AbACbdCEdfEf$ $\ldots$

The generating minimum braids for given s, with $l = 3s-2$, corresponding to KLs of the form $2\,1\,\ldots\,1\,2 = 2\,1^{3s-6}\,2$, where 1 occurs $3s-6$ times, can be obtained using only $(s+1)$-extension by addition. For $3 \leq s \leq 6$ we obtain: $AbAbCbC$, $AbAbCbCdCd$, $AbAbCbCdCdEdE$, $AbAbCbCdCdEdEfEf$ $\ldots$

Applying the same procedure, from A^3 we obtain the series A^3BaB, $A^3BaBCbC$, $A^3BaBCbCDcD$, $A^3BaBCbCDcDEdE$ $\ldots$, corresponding to the knots $3\,2$, $5\,2$, $7\,2$, $9\,2$ $\ldots$

Analogously, starting with $w_1 = AbAbCbdCd$ and using the $(s+1)$-extension by replacement, the generating minimum braids with $l = 2s+1$, corresponding to KLs of the form $2\,2\,1\,\ldots\,1\,2 = 2^2\,1^{2s-5}\,2$ are obtained for given s.

However, in order to exhaust all possibilities, all combinations of $(s+1)$-extending operations are used for derivation of reduced minimum braids.

Theorem 3.1 *Every generating algebraic minimum braid can be derived from AbAb by a recursive application of* $(s+1)$*-extending operations.*

The minimal generating braid words for $s \leq 5$ with their corresponding KLs are given in the following table:

$s=1$	$l=1$	A	1
$s=2$	$l=4$	$AbAb$	2 2
$s=3$	$l=6$	$AbACbC$	2 2 2
$s=3$	$l=7$	$AbAbCbC$	2 1 1 1 2
$s=4$	$l=8$	$AbACbdCd$	2 2 2 2
$s=4$	$l=9$	$AbAbCbdCd$	2 2 1 1 1 2
$s=4$	$l=10$	$AbAbCbCdCd$	2 1 1 1 1 1 1 2

$s = 5$	$l = 10$	$AbACbdCEdE$	2 2 2 2 2
$s = 5$	$l = 11$	$AbAbCbdCdEdE$	2 2 2 1 1 1 2
$s = 5$	$l = 11$	$AbACbCdCEdE$	2 2 1 1 1 2 2
$s = 5$	$l = 12$	$AbAbCbCdCEdE$	2 2 1 1 1 1 1 1 2
$s = 5$	$l = 12$	$AbAbCbdCdEdE$	2 1 1 1 2 1 1 1 2
$s = 5$	$l = 13$	$AbAbCbCdCdEcE$	2 1 1 1 1 1 1 1 1 1 2

In the case of polyhedral generating minimum braid words it is also possible to make generalizations. We have already considered the infinite class of generating polyhedral minimum braid words $(Ab)^n$ with $s = 2$. The first infinite class with $s = 3$ will be $(Ab)^{n-1}ACbC$, with the corresponding KLs of the form $(2n)^*2\,1\,0$.

Every BFR can be derived from a generating minimum braid by assigning a degree (that can be greater then 1) to every letter.

For $s = 3$ there are two generating alternating algebraic minimum braid words:

$AbACbC$, $l = 6$, with the corresponding link 2 2 2;
$AbAbCbC$, $l = 7$, with the corresponding knot 2 1 1 1 2,

that generate prime KLs.

From $AbACbC$ we derived 17 alternating BFRs and their corresponding families of KLs, given in the following table:

A^pbACbC	$p\,1\,2\,2$	$A^pbA^qCb^rC$	$(p,q)\,(r,2+)$
$AbACb^pC$	$p,2,2+$	$A^pb^qA^rCb^sC$	$(p,r)\,(q,2,s)$
A^pbACb^qC	$p\,1,q,2+$	$A^pb^qACb^rC^s$	$p\,1,q,s\,1,r$
A^pbA^qCbC	$p,q,2\,2$	$A^pbA^qCb^rC^s$	$(p,q)\,(r,s\,1+)$
A^pbACbC^q	$p\,1\,2\,1\,q$	$A^pbA^qC^rbC^s$	$(p,q)\,2\,(r,s)$
Ab^pACb^qC	$p,2,q,2$	$A^pb^qA^rCb^sC^t$	$(p,r)\,(q,t\,1,s)$
$A^pb^qACb^rC$	$p\,1,q,r,2$	$A^pbA^qC^rb^sC^t$	$(p,q),s,(t,r)+$
$A^pbACb^qC^r$	$p\,1,q,r\,1+$	$A^pb^qA^rC^sb^tC^u$	$(p,r),q,(u,s),t$
$A^pbA^qCbC^r$	$p,q,r\,1\,2$		

