

# SOME RECENT APPROACHES IN 4-DIMENSIONAL SURGERY THEORY \*

FRIEDRICH HEGENBARTH

*Department of Mathematics, University of Milano, Via C. Saldini 50,  
02130 Milano, Italy*

*E-mail: friedrich.hegenbarth@mat.unimi.it*

DUŠAN REPOVŠ †

*Institute of Mathematics, Physics and Mechanics, University of Ljubljana,  
P. O. Box 2964, 1001 Ljubljana, Slovenia*

*E-mail: dusan.repovs@fmf.uni-lj.si*

It is well-known that an  $n$ -dimensional Poincaré complex  $X^n$ ,  $n \geq 5$ , has the homotopy type of a compact topological  $n$ -manifold if the total surgery obstruction  $s(X^n)$  vanishes. The present paper discusses recent attempts to prove analogous result in dimension 4. We begin by reviewing the necessary algebraic and controlled surgery theory. Next, we discuss the key idea of Quinn's approach. Finally, we present some cases of special fundamental groups, due to the authors and to Yamasaki.

## 1. Introduction

Classical surgery methods of Browder–Novikov–Wall break down in dimension 4. The Wall groups, depending only on the fundamental group, do not seem to be strong enough as obstruction groups to completing the surgery. It is strongly believed that for free nonabelian fundamental groups of rank  $r \geq 2$  Wall groups are not sufficient [3]. Nevertheless, one can make progress using controlled surgery theory to produce controlled embeddings of 2-spheres needed for surgery, by using results of Quinn [16]. However, this works only if the control map satisfies the  $UV^1$ -condition (in fact, one

---

\* *MSC 2000*: 57R67, 57P99, 55N45.

*Keywords*: Poincaré complex, chain duality, controlled topology, controlled surgery, homology manifold.

† Work supported by the Slovenian Research Agency grant No. P1-0292-0101-04.

needs the  $UV^1(\delta)$ -condition, for sufficiently small  $\delta > 0$ ).

A map  $p : X \rightarrow B$  of  $X$  into a metric space  $B$  (with metric  $d$ ) satisfies the  $UV^1$ -condition if it is proper and surjective and if for every  $\delta > 0$  and every map  $\alpha : K \rightarrow B$  of any 2-complex  $K$  into  $B$  with a lifting  $\alpha_0 : L \rightarrow X$ , defined on a subcomplex  $L$  of  $K$ , there is a map  $\bar{\alpha} : K \rightarrow X$  such that (i)  $\bar{\alpha}|_L = \alpha_0$ , and (ii)  $d(p \circ \bar{\alpha}(x), \alpha(x)) < \delta$ , for all  $x \in K$ .

In our applications,  $X$  and  $B$  will be compact and connected, with chosen base points. In particular, the  $UV^1$ -condition implies that  $p$  induces an isomorphism of fundamental groups  $p_* : \pi = \pi_1(X) \rightarrow \pi_1(B)$ .

The obstructions belong to a controlled Wall group. Its construction is conceptual, done by means of Ranicki's machinery [10], [11], [20]–[25], [27], [29], [30]. The most important fact is that the controlled groups are homology groups, so they can in principle be calculated. The following is a basic result in surgery theory in dimension  $n \geq 5$ . It is due to Ranicki [20], see §2 below:

**Theorem 1.1.** *Suppose that  $X^n$  is a Poincaré  $n$ -complex,  $n \geq 5$ , with total surgery obstruction  $s(X^n) = 0$ . Then  $X^n$  is (simple) homotopy equivalent to a topological  $n$ -manifold.*

Quinn [19] proposed in 2004 a strategy to extend Theorem 1.1 to dimension  $n = 4$ . More precisely, he suggested how to prove the following conjecture:

**Conjecture 1.2.** *If  $X^4$  is a Poincaré 4-complex with  $s(X^4) = 0$ , then  $X^4$  is (simple) homotopy equivalent to an ANR homology 4-manifold.*

We shall outline the idea of Quinn's approach in §4, after having prepared the necessary preliminaries. We shall also prove a *stable* version of Conjecture 1.2:

**Theorem 1.3.** *If  $X^4$  is a Poincaré 4-complex with  $s(X^4) = 0$ , then  $X^4 \# \underset{1}{\#} S^2 \times S^2$  is (simple) homotopy equivalent to a topological 4-manifold.*

In the rest of §4 we shall prove special cases of Conjecture 1.2, when  $\pi_1(X^4)$  is free nonabelian. We begin by establishing some notations and results which are needed for our presentation. We acknowledge the referee for comments and suggestions.

