QUANTUM DUALITY PRINCIPLE FOR COISOTROPIC SUBGROUPS AND POISSON QUOTIENTS *

NICOLA CICCOLI
Università di Perugia, Dipartimento di Matematica,
Via Vanvitelli 1, I-06123 Perugia, ITALY
E-mail: ciccoli@dipmat.unipg.it

FABIO GAVARINI
Università di Roma "Tor Vergata", Dipartimento di Matematica,
Via della ricerca scientifica 1, I-00133 Roma, ITALY
E-mail: gavarini@mat.uniroma2.it

We develop a quantum duality principle for coisotropic subgroups of a (formal) Poisson group and its dual: namely, starting from a quantum coisotropic subgroup (for a quantization of a given Poisson group) we provide functorial recipes to produce quantizations of the dual coisotropic subgroup (in the dual formal Poisson group). By the natural link between subgroups and homogeneous spaces, we argue a quantum duality principle for Poisson homogeneous spaces which are Poisson quotients, i.e. have at least one zero-dimensional symplectic leaf.
Only bare results are presented, while detailed proofs can be found in [3].

1. Introduction

In the study of quantum groups, the natural semiclassical counterpart is the theory of deformation (or quantization) of Poisson groups: actually, Drinfeld himself introduced Poisson groups as the semiclassical limits of quantum groups. Therefore, it should be not surprising that the geometry of quantum groups turns more clear and comprehensible when its connection with Poisson geometry is more transparent. The same situation occurs when dealing with Poisson homogeneous spaces of Poisson groups.

In particular, in the study of Poisson homogeneous spaces a special role is

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played by *Poisson quotients*. By this we mean Poisson homogeneous spaces whose symplectic foliation has at least one zero-dimensional leaf: therefore, they can be seen as pointed Poisson homogeneous spaces, just like Poisson groups themselves are pointed by the identity element.

Poisson quotients form a natural subclass of Poisson homogeneous G–spaces (G a Poisson group) which is best adapted to the standard relation between homogeneous G–spaces and subgroups of G: to a given Poisson quotient, one associates the stabilizer subgroup of its distinguished point (the fixed zero-dimensional symplectic leaf). What characterizes such subgroups is *coisotropy*, with respect to the Poisson structure on G (see the definition in Section 2 later on). On the other hand, if a (closed) subgroup K of G is coisotropic, then the homogeneous G–space G/K is a Poisson quotient. So the two notions of Poisson quotient and coisotropic subgroup must be handled in couple. In particular, the quantization process for a Poisson G–quotient corresponds to a similar procedure for the attached coisotropic subgroup of G.
If one looks at quantizations of a Poisson homogeneous space, their existence is guaranteed only if the space is a quotient [8]; thus the notion of Poisson quotient shows up naturally as a necessary condition. On the other hand, let K be a subgroup of G, and assume that G has a quantization, inducing on it a Poisson group structure. If K itself also admits a quantization, which is "consistent" (in a natural sense) with the one of G, then K is automatically coisotropic in G. So also the related notion of coisotropic subgroup shows to be a necessary condition for the existence of quantizations. Of course an analogous description can be entirely carried out at an infinitesimal level, with conditions at the level of Lie bialgebras.
When dealing with quantizations of Poisson groups (or Lie bialgebras), a precious tool is the quantum duality principle (QDP). Roughly speaking, it claims that any quantized enveloping algebra can be turned — via a functorial recipe — into a quantum function algebra for the dual Poisson group; conversely, any quantum function algebra can be turned into a quantization of the enveloping algebra of the dual Lie bialgebra. To be precise, let $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ respectively be the category of all quantized universal enveloping algebras (QUEA) and the category of all quantized formal series Hopf algebras (QFSHA), in Drinfeld's sense. Then the QDP establishes [6, 11] a category equivalence between $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ via two functors, $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $(\ )^\vee \colon \mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$. Moreover, starting from a QUEA over a Lie bialgebra (resp. from a QFSHA over a Poisson group) the functor $(\ )'$ (resp. $(\ )^\vee$) gives a QFSHA (resp. a

QUEA) over the dual Poisson group (resp. the dual Lie bialgebra). In short, $U_\hbar(\mathfrak{g})' = F_\hbar[[G^*]]$ and $F_\hbar[[G]]^\vee = U_\hbar(\mathfrak{g}^*)$ for any Lie bialgebra $\mathfrak{g}$ and Poisson group G with $Lie(G) = \mathfrak{g}$. So from a quantization of any Poisson group this principle gets out a quantization of the dual Poisson group too.