The next generating alternating algebraic minimum braid $AbAbCbC$ of the length 7, with $s = 3$, gives the following results:

$A^pbAbCbC$	$p\,1\,1\,1\,1\,2$	$A^pb^qAbCb^rC^s$	$(p\,1,q)\,1\,(s\,1,r)$
$AbAbCb^pC$	$p,2\,1\,1,2$	$A^pbA^qb^rCb^sC$	$(p,q)\,1\,r\,(2,s)$
$AbAb^pCbC$	$2\,1\,p\,1\,2$	$A^pbA^qb^rCbC^s$	$p,q,s\,1\,1\,r\,1$
A^pb^qAbCbC	$p\,1,q,2\,1\,1$	$A^pbA^qbC^rbC^s$	$(p,q)\,1\,1\,1\,(r,s)$

A^pbA^qbCbC	$p,q,2\,1\,1\,1$	$A^pbA^qbCb^rC^s$	$(p,q)\,1\,1\,(r,s\,1)$
A^pbAb^qCbC	$p\,1\,1\,q\,1\,2$	$A^pbAb^qCb^rC^s$	$p\,1\,1\,q,r,s\,1$
$A^pbAbCbC^q$	$p\,1\,1\,1\,1\,1\,q$	$A^pbAbC^qb^rC^s$	$(p\,1\,1\,1,r)\,(q,s)$
$AbAb^pCb^qC$	$2\,1\,p,q,2$	$A^pb^qA^rbCb^sC$	$(p,r),q,(2,s)\,1$
Ab^pAbCb^qC	$(p,2)\,1\,(q,2)$	$A^pb^qAb^rCb^sC$	$(p\,1,q)\,r\,(2,s)$
$A^pb^qA^rbCbC$	$(p,r)\,(q,2\,1\,1)$	$A^pb^qA^rbCb^sC^t$	$(p,r),q,(t\,1,s)\,1$
$A^pb^qAb^rCbC$	$p\,1,q,2\,1\,r$	$A^pb^qAb^rCb^sC^t$	$(p\,1,q)\,r\,(t\,1,s)$
$A^pb^qAbCb^rC$	$(p\,1,q)\,1\,(r,2)$	$A^pbA^qb^rC^sbC^t$	$(p,q)\,1\,r\,1\,(s,t)$
$A^pbA^qbCb^rC$	$(p,q)\,1\,1\,(2,r)$	$A^pbA^qb^rCb^sC^t$	$(p,q)\,1\,r\,(s,t\,1)$
$A^pbA^qb^rCbC$	$p,q,2\,1\,r\,1$	$A^pbA^qbC^rb^sC^t$	$(p,q)\,1\,1,(t,r),s$
$A^pbA^qbCbC^r$	$p,q,r\,1\,1\,1\,1$	$A^pbAb^qC^rb^sC^t$	$(p\,1\,1\,q,s)\,(r,t)$
$A^pbAb^qCb^rC$	$p\,1\,1\,q,r,2$	$A^pb^qA^rb^sCb^tC$	$(p,r),q,(t,2)\,s$
$A^pbAb^qCbC^r$	$p\,1\,1\,q\,1\,1\,r$	$A^pb^qA^rb^sCb^tC^u$	$((p,r),q)\,s\,(u\,1,t)$
$A^pbAbCb^qC^r$	$p\,1\,1\,1,q,r\,1$	$A^pb^qA^rbC^sb^tC^u$	$((p,r),q)\,1\,((u,s),t)$
$Ab^pAb^qCb^rC$	$(p,2)\,q\,(r,2)$	$A^pbA^qb^rC^sb^tC^u$	$(p,q)\,1\,r\,((u,s),t)$
$A^pb^qA^rb^sCbC$	$(p,r)\,(q,2\,1\,s)$	$A^pb^qA^rb^sC^tb^uC^v$	$((p,r),q)\,s\,((u,t),v)$

Except $AbACbC$ and $AbAbCbC$, all generating minimum braids with $s=3$ are polyhedral.