## 2. Notations and basic results of algebraic surgery

Let  $\Lambda$  be some ring with anti-involution. Here we will only consider  $\Lambda = \mathbb{Z}[\pi_1]$ , the integral group ring of a fundamental group of a space with trivial orientation character. An  $n$ -quadratic chain complex is a pair  $(C_\#, \psi_\#)$ , where  $C_\#$  is a free  $\Lambda$ -module chain complex and  $\psi_\# = \{\psi_s \mid s = 0, 1, \dots\}$  is a collection of  $\Lambda$ -homomorphisms  $\psi_s : C^{n-r-s} \rightarrow C_r$  satisfying certain relations. Here  $C^\#$  denotes the  $\Lambda$ -dual cochain complex of  $C$  (as in [32]). The pair is a quadratic  $n$ -Poincaré complex if  $\psi_0 + \psi_0^\# : C^{n-\#} \rightarrow C_\#$  is a chain equivalence, where  $\psi_0^\#$  is the dual map of  $\psi_0$ .

There is the notion of  $(n+1)$ -quadratic (Poincaré) pairs, hence of *cobordism* between  $n$ -quadratic Poincaré complexes, which is an equivalence relation. Let  $L_n(\Lambda)$  be the set of equivalence classes of  $n$ -quadratic Poincaré complexes. It has a group structure induced by direct sum constructions.

If  $n = 2k$  then particular examples of quadratic Poincaré complexes are given by surgery kernels  $[K_k(f), \Lambda, \mu]$  of degree 1 normal maps  $(f, b) : M^n \rightarrow X^n$  (see [21], [22], [23], [26]). This gives an isomorphism of  $L_{2k}(\Lambda)$  with Wall groups  $L_{2k}(\pi_1)$ . There is also an isomorphism in odd dimensions in terms of formations. There is an  $\Omega$ -spectrum  $\mathbb{L}$  with  $\pi_n(\mathbb{L}) = L_n(\{1\})$  and with zeroth space  $\mathbb{L}_0 = G/TOP \times \mathbb{Z}$  (see [12]). The groups  $L_n(\{1\})$  were calculated in [8]. We denote by  $\dot{\mathbb{L}} \rightarrow \mathbb{L}$  the connected covering spectrum, so  $\pi_n(\dot{\mathbb{L}}) = \pi_n(G/TOP)$ .

If  $K$  is a simplicial complex, elements  $\xi \in H_n(K, \dot{\mathbb{L}})$  can be represented by equivalence classes of compatible collections of  $n$ -quadratic Poincaré complexes  $\{(C_\#(\sigma), \psi_\#(\sigma)) \mid \sigma \in K\}$ . Gluing these individual quadratic complexes together gives a *global*  $n$ -quadratic Poincaré complex  $(C_\#, \psi_\#)$  hence an element in  $L_n(\Lambda)$ . There results a homomorphism  $A : H_n(K, \dot{\mathbb{L}}) \rightarrow L_n(\pi_1(K))$ , called the assembly map. To define the structure set  $\mathbb{S}_{n+1}(K)$ , one considers compatible collections  $\{(C_\#(\sigma), \psi_\#(\sigma)) \mid \sigma \in K\}$  of  $n$ -quadratic Poincaré complexes which are *globally* contractible, i.e. the mapping cone of the map  $\psi_0 + \psi_0^\# : C^{n-\#} \rightarrow C_\#$  is contractible (see [24]).

Cobordism classes of such objects build the set  $\mathbb{S}_{n+1}(K)$ . It has a group structure arising from direct sum.

**Theorem 2.1.** *The assembly map fits into the exact sequence*

$$\begin{aligned} \cdots \rightarrow L_{n+1}(\pi_1(K)) \xrightarrow{\partial} \mathbb{S}_{n+1}(K) \rightarrow H_n(K, \dot{\mathbb{L}}) \xrightarrow{A} \\ \xrightarrow{A} L_n(\pi_1(K)) \xrightarrow{\partial} \mathbb{S}_n(K) \rightarrow \cdots \end{aligned}$$

For any geometric Poincaré duality complex  $X^n$  of dimension  $n$ , Ranicki defined its total surgery obstruction  $s(X^n) \in \mathbb{S}_n(X^n)$  (see [20]).