In this paper we establish a similar quantum duality principle for (closed) coisotropic subgroups of a Poisson group G, or equivalently for Poisson G–quotients, sticking to the formal approach (hence dealing with quantum groups à la Drinfeld). The starting point is that any formal coisotropic subgroup K of a Poisson group G has two possible algebraic descriptions via objects related to $U(\mathfrak{g})$ or $F[[G]]$, and similarly for the formal Poisson quotient $G\big/K$; thus the datum of K or equivalently of $G\big/K$ is described algebraically in four possible ways. By *quantization* of such a datum we mean a quantization of any one of these four objects, which has to be "consistent" — in a natural sense — with given quantizations $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ of G. Our "QDP" now is a bunch of functorial recipes to produce, out of a quantization of K or $G\big/K$ as before, a similar quantization of the so-called *complementary dual* of K, that is the coisotropic subgroup $K^\perp$ of G^* whose tangent Lie bialgebra is just $\mathfrak{k}^\perp$ inside $\mathfrak{g}^*$, or of the associated Poisson G^*–quotient, namely $G^*\big/K^\perp$. The basic idea is quite simple. The quantizations of coisotropic subgroups — or Poisson quotients — are sub-objects of quantizations of Poisson groups, and the recipes of the original QDP (for Poisson groups) apply to the latter objects. Then we simply "restrict", somehow, such recipes to the previously mentioned sub-objects.

In recent times, the general problem of quantizing coisotropic manifolds of a given Poisson manifold, in the context of deformation quantization, has raised quite some interest [1, 2]. It is then important to point out that ours is by no means an existence result: instead, it can be thought of as a *duplication result*, because it yields a new quantization — for a complementary dual object — out of one given from scratch (much like the QDP for quantum groups). On the other hand, we would better stress that our result is really *effective*, and calling for applications. A sample of application is presented in the extended version of this work [3]; see also Subsection 5.6.

2. The classical setting

2.1. *Formal Poisson groups*

We shall work in the setup of formal geometry. Recall that a formal variety is uniquely characterized by a tangent or cotangent space (at its unique point), and it is described by its "algebra of regular functions" — such as $F[[G]]$ below. This is a complete, topological local ring which can be realized as a $\Bbbk$–algebra of formal power series. Hereafter $\Bbbk$ is a field of zero characteristic.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$, and let $U(\mathfrak{g})$ be its universal enveloping algebra (with the natural Hopf algebra structure). We denote by $F[[G]]$ the algebra of functions on the formal algebraic group G associated to $\mathfrak{g}$ (which depends only on $\mathfrak{g}$ itself); this is a complete, topological Hopf algebra. Furthermore $F[[G]] \cong U(\mathfrak{g})^*$, so that there is a natural pairing of (topological) Hopf algebras — see below — between $U(\mathfrak{g})$ and $F[[G]]$.

In general, if H, K are Hopf algebras (even topological) over a ring R, a pairing $\langle\ ,\ \rangle : H \times K \longrightarrow R$ is called a *Hopf pairing* if $\big\langle x, y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x), y_1 \otimes y_2 \big\rangle$, $\big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle$, $\langle x, 1 \rangle = \epsilon(x)$, $\langle 1, y \rangle = \epsilon(y)$, $\big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle$ for all $x, x_1, x_2 \in H$, $y, y_1, y_2 \in K$. The pairing is called *perfect* if it is non-degenerate.