For $s=3$ and $l \leq 12$, the polyhedral generating braids and their corresponding KLs are given in the following table, with the notation for basic polyhedra with 12 crossings according to A. Caudron [3]:

$l=8$	$AbAbACbC$	.2 1	$l=11$	$AbAbAbAbCbC$	8*2 1 1
$l=8$	$AbCbAbCb$	.2 : 2	$l=11$	$AbAbAbCbCbC$	11***
			$l=11$	$AbAbACbAbCb$	10**.2 0
$l=9$	$AbAbCbAbC$	8*2 0	$l=11$	$AbAbACbACbC$	11**
$l=9$	$AbAbAbCbC$	.2 1 1	$l=11$	$AbAbCbACbCb$	11*
$l=9$	$AbACbACbC$	9*			
			$l=12$	$AbAbAbAbACbC$	10*2 1 0
$l=10$	$AbAbAbCbCb$	.2 1 2	$l=12$	$AbAbAbACbCbC$	12I
$l=10$	$AbAbAbACbC$	8*2 1 0	$l=12$	$AbAbAbCbACbC$	12F
$l=10$	$AbAbACbAbC$	9*.2	$l=12$	$AbAbACbAbCbC$	12H
$l=10$	$AbAbCbAbCb$	9*2	$l=12$	$AbAbCbAbACbC$	12G
$l=10$	$AbAbACbCbC$	10***	$l=12$	$AbAbCbAbCbCb$	12D
$l=10$	$AbAbCbACbC$	10**	$l=12$	$AbCbAbCbAbCb$	12C

From them, BFRs without duplications are derived. E.g., for $l=8$, the generating minimum braid .2 1 gives 70 BFRs, and .2 : 2 gives 19 BFRs. Overlapping of those families can occur only if all parameters are equal 2, i.e., for source braids and source KLs corresponding to them. According to the minimality criteria, all those source braids will belong to the first BFR. The generating minimum braid $.2\,1$ gives the following BFRs:

$A^pbAbACbC$	$.2\,1.p\,0$	$AbA^pbAC^qbC^r$	$.p:(q,r)1$
AbA^pbACbC	$.2\,1:p$	$AbA^pb^qACb^rC$	$.p.q.(2,r)$
$AbAbACb^pC$	$.(p,2)$	$AbA^pb^qACbC^r$	$.r\,1\ 1.q.p$
$AbAbACbC^p$	$.p\,1\,1$	$AbAb^pACb^qC^r$	$.p.(r\,1,q)$
$AbAb^pACbC$	$.2\,1.p$	$AbAb^pAC^qbC^r$	$.p.(r,q)\,1$
A^pbA^qbACbC	$.2\,1.p\,0.q$	$Ab^pA^qb^rACbC$	$2\,1.p.r\,0.q\,0$
A^pbAbA^qCbC	$2\,1\,0:p\,0:q\,0$	$Ab^pAb^qACb^rC$	$p:q:(2,r)\,0$
$A^pbAbACb^qC$	$.(2,q).p\,0$	$Ab^pAb^qACbC^r$	$q:p:r\,1\,1\,0$
$A^pbAbACbC^q$	$.q\,1\,1.p\,0$	$A^pbA^qbA^rCb^sC$	$(2,s).p\,0.r.q\,0$
A^pbAb^qACbC	$.q.2\,1.p\,0$	$A^pbA^qbA^rCbC^s$	$s\,1\,1\,0:r\,0.q.p\,0$
$A^pb^qAbACbC$	$2\,1:p:q\,0$	$A^pbA^qbACb^rC^s$	$.(s\,1,r).p\,0.q$
AbA^pbACb^qC	$.(2,q):p$	$A^pbA^qbAC^rbC^s$	$(s,r)\,1:p.q\,0$
AbA^pbACbC^q	$.q\,1\,1:p$	$A^pbA^qb^rA^sCbC$	$q.r.s.2\,1\,0.p$
AbA^pb^qACbC	$.2\,1.q.p$	$A^pbA^qb^rACb^sC$	$.(s,2).r.q.p\,0$
$AbAbACb^pC^q$	$.(q\,1,p)$	$A^pbA^qb^rACbC^s$	$.s\,1\,1.r.q.p\,0$
$AbAbAC^pbC^q$	$.(q,p)\,1$	$A^pbAbA^qCb^rC^s$	$(r,s\,1)\,0:q\,0:p\,0$
$AbAb^pACb^qC$	$.(q,2).p$	$A^pbAbA^qC^rbC^s$	$(s,r)\,1\,0:q\,0:p\,0$
$AbAb^pACbC^q$	$.q\,1\,1.p$	$A^pbAb^qA^rCb^sC$	$q.(2,s).r\,0.p$
Ab^pAb^qACbC	$p:q:2\,1\,0$	$A^pbAb^qA^rCbC^s$	$q.s\,1\,1.r\,0.p$
$A^pbA^qbA^rCbC$	$.2\,1.p\,0.q:r$	$A^pbAb^qACb^rC^s$	$.q.(s\,1,r).p\,0$
$A^pbA^qbACb^rC$	$.q.p\,0.(r,2)$	$A^pbAb^qAC^rbC^s$	$.q.(s,r)\,1.p\,0$
$A^pbA^qbACbC^r$	$.r\,1\,1.p\,0.q$	$A^pb^qA^rbACb^sC$	$r.q.p.(2,s)\,0$
$A^pbA^qb^rACbC$	$.2\,1.r.q.p\,0$	$A^pb^qA^rbACbC^s$	$r.q.p.s\,1\,1\,0$
$A^pbAbA^qCb^rC$	$p\,0:q\,0:(r,2)\,0$	$A^pb^qA^rb^sACbC$	$s.r.q.p.210$
$A^pbAbA^qCbC^r$	$r\,1\,1\,0:q\,0:p\,0$	$A^pb^qAbACb^rC^s$	$(s\,1,r):p:q\,0$
$A^pbAbACb^qC^r$	$.p.(r\,1,q)\,0$	$A^pb^qAbAC^rbC^s$	$(s,r)\,1:p:q\,0$
$A^pbAbAC^qbC^r$	$.p.(r,q)\,1\,0$	$A^pb^qAb^rA^sCbC$	$s.r.q.p.2\,1\,0$
$A^pbAb^qA^rCbC$	$q.2\,1.r\,0.p$	$A^pb^qAb^rACb^sC$	$p\,0.(s,2).q.r\,0$
$A^pbAb^qACb^rC$	$.q.(2,r).p\,0$	$A^pb^qAb^rACbC^s$	$p\,0.s\,1\,1.q.r\,0$
$A^pbAb^qACbC^r$	$.q.r\,1\,1.p\,0$	$AbA^pb^qACb^rC^s$	$.(r,s\,1).q.p$
$A^pb^qA^rbACbC$	$r.q.p.2\,1\,0$	$AbA^pb^qAC^rbC^s$	$.(s,r)\,1.q.p$
$A^pb^qAbACb^rC$	$p:(r,2):q\,0$	$Ab^pA^qb^rACb^sC$	$(s,2).p.r\,0.q\,0$
$A^pb^qAbACbC^r$	$r\,1\,1:p:q\,0$	$Ab^pA^qb^rACbC^s$	$s\,1\,1.r.p\,0.q\,0$
$A^pb^qAb^rACbC$	$p\,0.2\,1.q.r0$	$Ab^pAb^qACb^rC^s$	$q:p:(s\,1,r)\,0$
$AbA^pbACb^qC^r$	$.p:(r\,1,q)$	$Ab^pAb^qAC^rbC^s$	$p:q:(r,s)\,1\,0$

The generating minimum braid $.2:2$ gives the following BFRs:

$A^pbCbAbCb$	$.p\,1:2$	$A^pb^qCbAbC^rb$	$.r\,1.q\,0.p\,1$
$Ab^pCbAbCb$	$.2.p\,0.2$	$A^pb^qCb^rAbCb$	$2.q\,0.r.p\,1\,0$
$A^pb^qCbAbCb$	$.p\,1.q\,0.2$	$A^pbCbA^qbC^rb$	$.r\,1:(p,q)$
A^pbCbA^qbCb	$.(p,q):2$	$Ab^pCbAb^qCb^r$	$2.p\,0.r.2\,0.q$
$A^pbCbAbC^qb$	$.p\,1:q\,1$	$A^pb^qCbA^rb^sCb$	$.(p,r).s\,0.2.q\,0$
Ab^pCbAb^qCb	$.p.2\,0.q.2\,0$	$A^pb^qCbA^rbC^sb$	$.s\,1.q\,0.(p,r)$
$Ab^pCbAbCb^q$	$2.p\,0.q.2\,0$	$A^pb^qCbA^rbCb^s$	$(r,p).q\,0.s.2\,0$

$A^p b^q CbA^r bCb$	$.2.q\,0.(p,r)$	$A^p b^q CbAb^r C^s b$	$.p\,1.r\,0.s\,1.q\,0$
$A^p b^q CbAb^r Cb$	$.p\,1.r\,0.2.q\,0$	$A^p b^q CbAbC^r b^s$	$p\,1.s\,0.q.r\,1\,0$
$A^p b^q CbAbCb^r$	$p\,1.q\,0.r.2\,0$		

In the same way, from all generating minimum braid words with $s = 3$, it is possible to derive alternating and non-alternating BFRs and their corresponding families of KLs.