**Theorem 2.2.** *If  $n \geq 5$  then  $s(X^n) = 0$  if and only if  $X^n$  is (simple) homotopy equivalent to a topological  $n$ -manifold.*

The element  $s(X)$  can be described as follows: Suppose that  $X$  is triangulated. The fundamental cycle  $[X] \in C_n(X)$  defines a simple chain equivalence

$$-\cap [X]: C^{n-\#}(\tilde{X}) \rightarrow C_{\#}(\tilde{X}), \quad \tilde{X} \rightarrow X$$

is the universal covering, in particular the desuspension of the algebraic mapping cone  $S^{-1}C(-\cap[X])$  of  $-\cap[X]$  is contractible as a  $\Lambda$ -chain complex.

Let  $\sigma^*$  be the dual cell of the simplex  $\sigma \in X$  with respect of its barycentric subdivision. The global fundamental cycle  $[X]$  defines local cycles  $[X(\sigma)] \in C_{n-|\sigma|}(\sigma^*, \partial\sigma^*)$ , hence it maps

$$-\cap [X(\sigma)]: C^{n-r-|\sigma|}(\sigma^*) \rightarrow C_r(\sigma^*, \partial\sigma^*).$$

The collection

$$\{D_{\#}(\sigma) = S^{-1}C(-\cap [X(\sigma)]) \mid \sigma \in X\}$$

assembles to  $D = S^{-1}C(-\cap[X])$ . There are  $(n-1-|\sigma|)$ -quadratic Poincaré structures  $\psi_{\#}(\sigma)$  on  $D(\sigma)$ , giving rise to an  $(n-1)$ -quadratic structure on  $D$ . Then  $s(X)$  is represented by the class of the compatible collection  $\{(D_{\#}(\sigma), \psi_{\#}(\sigma)) \mid \sigma \in X\}$ .

### 3. Controlled $L$ -groups of geometric quadratic complexes and the controlled surgery sequence

Geometric modules were introduced by Quinn [14], [15], [17], [18] (see also [28]). We introduce here the simple version that locates bases at points in a control space  $K$  over  $B$ , and morphisms without incorporating paths.

Let  $K$  be a space,  $p: K \rightarrow B$  a (continuous) map to a finite-dimensional compact metric ANR space. We assume this already here since the surgery sequence of [10] requires these properties. Let  $d: B \times B \rightarrow \mathbb{R}$  be a metric. Let  $\Lambda$  be a ring with involution and  $1 \in \Lambda$ , for example,  $\Lambda = \mathbb{Z}[\pi]$  is a group ring with involution coming from  $g \rightarrow g^{-1}$ , for  $g \in \pi$ .

A geometric module over  $K$  is a free  $\Lambda$ -module  $M = \Lambda[S]$ ,  $S$  a basis together with a map  $\varphi: S \rightarrow K$ . It is required that for any  $x \in K$ ,  $\varphi^{-1}(x) \subset S$  is finite. A morphism  $f: M = \Lambda[S] \rightarrow N = \Lambda[T]$  is a collection  $f_{st}: M_s \rightarrow N_t$

where  $M_s = \Lambda[s]$ ,  $N_t = \Lambda[t]$  such that for a fixed  $s \in S$  only finitely many  $f_{st} \neq 0$ . The dual is  $M^* = \Lambda^*[S]$ , where  $\Lambda^* = \text{Hom}_\Lambda(\Lambda, \Lambda)$ , so it is essentially the same. However, if  $f: M \rightarrow N$  is a geometric morphism, its dual is a geometric morphism  $f^*: N^* \rightarrow M^*$ .

Composition of geometric modules is defined in the obvious way. We define the radius of  $f$  by  $\text{rad}(f) = \max\{d(p\varphi(s), p\psi(t)) \mid f_{st} \neq 0\}$ . Here  $\psi: T \rightarrow K$  belongs to the geometric module  $N$ . The map  $f$  is an  $\varepsilon$ -morphism if  $\text{rad}(f) < \varepsilon$ . The composition of an  $\varepsilon$ - and a  $\delta$ -morphism is an  $(\varepsilon + \delta)$ -morphism. The sum of an  $\varepsilon$ - and a  $\delta$ -morphism is a  $\max\{\varepsilon, \delta\}$ -morphism. See [28] for further properties.