Assume G is a formal *Poisson* (algebraic) group. Then $\mathfrak{g}$ is a Lie bialgebra, $U(\mathfrak{g})$ is a co-Poisson Hopf algebra, $F[[G]]$ is a topological Poisson Hopf algebra, and the Hopf pairing above respects these additional co-Poisson and Poisson structures. Furthermore, the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Lie bialgebra as well, so a dual formal Poisson group G^* exists.

Notation: hereafter, the symbol $\dot{\trianglelefteq}$ stands for "coideal", $\leq^1$ for "unital subalgebra", $\dot{\leq}$ for "subcoalgebra", $\leq_{\mathcal{P}}$ for "Poisson subalgebra", $\dot{\trianglelefteq}_{\mathcal{P}}$ for "Poisson coideal", $\leq_{\mathcal{H}}$ for "Hopf subalgebra", $\trianglelefteq_{\mathcal{H}}$ for "Hopf ideal", and the subscript ℓ stands for "left" (everything in topological sense if necessary).

2.2. *Subgroups and homogeneous G–spaces*

A homogeneous left G–space M corresponds to a conjugacy class of closed subgroups $K = K_M$, which we assume connected, of G, such that $M \cong G/K$. In formal geometry K may be replaced by $\mathfrak{k} := Lie(K)$. The whole geometric setting given by the pair $\big(K, G/K\big)$ then is encoded by any one of the following data:

(a) the set $\mathcal{I} = \mathcal{I}(K) \equiv \mathcal{I}(\mathfrak{k})$ of (formal) functions vanishing on K, that is to say $\mathcal{I} = \big\{ \varphi \in F[[G]] \,\big|\, \varphi(K) = 0 \big\}$; note that $\mathcal{I} \trianglelefteq_{\mathcal{H}} F[[G]]$;

(b) the set of left $\mathfrak{k}$–invariant functions, namely $\mathcal{C} = \mathcal{C}(K) \equiv \mathcal{C}(\mathfrak{k}) = F[[G]]^K$; note that $\mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell F[[G]]$;

(c) the set $\mathfrak{I} = \mathfrak{I}(K) \equiv \mathfrak{I}(\mathfrak{k})$ of left-invariant differential operators on $F[[G]]$ which vanish on $F[[G]]^K$, that is $\mathfrak{I} = U(\mathfrak{g}) \cdot \mathfrak{k}$ (via standard identification of the set of left-invariant differential operators with $U(\mathfrak{g})$);

note that $\mathfrak{I}(\mathfrak{k}) = \mathfrak{I} \trianglelefteq_\ell \dot{\trianglelefteq}\, U(\mathfrak{g})$;

(d) the universal enveloping algebra of $\mathfrak{k}$, denoted $\mathfrak{C} = \mathfrak{C}(K) \equiv \mathfrak{C}(\mathfrak{k}) := U(\mathfrak{k})$; note that $\mathfrak{C}(\mathfrak{k}) = \mathfrak{C} \leq_{\mathcal{H}} U(\mathfrak{g})$.

In this way any formal subgroup K of G , or the associated homogeneous G–space $G\big/K$, is characterized by any of the following objects:

$$\textit{(a)}\ \mathcal{I} \trianglelefteq_{\mathcal{H}} F[[G]] \quad \textit{(b)}\ \mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell\, F[[G]] \quad \textit{(c)}\ \mathfrak{I} \trianglelefteq_\ell \dot{\trianglelefteq}\, U(\mathfrak{g}) \quad \textit{(d)}\ \mathfrak{C} \leq_{\mathcal{H}} U(\mathfrak{g})$$

These four data are all equivalent to each other, as we now explain.
For any Hopf algebra H , with counit ϵ , and every submodule $M \subseteq H$, we set: $M^+ := M \cap Ker(\epsilon)$ and $H^{coM} := \big\{\, y \in H \,\big|\, \big(\Delta(y) - y \otimes 1\big) \in H \otimes M \,\big\}$ (the set of M–*coinvariants* of H). Letting $\mathbb{A}$ be the set of all subalgebras left coideals of H and $\mathbb{K}$ be the set of all coideals left ideals of H , we have well-defined maps $\mathbb{A} \longrightarrow \mathbb{K}$, $A \mapsto H \cdot A^+$, and $\mathbb{K} \longrightarrow \mathbb{A}$, $K \mapsto H^{coK}$ (see for instance Masuoka's work [15]). Then the above equivalence stems from