As the example, the following table contains non-alternating BFRs with at most two parameters, derived from the minimum reduced braid $AbACbC$:

$A^p BacBc$	$(p-1)\,3\,2$	$A^p ba^q CbC$	$p, 2\,2, -q$
$A^p BacBc^q$	$(p-1)\,3\,1\,q$	$A^p BA^q cBc$	$p, 2\,1\,1, -(q-1)\,1$
$A^p bACB^q C$	$p\,1, (q-1)\,1, 2$	$Ab^p ACB^q C$	$p, 2, 2, -q$
$A^p BacB^q c$	$p-1, q, 2++$	$Ab^p AcB^q c$	$p, 2, -q, -2$
$A^p bACB^q C$	$p\,1, (q-1)\,1, 2$	$Ab^p Acb^q c$	$p, q, 2, -2$
$A^p bACbc^q$	$p\,1\,3\,(q-1)$	$AB^p ACB^q C$	$p, q, -2, -2$
$A^p BacBC^q$	$(p-1)\,4\,(q-1)$	$AB^p ACb^q C$	$p, 2, q, -2$

From the generating minimum braid word $W = (Ab)^n$ $(n \geq 2)$, that defines the family of basic polyhedra $(2n)^*$, by word extension $w_1 = CbACbC$ we obtain the second family of basic polyhedra 9^* ($AbACbACbC$), 10^{**} ($AbAbCbACbC$), 11^{**} ($AbAbACbACbC$), 12F ($AbAbAbCbACbC$), *etc.*

The third family of basic polyhedra 10^{***} ($AbAbACbCbC$), 11^{***} ($AbAbAbCbCbC$), 12I ($AbAbAbACbCbC$), *etc.*, is derived from $W = (Ab)^n$ $(n \geq 3)$ for $w_1 = CbCbC$.

In the same way, for $W = (Ab)^n$ $(n \geq 1)$, $w_1 = CbAbCbAbCb$ the family of basic polyhedra beginning with 12C ($AbCbAbCbAbCb$) is obtained;

for $W = (Ab)^n$ $(n \geq 2)$, $w_1 = CbAbCbCb$ the family of basic polyhedra beginning with 12D ($AbAbCbAbCbCb$) is obtained;

for $W = (Ab)^n$ $(n \geq 2)$, $w_1 = CbAbACbC$ the family of basic polyhedra beginning with 12G ($AbAbCbAbACbC$) is obtained;

for $W = (Ab)^n$ $(n \geq 2)$, $w_1 = CbAbCbC$ the family of basic polyhedra beginning with 12H ($AbAbACbAbCbC$) is obtained, *etc.*

Among them it is possible to distinguish subfamilies obtained using extensions by replacing or by adding.

Theorem 3.2 *For $s = 4$ generating algebraic minimum braids are:*

$AbACbdCd$, $l = 8$, with the corresponding link $2\,2\,2\,2$,

$AbAbCbdCd$, $l = 9$, with the corresponding link $2\,2\,1\,1\,1\,2$,

AbAbCbCdCd, $l = 10$, *with the corresponding knot* 2 1 1 1 1 1 1 2.

All other generating minimum braid words with $s = 4$ *are polyhedral.*

For $s = 4$ and $l \leq 12$, the polyhedral generating braids and their corresponding KLs are given in the following table, with the notation for basic polyhedra with 12 crossings according to A. Caudron [3]:

$l = 10$	*AbAbACbdCd*	.2 2 1	$l = 12$	*AbAbACbdCdCd*	12J
$l = 10$	*AbACbCbdCd*	.2 1.2 1	$l = 12$	*AbAbACdCbCdC*	11*** : .2 0
$l = 10$	*AbACbdCbdC*	.2 1 : 2 1 0	$l = 12$	*AbAbCbAbdCbd*	9*2 2
$l = 10$	*AbACdCbCdC*	.22 : 2	$l = 12$	*AbAbCbCdCbCd*	8*2 1 1 :: 2 0
			$l = 12$	*AbAbCbdCbCdC*	8*2 1 1 0 : .2 0
$l = 11$	*AbAbACbCdCd*	.2 1 1 1 1	$l = 12$	*AbAbCbdCbdCd*	9*2 1 1
$l = 11$	*AbAbCbCbdCd*	.2 1 1.2 1 0	$l = 12$	*AbAbCdCbCdCd*	8*2 1 1 1 0
$l = 11$	*AbAbCbdCbdC*	.2 1 1 : 2 1	$l = 12$	*AbACbAdCbdCd*	12L
$l = 11$	*AbAbCdCbCdC*	.2 1 1 1 : 2	$l = 12$	*AbACbCbCbdCd*	8*2 1 0.2 1 0
$l = 11$	*AbACbACbdCd*	9*2 1 0	$l = 12$	*AbACbCbdCbCd*	9*.2 1 : .2
$l = 11$	*AbACbCdCbCd*	8*2 1 0 :: 2 0	$l = 12$	*AbACbCbdCbdC*	8*2 1 0 : .2 1 0
$l = 11$	*AbACbCdCdCd*	.2 2 1 1	$l = 12$	*AbACbCdCbCdC*	9*2 1 : 2
$l = 11$	*AbACbdCbCdC*	8*2 1 : .2 0	$l = 12$	*AbACbCdCbdCd*	10** : 2 1 0
$l = 11$	*AbACdCbCdCd*	8*2 2 0	$l = 12$	*AbACbdCbCdCd*	10**.2 1
			$l = 12$	*AbACbdCbdCdC*	10** : 2 1
$l = 12$	*AbAbAbACbdCd*	8*2 2 1 0	$l = 12$	*AbCbAbCdCbCd*	10** : 20 :: .2 0
$l = 12$	*AbAbACbAbdCd*	9*.2 2	$l = 12$	*AbCbACbdCbCd*	10**2 0 :: .2 0

For $W = (Ab)^n$ $(n \geq 2)$, $w_1 = ACbdCdCd$ the family of basic polyhedra beginning with 12J ($AbAbACbdCdCd$) is obtained, and for $W = (Ab)^n$ $(n \geq 1)$, $w_1 = ACbAdCbdCd$ the family of basic polyhedra beginning with 12L ($AbACbAdCbdCd$) is obtained.

4. Applications of Minimum Braids and *BFR*s

4.1. *Graph Trees*

A rational KL in Conway notation is any sequence of natural numbers not beginning or ending with 1, where each sequence is identified with its inverse. From this definition is computed the number of rational KLs with n crossings. It is given by the formula

$$2^{n-4} + 2^{[n/2]-2}$$

that holds for every $n \geq 4$. This very simple formula is derived first by C. Ernst and D.W. Sumners in another form [8], and later independently by S. Jablan [9,10]. For $n \geq 4$ we can compute the first 20 numbers of this sequence. After prepending to it the first three numbers 1 for $n = 1, 2, 3$, the result is the sequence: 1, 1, 1, 2, 3, 6, 10, 20, 36, 72, 136, 272, 528, 1056, 1080, 4160, 8256, 16512, 32986, 65792, 131328, 262656, 524800, ... This sequence is included in *On-Line Encyclopedia of Integer Sequences* (http://www.research.att.com/~njas/sequences/) as the sequence $A005418$. The number of rational knots with n crossings ($n \geq 3$) is given by the formula

$$\frac{2^{n-3} + 2^{[\frac{n}{2}]-2^{(n-1)} \pmod 2} + (-1)^{(n-1)[\frac{n}{2}]} \pmod 2}{3}$$

so we can simply derive the formula for the number of rational links with n crossings as well.

A graph-theoretical approach to knot theory is proposed by A. Caudron [3]. T. Gittings established a mapping between minimum braids with s strands and trees with $s+1$ vertices and conjectured that the number of graph trees of n vertices with alternating minimum braids is equal to the number of rational KLs with n crossings [5, Conjecture 1].