Chain complexes of geometric modules are then defined as pairs  $(C_\#, \partial_\#)$  where  $\partial_n$  are geometric morphisms.  $(C_\#, \partial_\#)$  is an  $\varepsilon$ -chain complex if all  $\partial_n$  are  $\varepsilon$ -morphisms. A chain map  $f: (C_\#, \partial_\#) \rightarrow (C'_\#, \partial'_\#)$  of geometric chain complexes is a  $\delta$ -chain equivalence if there is a  $\delta$ -chain map  $g: (C'_\#, \partial'_\#) \rightarrow (C_\#, \partial_\#)$  and chain homotopies  $\{h_n: C_n \rightarrow C_{n+1}\}$ ,  $\{h'_n: C'_n \rightarrow C'_{n+1}\}$  of  $g \circ f$  and  $f \circ g$  with  $\text{rad } h_n < \delta$  and  $\text{rad } h'_n < \delta$ . The composition of a  $\delta$ -chain equivalence with a  $\delta'$ -chain equivalence is a  $(\delta + \delta')$ -chain equivalence. Note that the dual  $f^*$  of  $f$  has the same radius.

A chain equivalence is  $\varepsilon$ -contractible if it is  $\varepsilon$ -chain equivalent to the zero complex. If  $f$  is an  $\varepsilon$ -chain equivalence then its mapping cone is  $3\varepsilon$ -contractible. An  $\varepsilon$ -mapping  $f$  is a  $2\varepsilon$ -equivalence if the mapping cone is  $\varepsilon$ -contractible. A pair  $(C_\#, \psi_\#)$  is called an  $\varepsilon$ -quadratic geometric  $\Lambda$ -module complex if the maps  $\psi_s: C^{n-r-s} \rightarrow C_r$  have radius  $< \varepsilon$ . A pair  $(C_\#, \psi_\#)$  is  $\varepsilon$ -Poincaré if the mapping cone  $\psi_0 + \psi_0^\#$  is  $4\varepsilon$ -contractible. An  $(n+1)$ -dimensional  $\varepsilon$ -quadratic pair  $(C_\# \rightarrow D_\#, \psi_\#, \delta\psi_\#)$  is a quadratic pair such that  $\delta\psi_s: D^{n+1-r-s} \rightarrow D_r$  and  $\psi_s: C^{n-r-s} \rightarrow C_r$  have radius  $< \varepsilon$ . It is  $\varepsilon$ -Poincaré if  $C(f)^{n+1-r} = D^{n+1-r} \oplus C^{n-r} \rightarrow D_r$ , given by  $(\delta\psi_0 + \delta\psi_0^\#, f \circ (\psi_0 + \psi_0^\#))$  has  $4\varepsilon$ -contractible algebraic mapping cone (note that it is a  $2\varepsilon$ -chain map).

Let  $\delta \geq \varepsilon > 0$ . Then  $L_n(p: K \rightarrow B, \varepsilon, \delta)$  is the set of equivalence classes of  $n$ -dimensional  $\varepsilon$ -quadratic  $\varepsilon$ -Poincaré  $\Lambda$ -chain complexes on  $p: K \rightarrow B$ . The equivalence relation is generated by  $\delta$ -bordism defined in the obvious way. Actually, Theorem 3.1 below implies that  $\delta$ -bordism is an equivalence relation if  $\delta$  is sufficiently small (see Ch.4 of [15]) for analogous results. The set  $L_n(p: K \rightarrow B, \varepsilon, \delta)$  has a natural abelian group structure given by direct sums. As defined above, these  $\varepsilon$ - $\delta$ - $L$ -groups seem to not be calculable.

However, there are deep results identifying these groups with homology groups in certain spectra [10], [11], [29]. These spectra were constructed by

Quinn in [15] (see also [30]). Here is a special case, due to Pedersen–Yamasaki [11] and Ranicki–Yamasaki [29], which is most useful in 4–dimensional surgery:

**Theorem 3.1.** *Consider  $\Lambda = \mathbb{Z}$ . Suppose that  $p: K \rightarrow B$  is a fibration with simply connected fibers. Then for all  $n \geq 0$ ,  $L_n(p: K \rightarrow B, \varepsilon, \delta) \cong H_n(B, \mathbb{L})$ , for sufficiently small  $\varepsilon$  and  $\delta$ .*

Recall that  $\mathbb{L}$  is the 4–periodic (nonconnected) surgery spectrum with  $\pi_n(\mathbb{L}) = L_n(\{1\})$ , if  $n \geq 0$ . The  $\Omega$ –spectrum  $\mathbb{L} = \{L_k \mid k \in \mathbb{Z}\}$  is algebraically defined in Ch.14 of [24]. However, the controlled surgery obstruction depends only on the  $L_k$  with  $k > -1$ , as can easily be deduced from the Atiyah–Hirzebruch spectral sequence. A particular case is  $p = \text{Id}: B \rightarrow B$ , proved by Pedersen–Quinn–Ranicki [10] (an alternative proof was given by Ferry [2]):