— *(1) orthogonality relations* — w.r.t. the natural pairing between $F[[G]]$ and $U(\mathfrak{g})$ — namely $\mathcal{I} = \mathfrak{C}^\perp$, $\mathfrak{C} = \mathcal{I}^\perp$, and $\mathcal{C} = \mathfrak{I}^\perp$, $\mathfrak{I} = \mathcal{C}^\perp$;

— *(2) subgroup-space correspondence*, namely $\mathcal{I} = F[[G]] \cdot \mathcal{C}^+$, $\mathcal{C} = {F[[G]]}^{co\mathcal{I}}$, and $\mathfrak{I} = U(\mathfrak{g})\, \mathfrak{C}^+$, $\mathfrak{C} = {U(\mathfrak{g})}^{co\mathfrak{I}}$. Moreover, the maps $\mathbb{A} \longrightarrow \mathbb{K}$ and $\mathbb{K} \longrightarrow \mathbb{A}$ above are inverse to each other in the formal setting.

2.3. *Coisotropic subgroups and Poisson quotients*

Assume now that G is a formal Poisson group. A closed formal subgroup K of G with Lie algebra $\mathfrak{k}$ is called *coisotropic* if its defining ideal $\mathcal{I}(\mathfrak{k})$ is a topological Poisson subalgebra of $F[[G]]$. The following are equivalent [13, 14]:

(C-i) K is a coisotropic formal subgroup of G ;

(C-ii) $\delta(\mathfrak{k}) \subseteq \mathfrak{k} \wedge \mathfrak{g}$, that is $\mathfrak{k}$ is a Lie coideal of $\mathfrak{g}$;

(C-iii) $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$

Clearly, conditions *(C-i,ii,iii)* characterize coisotropic subgroups.
As to homogeneous spaces, a formal Poisson manifold (M, ω_M) is a *Poisson homogeneous G–space* if it carries a homogeneous action $\phi\colon\ G \times M \to M$ which is a smooth Poisson map. In addition, (M, ω_M) is said to be *of group type* (after Drinfeld [7]), or simply a *Poisson quotient*, if there is a coisotropic closed Lie subgroup K_M of G such that $G\big/K_M \simeq M$ and the natural projection map $\pi\colon\ G \longrightarrow G\big/K_M \simeq M$ is a Poisson map.
The following is a characterization of Poisson quotients [17]:

(PQ-i) there exists $x_0 \in M$ whose stabilizer G_{x_0} is coisotropic in G ;

(PQ-ii) there exists $x_0 \in M$ such that $\phi_{x_0} : G \longrightarrow M$, $g \mapsto \phi(g, x_0)$, is a Poisson map, that is M is a Poisson quotient;

(PQ-iii) there exists $x_0 \in M$ such that $\omega_M(x_0) = 0$.

It is important to remark that in Poisson geometry, the usual relationship between closed subgroups of G and G–homogeneous spaces does not hold anymore: in fact, in the *same* conjugacy class one can have Poisson subgroups, coisotropic subgroups *and* even non-coisotropic subgroups. Now, the above characterization means exactly that the Poisson quotients are just those Poisson homogeneous spaces in which (at least) one of the stabilizers is coisotropic. Moreover, in general the correspondence between homogeneous spaces and subgroups is somewhat ambiguous, because it passes through the choice of a distinguished point of the space (whose stabilizer is the subgroup). In the case of Poisson quotients this ambiguity is cleared off, as we do fix as distinguished point on the space the zero-dimensional symplectic leaf that it has for sure — although it is non-unique, a priori.
In addition, passing through coisotropic subgroups allows us to introduce a good notion of (Poisson) *duality* for our objects, namely

Definition 2.1. (notation of Subsection 2.1)

(a) If K is a formal coisotropic subgroup of G, we call *complementary dual* of K the formal subgroup $K^\perp$ of G^* whose tangent Lie algebra is $\mathfrak{k}^\perp$.