4.2. *Amphicheiral KLs*

KL is *achiral* (or *amphicheiral*) if its "left" and "right" forms are equivalent, meaning that one can be transformed to the other by an ambient isotopy. If an oriented knot or link L can be represented by an antisymmetrical vertex-bicolored graph on a sphere, whose vertices with the sign $+1$ are white, and vertices with the sign -1 are black, it is achiral. In this case, for an oriented knot or link L there exists an antisymmetry (sign-changing symmetry) switching orientations of vertices, i.e., mutually exchanging vertices with the signs $+1$ and -1 [9,10]. In the language of braid words, this means that its corresponding braid word is antisymmetric (or palindromic): there exist a mirror antisymmetry transforming one letter to another and *vice versa* and changing their case (i.e., transforming capital to lower case letters and *vice versa*). For example, the reduced braid words $Ab|Ab$ or $ABac|BDcd$ are palindromic, where the anti-mirror is denoted by $|$. Hence, we believe that the origin of all oriented achiral KLs are palindromic reduced braids.

Conjecture An oriented KL is achiral *iff* it can be obtained from a palindromic reduced braid by a symmetric assigning of degrees.

For $s=2$ all alternating BFRs are of the form $(Ab)^n$ ($n \geq 2$), defining a series of the basic polyhedra $(2n)^*$, beginning with $2\,2$, $.1 = 6^*$, 8^*, 10^*, 12^*, *etc.* All of them are achiral KLs, representing a source of other achiral KLs. From 4:1-01 $AbAb$ ($2\,2$ or 4_1) by a symmetric assigning of degrees we can derive achiral alternating knots with $n \leq 10$ crossings: 6:1-02 A^2bAb^2 (2 1 1 2 or 6_3), 8:1-05 A^3bAb^3 (3 1 1 3 or 8_9), 10:1-017 $A^3b^2A^2b^3$ ((3, 2) (3, 2) or 10_{79}), and one achiral alternating link with $n \leq 9$ crossings: 8:3-05a $A^2b^2A^2b^2$ ((2, 2) (2, 2) or 8^3_4), *etc.* In general, from $AbAb$ the following families of achiral alternating KLs are derived:

A^pbAb^p	$p\,1\,1\,p$	$A^pb^qA^qb^p$	$(p,q)\,(p,q)$

Borromean rings 6:3-02 $AbAbAb$ (.1 = 6^* or 6^3_2) are the origin of achiral alternating knots 8:1-07 A^2bAbAb^2 (.2.2 or 8_{17}), 10:1-020 $A^2bA^2b^2Ab^2$ (.2.2.2 0.2 0 or 10_{99}), 10:1-022 $A^2b^2AbA^2b^2$ (2.2.2.2 or 10_{109}), and of the link 8:3-04a Ab^2AbA^2b (.2 : 2 0 or 8^3_6), *etc.* In general, from $AbAbAb$ the following families of achiral alternating KLs are derived:

A^pbAbAb^p	$.p.p$	$A^pbA^qb^qAb^p$	$.p.p.q\,0.q\,0$
Ab^pAbA^pb	$.p : p\,0$	$A^pb^qA^rb^rA^qb^p$	$p.q.r.r.q.p$
$A^pb^qAbA^qb^p$	$p.q.q.p$		

Achiral basic polyhedron $AbAbAbAb$ (8^*) is the origin of the following families of alternating achiral KLs:

$A^pbAbAbAb^p$	$8^*p.p$	$A^pbAb^qA^qbAb^p$	$8^*p.q : .q.p$
AbA^pbAb^pAb	$8^*p : .p$	$A^pb^qA^rbAb^rA^qb^p$	$8^*p.q.r.r.q.p$
$A^pb^qAbAbA^qb^p$	$8^*p.q.q.p$	$A^pbA^qb^rA^rb^qAb^p$	$8^*p.q.q.p : r.r$
$A^pbA^qbAb^qAb^p$	$8^*.p : q.q : p$		

In the same way it is possible to derive achiral alternating KLs from all achiral basic polyhedra $(Ab)^n$ for $n \geq 5$.

From the antisymmetry condition it follows that every palindromic braid has an even number of strands. For $s = 4$ and $l \leq 12$ palindromic algebraic generating braids are:

$AbACbdCd$, $l = 8$ with the corresponding achiral link 2 2 2 2,

$AbAbCbCdCd$, $l = 10$, with the corresponding achiral knot 2 1 1 1 1 1 1 2.