**Theorem 3.2.** *Let  $B$  be as above. Then there exists an (assembly) isomorphism  $L_n(B, \varepsilon, \delta) \cong H_n(B, \mathbb{L})$ .*

The following is the main result of controlled surgery theory with locally trivial fundamental groups (see [10]). By results of Quinn (see Ch.3.2 of [16]) it also holds in dimension 4:

**Theorem 3.3.** *Suppose that  $B$  is as above. Then there is  $\varepsilon_0 > 0$  such that for any  $\varepsilon_0 > \varepsilon > 0$  there is  $\delta > 0$  with the following property: If  $X^n$  is a  $\delta$ –Poincaré complex with respect to a  $UV^1(\delta)$  map  $p: X^n \rightarrow B$  and  $n \geq 4$ , then there is a controlled surgery exact sequence*

$$H_{n+1}(B, \mathbb{L}) \rightarrow \mathcal{S}_{\varepsilon, \delta}(X^n) \rightarrow [X^n, G/_{TOP}] \xrightarrow{\Theta} H_n(B, \mathbb{L}).$$

Here we must additionally assume, that there is a *TOP* reduction of  $\nu_X$ . Recall that  $\mathcal{S}_{\varepsilon, \delta}(X)$  consists of pairs  $(M, f)$  where  $M^n$  is an  $n$ –manifold,  $f: M \rightarrow X$  a  $\delta$ –homotopy equivalence over  $p: X \rightarrow B$ , modulo the equivalence relation:  $(M, f) \sim (M', f')$  if there is a homeomorphism  $h: M \rightarrow M'$  such that  $f$  and  $f' \circ h$  are  $\varepsilon$ –homotopic over  $B$ . To define the Wall realization map  $H_{n+1}(B, \mathbb{L}) \rightarrow \mathbb{S}_{\varepsilon, \delta}(X)$ , one needs  $\mathbb{S}_{\varepsilon, \delta}(X) \neq \emptyset$ .

**Remark 3.1.** Ranicki and Yamasaki worked out, in a conceptual way, the controlled surgery obstruction, using a controlled version of the quadratic construction [27].

**Summary.** Consider a surgery problem  $(f, b): M \rightarrow X$  with control map  $p: X \rightarrow B$ . If  $X$  is a  $\delta$ –Poincaré complex for sufficiently small  $\delta > 0$

over  $B$ , then one can construct as above the controlled surgery obstruction belonging to  $L_n(B, \varepsilon, \delta)$ . If  $p$  is additionally  $UV^1(\delta)$  for sufficiently small  $\delta > 0$ , then the controlled surgery sequence holds.

#### 4. Some conclusions and comments

In this section we present Quinn's approach and then we consider Poincaré 4-complexes with free fundamental groups. We mentioned in the introduction Ranicki's main result in high-dimensional surgery theory: If  $X^n$  is a Poincaré  $n$ -complex with vanishing total surgery obstruction  $s(X^n)$ , then  $X^n$  is (simple) homotopy equivalent to topological  $n$ -manifold  $M^n$ , provided that  $n \geq 5$ . One of the main objectives is to extend this result to dimension 4.

Here are the key ideas of Quinn's approach [19]: Let  $X^4$  be a 4-dimensional Poincaré complex.

- (1) We investigate the algebraic surgery sequence explained in §2.

$$\cdots \rightarrow L_4(\pi_1(X^4)) \rightarrow \mathbb{S}_4(X^4) \rightarrow H_3(X^4, \mathbb{L}) \rightarrow \cdots$$

with  $s(X^4) \in \mathbb{S}_4(X^4)$ .

- (2) We consider the image of  $s(X^4)$  under the composite map

$$\mathbb{S}_4(X^4) \rightarrow H_3(X^4, \mathbb{L}) \rightarrow H_3(X^4, \mathbb{L})$$

and use the identification (§3)

$$H_3(X^4, \mathbb{L}) \cong L_3(X^4, \varepsilon, \delta), \forall \varepsilon < \varepsilon_0.$$

Thus  $s(X^4)$  determines an element  $[s(X^4)] \in L_3(X^4, \varepsilon, \delta)$ , i.e.  $(D_\#, \psi_\#) = (S^{-1}C(- \cap [X^4]), \psi_\#)$  as described in §2, carries an  $\varepsilon$ -quadratic Poincaré structure, unique up to  $\delta$ -bordism.