(b) If $M \cong G\big/K_M$ is a formal Poisson G–quotient, with K_M coisotropic, we call $M^\perp := G^*\big/K_M^\perp$ the *complementary dual* of M.

Here the key point is that — by *(C-iii)* in Subsection 2.3 — a subset $\mathfrak{k}$ of $\mathfrak{g}$ is a Lie coideal if and only if $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$. Even more, by *(C-i,ii,iii)*, the complementary dual subgroup to a coisotropic subgroup is *coisotropic* too, and taking twice the complementary dual gives back the initial subgroup. Similarly, the Poisson homogeneous space which is complementary dual to a Poisson homogeneous space of group type is in turn *of group type* too, and taking twice the complementary dual gives back the initial manifold. At the level of Poisson homogeneous spaces, one should think of $\big(K, G/K^\perp\big)$ and $\big(G/K, K^\perp\big)$ as mutually dual pairs; if $K = \{e\}$, one recovers the usual couple of Poisson groups G, G^* . When using these pairs a price is paid: one object (the subgroup) is not a Poisson manifold and the other (the homogeneous spaces) is not a group anymore.

2.4. *Remarks*

(a) The notion of Poisson homogeneous G–spaces of group type was in-

troduced by Drinfeld [7], who also explained the relation between such G–spaces and Lagrangian subalgebras of Drinfeld's double $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$. This was further developed by Evens and Lu [10], who gave a Poisson structure on the algebraic variety of Lagrangian subalgebra. It is an open problem to quantize this sort of *universal moduli space* of Poisson homogeneous G–spaces.

(b) As a matter of notation, we denote by $co\mathcal{S}(G)$ the set of all formal coisotropic subgroups of G, which is as well described by the set of all Lie subalgebras, Lie coideals of $\mathfrak{g}$. Since it is ordered by inclusion, this set will be also considered as a category.

2.5. *Algebraic characterization of coisotropy*

Let K be a formal coisotropic subgroup of G. In terms of Subsection 2.2, coisotropy corresponds to

(a) $\mathcal{I} \leq_{\mathcal{P}} F[[G]]$ *(b)* $\mathcal{C} \leq_{\mathcal{P}} F[[G]]$ *(c)* $\mathfrak{I} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g})$ *(d)* $\mathfrak{C} \,\dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g})$

Thus a formal coisotropic subgroup of G is identified by any one of

(a) $\mathcal{I} \trianglelefteq_{\mathcal{H}} \leq_{\mathcal{P}} F[[G]]$ *(b)* $\mathcal{C} \leq^1 \dot{\trianglelefteq}_\ell \leq_{\mathcal{P}} F[[G]]$

(c) $\mathfrak{I} \trianglelefteq_\ell \dot{\trianglelefteq}\, \dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g})$ *(d)* $\mathfrak{C} \leq_{\mathcal{H}} \dot{\trianglelefteq}_{\mathcal{P}}\, U(\mathfrak{g})$

3. The quantum setting

3.1. *Topological* $\Bbbk[[\hbar]]$*–modules and tensor structures*

Let $\Bbbk[[\hbar]]$ be the topological ring of formal power series in the indeterminate $\hbar$. If X is any $\Bbbk[[\hbar]]$–module, we set $X_0 := X\big/\hbar X = \Bbbk \otimes_{\Bbbk[[\hbar]]} X$, the *specialization* of X at $\hbar = 0$, or *semiclassical limit* of X. We are interested in certain families of $\Bbbk[[\hbar]]$–modules, complete with respect to suitable topologies, and for which a good notion of tensor product be available. The exact notion of topology or of tensor product to choose depends on whether one looks for quantizations of universal enveloping algebras or of formal series Hopf algebras: this leads to two different setups, as follows.