The palindromic polyhedral generating braids are:

$AbACbCbdCd$, $l = 10$, with the corresponding achiral knot .2 1.2 1,

$AbAbACbdCdCd$, $l = 12$, with the corresponding achiral link 12J,

$AbACbAdCbdCd$, $l = 12$, with the corresponding achiral knot 12L,

$AbACbCbCbdCd$, $l = 12$, with the corresponding achiral link 8^*2 1 0.2 1 0,

$AbCbAbCdCbCd$, $l = 12$, with the corresponding achiral knot 10^{**} : 20 :: .2 0,

$AbCbACbdCbCd$, $l = 12$, with the corresponding achiral knot 10^{**}2 0 :: .2 0.

From the generating braid $AbACbdCd$ following families of alternating achiral KLs are derived:

$A^pbACbdCd^p$	$p\,1\,2\,2\,1\,p$	$A^pbA^qC^rb^rd^qCd^p$	$(((p,q),r)+)\,(((p,q),r)+)$
$AbAC^pb^pdCd$	$(p,2+)\,(p,2+)$	$A^pb^qAC^rb^rdC^qd^p$	$(q,p\,1,r)\,(q,p\,1,r)$
$Ab^pAC^qb^qdC^pd$	$(p,q,2)\,(p,q,2)$	$A^pb^qA^rC^sb^sd^rC^qd^p$	$(q,(p,r),s)\,(q,(p,r),s)$

From the same palindromic non-alternating generating braid the following families of achiral KLs are obtained:

$A^pBacBDcd^p$	$p\,p$	$A^pBacBDcd^p$	$2\,p\,p\,2$
$AbAc^pB^pdCd$	$(p,2)\,(q,2)$	$A^pbAc^qB^qdCd^p$	$(p\,1,q)\,(p\,1,q)$

In the same way is possible to continue the derivation of achiral KLs from other palindromic reduced braids.

The family of achiral odd crossing number knots discovered by J. Hoste, M. Thistlethwaite and J. Weeks in 1998 [11] can be extended to the two-parameter BFR defined by the palindromic braid $ABaB^qC^pBAdcb^pc^qDcd$ corresponding to the family of non-alternating achiral odd-crossing knots with $n=7+4p+4q$ crossings

$$10^{**}(-2p)\,0.-1.-2\,0.(2q):(-2p)\,0.-1.-2\,0.(2q).$$

4.3. *Unlinking Numbers and Unlinking Gap*

T. Gittings [5] noticed that it might be possible to calculate unlinking numbers from minimum braids. Unfortunately, this is true only for KLs with $n\leq 10$ crossings, including the link $4\,1\,4$ (9^2_4) and the Nakanishi-Bleiler example $5\,1\,4$ (10_8) with an unlinking gap [12].

Definition 4.1 The *minimum braid unlinking gap* is the positive difference between the unlinking number obtained from a minimum braid $u_B(L)$ and unlinking number $u(L)$ of a link L, i.e.,

$$\delta_B = u_B(L) - u(L) > 0.$$

The unlinking gap [12] for minimum braids appears for $n=11$. The following alternating links given in Conway notation, followed by their minimum braids have the minimum braid unlinking gap:

$.5.2$	A^5bAbAb^2	$8^*3.2$	$A^3bAbAbAb^2$
$.3.4$	A^4bAbAb^3	$8^*3:2$	A^3bA^2bAbAb
8^*4	$A^4bAbAbAb$	$8^*2.2:.2$	$A^2bAbA^2bAb^2$
$.2.3.3\,0$	$A^3bA^3bAb^2$	10^*2	$A^2bAbAbAbAb$

For the links .5.2, .3.4 the value of minimum braid unlinking gap is $\delta_B = 2$, and for other links from this list $\delta_B = 1$. Hence, minimum braid unlinking number is different from the unlinking number and represents a new KL invariant.

4.4. *Periodic Tables of KLs*

Periodic tables of KLs can be established in three ways: starting with families of KLs given in Conway notation [9,10,13], with minimum braids [5], or with BFRs. Since we have established correspondence between BFRs and KLs in Conway notation, it follows that the same patterns (with regard to all KL polynomial invariants and KL properties) will appear in all cases. For example, for every family of KLs is possible to obtain a general formula for Alexander polynomials, with coefficients expressed by numbers denoting tangles in Conway symbols, or from their corresponding parameters from minimum braids or from BFRs. The same holds not only for KL polynomials, but for all other properties of KLs: writhe, amphicheirality, number of projections, unlinking number, signature, periods, *etc.* [9,10,13].

5. Acknowledgements

We would like to express our gratitude to Thomas Gittings for his critical reading of the manuscript, corrections, advice and suggestions.

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