- (3) If  $[s(X^4)] = 0$ , there is a  $\delta$ -null-bordism  $(D_\#, \psi_\#) \rightarrow (E_\#, \delta\psi_\#)$ . In fact, since we have assumed  $s(X^4) = 0$ ,  $E_\#$  is contractible.
- (4) This bordism can be topologically realized by a  $\delta'$ -homotopy equivalence  $X'^4 \rightarrow X^4$ , where  $X'^4$  is an  $\varepsilon'$ -Poincaré 4-complex. Here,  $(\delta', \varepsilon')$  depends on  $(\delta, \varepsilon)$ , and becomes arbitrary small as  $(\delta, \varepsilon)$  becomes small. Ideas of surgery on Poincaré and normal spaces are used here (see [13]).
- (5) Choose a sequence  $\{\varepsilon_n\} \rightarrow 0$  and iterate the above construction to produce a sequence  $\{X_n'^4 \rightarrow X_{n-1}^4\}_n$ . Its limit in the sense of [1] is an ANR homology 4-manifold  $X'^4$  which is homotopy equivalent to  $X^4$ .

This approach can be summarized as follows: Suppose  $X^4$  is a Poincaré 4-complex with  $s(X^4) = 0$ . Then  $X^4$  is (simple) homotopy equivalent to an ANR homology 4-manifold  $X'^4$ .

**Remark 4.1.** The topological realization step (4) requires a highly  $\delta$ -connected null bordism  $(E_\#, \delta\psi_\#)$ , which is *not* guaranteed when  $n$  is even.

**Remark 4.2.** Starting with a relative Poincaré complex  $(X^4, \partial X^4)$  such that  $\partial X^4$  is a topological 4-manifold, the 4-dimensional resolution theorem [16] implies that  $X'^4$  is a topological 4-manifold.

For the rest of this section we consider Poincaré 4-complexes  $X^4$  with free nonabelian fundamental groups, i.e.  $\pi_1(X^4) \cong \frac{p}{*} \mathbb{Z}$ . We benefit from the special topology of such complexes, in particular:

**Theorem 4.1.**

- (a)  $X^4$  is (simple) homotopy equivalent to  $\{\underset{1}{\vee}^p(S^1 \vee S^3) \vee (\underset{1}{\vee}^q S^2)\} \cup_\varphi D^4$ ;  
and
- (b) If the  $\Lambda$ -intersection form

$$\lambda_\Lambda: H_2(X^4, \Lambda) \times H_2(X^4, \Lambda) \rightarrow \Lambda$$

is induced by the  $\mathbb{Z}$ -intersection form

$$\lambda_\mathbb{Z}: H_2(X^4, \mathbb{Z}) \times H_2(X^4, \mathbb{Z}) \rightarrow \mathbb{Z}$$

by extension of scalars  $\mathbb{Z} \rightarrow \Lambda$ , then  $X^4$  is (simple) homotopy equivalent to  $Q^4 \# M'^4$ , where  $Q^4 = \frac{p}{\#} (S^1 \times S^3)$ , and  $M'^4$  is a simply connected topological 4-manifold determined by  $\lambda_\mathbb{Z}$ .

For proofs see [4], [6], [7], [9]. We note here that the first Postnikov invariant for  $X^4$  vanishes. Theorem 4.1 implies (what is much easier to see) that there is a degree 1 map  $p: X^4 \rightarrow Q^4$ .

**Lemma 4.2.** *The assembly maps satisfy the following properties:*

- (a)  $A: H_4(X^4, \dot{\mathbb{L}}) \rightarrow L_4(\pi_1(X^4))$  is onto; and
- (b)  $A: H_3(X^4, \dot{\mathbb{L}}) \rightarrow L_3(\pi_1(X^4))$  is injective.