First, let $\mathcal{T}_{\widehat{\otimes}}$ be the category whose objects are all topological $\Bbbk[[\hbar]]$–modules which are topologically free and whose morphisms are the $\Bbbk[[\hbar]]$–linear maps (which are automatically continuous). It is a tensor category for the tensor product $T_1 \,\widehat{\otimes}\, T_2$ defined as the separated $\hbar$–adic completion of the algebraic tensor product $T_1 \otimes_{\Bbbk[[\hbar]]} T_2$ (for all $T_1, T_2 \in \mathcal{T}_{\widehat{\otimes}}$). We denote by $\mathcal{HA}_{\widehat{\otimes}}$ the subcategory of $\mathcal{T}_{\widehat{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{T}_{\widehat{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{T}_{\widehat{\otimes}}$.

Second, let $\mathcal{P}_{\widetilde{\otimes}}$ be the category whose objects are all topological $\Bbbk[[\hbar]]$–modules isomorphic to modules of the type $\Bbbk[[\hbar]]^E$ (with the product topology) for some set E, and whose morphisms are the $\Bbbk[[\hbar]]$–linear continuous maps. Again, this is a tensor category w.r.t. the tensor product $P_1 \mathbin{\widetilde{\otimes}} P_2$ defined as the completion of the algebraic tensor product $P_1 \otimes_{\Bbbk[[\hbar]]} P_2$ w.r.t. the weak topology: thus $P_i \cong \Bbbk[[\hbar]]^{E_i}$ $(i = 1, 2)$ yields $P_1 \mathbin{\widetilde{\otimes}} P_2 \cong \Bbbk[[\hbar]]^{E_1 \times E_2}$ (for $P_1, P_2 \in \mathcal{P}_{\widetilde{\otimes}}$). We call $\mathcal{HA}_{\widetilde{\otimes}}$ the subcategory of $\mathcal{P}_{\widetilde{\otimes}}$ with objects the Hopf algebras in $\mathcal{P}_{\widetilde{\otimes}}$, and with morphisms the Hopf algebra morphisms in $\mathcal{P}_{\widetilde{\otimes}}$.
A *quantum group* for us will be a Hopf algebra, in either of $\mathcal{HA}_{\widehat{\otimes}}$ or $\mathcal{HA}_{\widetilde{\otimes}}$, having a special semiclassical limit. The exact definition is the following:

Definition 3.1. (cf. [7], § 7)

(a) We call QUEA any $H \in \mathcal{HA}_{\widehat{\otimes}}$ such that $H_0 := H \big/ \hbar H$ is a co-Poisson Hopf algebra isomorphic to $U(\mathfrak{g})$ for some finite dimensional Lie bialgebra $\mathfrak{g}$ (over $\Bbbk$); then we write $H = U_\hbar(\mathfrak{g})$, and say H is a *quantization* of $U(\mathfrak{g})$. We call $\mathcal{QUEA}$ the full tensor subcategory of $\mathcal{HA}_{\widehat{\otimes}}$ whose objects are QUEA, relative to all possible $\mathfrak{g}$.

(b) We call QFSHA any $K \in \mathcal{HA}_{\widetilde{\otimes}}$ such that $K_0 := K \big/ \hbar K$ is a topological Poisson Hopf algebra isomorphic to $F[[G]]$ for some finite dimensional formal Poisson group G (over $\Bbbk$); then we write $H = F_\hbar[[G]]$, and say K is a *quantization* of $F[[G]]$. We call $\mathcal{QFSHA}$ the full tensor subcategory of $\mathcal{HA}_{\widetilde{\otimes}}$ whose objects are QFSHA, relative to all possible G.

3.2. *Remarks*

If $H \in \mathcal{HA}_{\widehat{\otimes}}$ is such that its semiclassical limit $H_0 := H \big/ \hbar H$ as a Hopf algebra is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, then $H_0 = U(\mathfrak{g})$ is also a *co-Poisson* Hopf algebra w.r.t. the Poisson cobracket δ defined as follows: if $x \in H_0$ and $x' \in H$ gives $x = x' + \hbar H$, then $\delta(x) := \big(\hbar^{-1}\left(\Delta(x') - \Delta^{\mathrm{op}}(x')\right)\big) + \hbar H \mathbin{\widehat{\otimes}} H$. Thus, in particular — by Theorem 2 in §3 of Drinfeld's paper [6] — the restriction of δ makes $\mathfrak{g}$ into a Lie bialgebra. In Definition 3.1*(a)*, the co-Poisson structure considered on H_0 is nothing but the one arising in this way. Similarly, if $K \in \mathcal{HA}_{\widetilde{\otimes}}$ is such that its semiclassical limit $K_0 := K \big/ \hbar K$ is a topological Poisson Hopf algebra isomorphic to $F[[G]]$ for some formal group G then $K_0 = F[[G]]$ is also a topological *Poisson* Hopf algebra w.r.t. the Poisson bracket $\{\ ,\ \}$ defined as follows: if $x, y \in K_0$ and $x', y' \in K$ give $x = x' + \hbar K$ and $y = y' + \hbar K$, then $\{x, y\} := \big(\hbar^{-1}(x'\,y' - y'\,x')\big) + \hbar K$. Then, in particular, G is a *Poisson* formal group. And again, in Definition 3.1*(b)*, the Poisson structure considered on K_0 is exactly the one that arises from this construction.

3.3. *Drinfeld's functors*

Let H be a (topological) Hopf algebra over $\Bbbk[[\hbar]]$. Letting $J_H := Ker(\epsilon_H)$ and $I_H := \epsilon_H^{-1}\big(\hbar\,\Bbbk[[\hbar]]\big) = J_H + \hbar\, H$, we set

$$H^\times := \textstyle\sum_{n\geq 0} \hbar^{-n} {I_H}^n = \sum_{n\geq 0} \big(\hbar^{-1} I_H\big)^n = \bigcup\limits_{n\geq 0} \big(\hbar^{-1} I_H\big)^n = \sum_{n\geq 0} \hbar^{-n} {J_H}^n$$

which is a subspace of $\Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} H$. Then we define

$$H^\vee := \hbar\text{–adic completion of the } \Bbbk[[\hbar]]\text{–module } H^\times .$$

On the other hand, for each $n \in \mathbb{N}_+$, define $\Delta^n \colon H \longrightarrow H^{\otimes n}$ by $\Delta^1 := id_H$ and $\Delta^n := \big(\Delta \otimes id_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n \geq 2$, and set $\delta_0 := \delta_\emptyset$, and $\delta_n := (id_H - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}_+$. Then we define

$$H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in h^n H^{\otimes n} \;\; \forall\, n \in \mathbb{N} \,\big\} \qquad \big(\subseteq H\,\big) .$$

Note that the definition of $H^\vee$ is pretty direct. In particular, we specify how it can be generated (topologically), namely it is the (complete topological) unital $\Bbbk[[\hbar]]$–subalgebra of $\Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} H$ generated by $\hbar^{-1} J_H$ or $\hbar^{-1} I_H$. In contrast, the definition of H' is quite implicit: roughly speaking, it is the set of solution of a system with countably many equations (specified in terms of $\hbar$–adic valuation). Nevertheless, the two definitions are strictly related, in a sense made explicit by Proposition 2.6 below.
Now we state the Quantum Duality Principle (=QDP) for quantum groups:

Theorem 3.1. *(see [6], and [11] for a proof) The assignments $H \mapsto H^\vee$ and $H \mapsto H'$, respectively, define tensor functors $\mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$ and $\mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$, which are inverse to each other. Indeed, for all $U_\hbar(\mathfrak{g}) \in \mathcal{QUEA}$ and all $F_\hbar[[G]] \in \mathcal{QFSHA}$ one has*

$$U_\hbar(\mathfrak{g})' \Big/ \hbar\, U_\hbar(\mathfrak{g})' = F[[G^*]] \qquad\qquad F_\hbar[[G]]^\vee \Big/ \hbar\, F_\hbar[[G]]^\vee = U(\mathfrak{g}^*)$$

that is, if $U_\hbar(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ then $U_\hbar(\mathfrak{g})'$ is one of $F[[G^*]]$, and if $F_\hbar[[G]]$ is a quantization of $F[[G]]$ then $F_h[[G^*]]^\vee$ is one of $U(\mathfrak{g}^*)$.