**Proof.** Assembly is a natural construction so we have the commutative diagram



$$\begin{array}{ccc}
 H_l(X, \dot{\mathbb{L}}) & \xrightarrow{p_*} & H_l(Q, \dot{\mathbb{L}}) \\
 \searrow A & & \downarrow A \\
 & & L_l(\pi_1(X))
 \end{array}$$

A spectral sequence argument shows that  $p_*: H_4(X^4, \dot{\mathbb{L}}) \rightarrow H_4(Q^4, \dot{\mathbb{L}})$  is onto, and  $p_*: H_3(X^4, \dot{\mathbb{L}}) \rightarrow H_3(Q^4, \dot{\mathbb{L}})$  is an isomorphism. If  $B = B(\pi_1(X^4))$  is the classifying space,  $c: Q^4 \rightarrow B$  the classifying map, then  $c_*: H_l(Q^4, \dot{\mathbb{L}}) \rightarrow H_l(B, \dot{\mathbb{L}})$  is an isomorphism for  $l = 3, 4$  by similar arguments. However, for free fundamental groups,  $A: H_l(B, \dot{\mathbb{L}}) \rightarrow L_l(\pi_1(X))$  is an isomorphism. This proves the lemma.

**Corollary 4.3.** *If  $X^4$  is a Poincaré 4-complex, then  $s(X^4)$  is zero. Moreover, the algebraic structure set  $\mathbb{S}_4(X^4)$  is trivial.*

**Proof.** This follows from Lemma 4.2 and the algebraic surgery sequence

$$\begin{aligned}
 &\rightarrow H_4(X^4, \dot{\mathbb{L}}) \xrightarrow{A} L_4(\pi_1(X^4)) \rightarrow \mathbb{S}_4(X^4) \rightarrow \\
 &\rightarrow H_3(X^4, \dot{\mathbb{L}}) \xrightarrow{A} L_3(\pi_1(X^4)) \rightarrow \dots
 \end{aligned}$$

By the discussion above it is plausible to conjecture:

**Conjecture 4.4.** *Any Poincaré 4-complex  $X^4$ , such that  $\pi_1(X^4) \cong \begin{smallmatrix} p \\ * \\ 1 \end{smallmatrix} \mathbb{Z}$ , is (simple) homotopy equivalent to an ANR homology 4-manifold.*

Part (b) of the above theorem confirms Conjecture 4.4 for the case when  $\lambda_\Lambda$  is extended from  $\lambda_{\mathbb{Z}}$ . Indeed, in this case  $X^4$  is homotopically a manifold. In general case we obtain a *stable* result:

**Corollary 4.5.** *If  $X^4$  has a free nonabelian fundamental group, then  $X^4 \# \begin{smallmatrix} r \\ \# \\ 1 \end{smallmatrix} S^2 \times S^2$  is (simple) homotopy equivalent to a topological 4-manifold.*

**Proof.** Since  $s(X^4) = 0$ , there is a degree 1 normal map  $(f, b): M^4 \rightarrow X^4$  whose Wall obstruction is zero. This means that  $(K_2(f), \lambda, \mu)$  is stably hyperbolic. The result then follows from [6].

**Remark 4.3.**  $X^4 \# \begin{smallmatrix} r \\ \# \\ 1 \end{smallmatrix} S^2 \times S^2$  is the connected sum made inside a 4-cell in  $X^4$ .

The controlled surgery method also works for certain other fundamental groups. We have proved this for those Poincaré complexes whose fundamental group is that of a torus knot [5]:

**Theorem 4.6.** *Let  $X^4$  be a 4-dimensional Poincaré complex such that  $\pi_1(X^4) \cong \pi_1(S^3 \setminus K)$ , where  $K \subset S^3$  is a torus knot, and suppose that  $s(X^4) = 0$ . Then  $X^4$  is (simple) homotopy equivalent to a closed topological 4-manifold.*

Yamasaki [31] has recently proved that Theorem 4.6 holds also for hyperbolic knots  $K \subset S^3$ . Note that in order to verify Theorem 4.6 for all knots  $K \subset S^3$  it would suffice, by Thurston's theorem, to answer in affirmative the following question:

**Question** Does Theorem 4.6 hold also if  $K \subset S^3$  is a satellite knot?

## References

1. J. L. Bryant, S. Ferry, W. Mio and S. Weinberger, *Topology of homology manifolds*, Ann. of Math. (2) **143** (1996), 435–467.
2. S. T. Ferry, *Epsilon-delta surgery over  $\mathbb{Z}$* , preprint, Rutgers Univ., Brunswick, N. Y., 2004.
3. M. Freedman and F. Quinn, *The Topology of 4-Manifolds*, Princeton Univ. Press, Princeton, NJ. 1990.
4. F. Hegenbarth and S. Piccarreta, *On Poincaré four complexes with free fundamental groups*, Hiroshima Math. J. **32** (2002), 145–154.
5. F. Hegenbarth and D. Repovš, *Applications of controlled surgery in dimension 4: Examples*, J. Math. Soc. Japan **58** (2006), 1151–1162.
6. F. Hegenbarth, D. Repovš and F. Spaggiari, *Connected sums of 4-manifolds*, Topol. Appl. **146–147** (2005), 209–225.
7. J. Hillman,  *$PD_4$ -complexes with free fundamental groups*, Hiroshima Math. J. **34** (2004), 295–306.
8. M. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Ann. of Math. (2) **77** (1963), 504–537.
9. T. Matumoto and A. Katanga, *On 4-dimensional closed 4-manifolds with free fundamental groups*, Hiroshima Math. J. **25** (1995), 367–370.
10. E. Pedersen, F. Quinn and A. Ranicki, *Controlled surgery with trivial local fundamental groups*, In: High-Dimensional Manifold Topology, F. T. Farrell and W. Lueck, Eds., World Scientific, Singapore, (2003), 421–426.
11. E. Pedersen and M. Yamasaki, *Stability in controlled L-theory*, In: Geometry and Topology Monographs, Volume 9: Exotic Homology Manifolds - Oberwolfach 2003, Frank Quinn and Andrew Ranicki, Eds., (2006), 67–86.
12. F. Quinn, *A geometric formulation of surgery*, Topology of Manifolds, Proc. Georgia Topol. Conf. 1969, Markham Press, Chicago, (1970), 500–511.

13. F. Quinn, *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc. **78** (1972), 263–267.
14. F. Quinn, *Ends of maps, I*, Ann. of Math. (2) **110** (1979), 275–331.
15. F. Quinn, *Ends of Maps, II* Invent. Math. **68** (1982), 353–424.
16. F. Quinn, *Ends of maps III, dimensions 4 and 5*, J. Diff. Geom. **17** (1982), 508–521.
17. F. Quinn, *Resolutions of homology manifolds, and the topological characterization of manifolds*, Invent. Math. **72** (1983), 267–284.
18. F. Quinn, *Geometric Algebra*, In: Proc. Conference on Alg. and Geom. Topol. Rutgers Univ. 1983, Lect. Notes Math. **1126**, Springer-Verlag, Berlin, (1985), 182–198.
19. F. Quinn, *Controlled and low-dimensional topology, and a theorem of Keldysh*, Plenary talk, Int. Conf. Geom. Topol., Discr. Geom. and Set Theory, Moscow, August 24 - 28, (2004).
20. A. Ranicki, *The total surgery obstruction*, In: Proc. Topol. Conf. Aarhus 1978, Lect. Notes Math. **763**, Springer-Verlag, Berlin, (1979), 275–316.
21. A. Ranicki, *The algebraic theory of surgery I, Foundations*, Proc. London Math. Soc. (3) **40** (1980), 87–192.
22. A. Ranicki, *The algebraic theory of surgery II, Applications to topology*, Proc. London Math. Soc. (3) **40** (1980), 193–283.
23. A. Ranicki, *Exact Sequences in the Algebraic Theory of Surgery*, Math. Notes **26**, Princeton Univ. Press, Princeton, NJ., 1981.
24. A. Ranicki, *Algebraic L-Theory and Topological Manifolds*, Cambridge Tracts in Math. **102**, Cambridge Univ. Press, Cambridge, 1992.
25. A. Ranicki, *Singularities, double points, controlled topology and chain duality*, Documenta Mat. **4** (1999), 1–59.
26. A. Ranicki, *An introduction to algebraic surgery*, Surveys on Surgery Theory, Vol. 2, Ann. of Math. Stud., 149, Princeton Univ. Press, Princeton, NJ, (2001), 81–163.
27. A. Ranicki and M. Yamasaki, *Symmetric and quadratic complexes with geometric control*, Proc. TGRC-KOSEF **3** (1993), 139–152.
28. A. Ranicki and M. Yamasaki, *Controlled K-theory*, Topol. Appl. **61** (1995), 1–59.
29. A. Ranicki and M. Yamasaki, *Controlled L-theory*, In: Geometry and Topology Monographs, Volume 9: Exotic Homology Manifolds - Oberwolfach 2003, Frank Quinn and Andrew Ranicki, Eds., (2006), 105–153.
30. M. Yamasaki, *L-groups of crystallographic groups*, Invent. Math. **88** (1987), 571–602.
31. M. Yamasaki, *Hyperbolic knots and 4-dimensional surgery*, preprint, Okayama Science Univ., Okayama, (2006).
32. C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, New York, 1971.