In addition, Drinfeld's functors respect Hopf duality, in the following sense:

Proposition 3.1. *(see [11], Proposition 2.2) Let $U_\hbar \in \mathcal{QUEA}$, $F_\hbar \in \mathcal{QFSHA}$ and let $\pi : U_\hbar \times F_\hbar \longrightarrow \Bbbk[[\hbar]]$ be a perfect Hopf pairing whose specialization at $\hbar = 0$ is perfect as well. Then π induces — by restriction on l.h.s. and scalar extension on r.h.s. — a perfect Hopf pairing ${U_\hbar}' \times {F_\hbar}^\vee \longrightarrow \Bbbk[[\hbar]]$ whose specialization at $\hbar = 0$ is again perfect.*

In other words, the above result ensures that, if one starts from a pair made by a QUEA and a QFSHA which are dual to each other, and then applies Drinfeld's functors to both terms of the pair, then one obtains another pair — now with QFSHA first and QUEA second — with the same property. In this sense, the two Drinfeld's functors are "dual to each other".

3.4. *Quantum subgroups and quantum homogeneous spaces*

From now on, let G be a formal Poisson group, $\mathfrak{g} := \mathit{Lie}(G)$ its tangent Lie bialgebra. We assume a quantization of G is given, in the sense that a QFSHA $F_\hbar[[G]]$ quantizing $F[[G]]$ and a QUEA $U_\hbar(\mathfrak{g})$ quantizing $U(\mathfrak{g})$ are given such that, in addition, $F_\hbar[[G]] \cong U_\hbar(\mathfrak{g})^* := \mathit{Hom}_{\Bbbk[[\hbar]]}\big(U_\hbar(\mathfrak{g}), \Bbbk[[\hbar]]\big)$ as topological Hopf algebras; the latter requirement is equivalent to fixing a perfect Hopf algebra pairing between $F_\hbar[[G]]$ and $U_\hbar(\mathfrak{g})$ whose specialization at $\hbar = 0$ be perfect too. This assumption is not restrictive, because [8] such a $U_\hbar(\mathfrak{g})$ always exists, and then one can take $F_\hbar[[G]] := U_\hbar(\mathfrak{g})^*$. We denote by $\pi_{F_\hbar} : F_\hbar[[G]] \relbar\joinrel\twoheadrightarrow F[[G]]$ and $\pi_{U_\hbar} : U_\hbar(\mathfrak{g}) \relbar\joinrel\twoheadrightarrow U(\mathfrak{g})$ the specialization maps, and we set $F_\hbar := F_\hbar[[G]]$, $U_\hbar := U_\hbar(\mathfrak{g})$.
Let K be a formal subgroup of G, and $\mathfrak{k} := \mathit{Lie}(K)$. As quantization of K and/or of G/K, we mean a quantization of any one of the four algebraic objects $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ and $\mathfrak{C}$ associated to them in Subsection 2.2, that is either of

$$
\begin{aligned}
&(a)\quad \text{a left ideal, coideal } \mathcal{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq}\, F_\hbar[[G]] \text{ such that} \\
&\qquad\qquad \mathcal{I}_\hbar\big/\hbar\,\mathcal{I}_\hbar \,\cong\, \pi_{F_\hbar}(\mathcal{I}_\hbar) \,=\, \mathcal{I} \\
&(b)\quad \text{a subalgebra, left coideal } \mathcal{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell\, F_\hbar[[G]] \text{ such that} \\
&\qquad\qquad \mathcal{C}_\hbar\big/\hbar\,\mathcal{C}_\hbar \,\cong\, \pi_{F_\hbar}(\mathcal{C}_\hbar) \,=\, \mathcal{C} \\
&(c)\quad \text{a left ideal, coideal } \mathfrak{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq}\, U_\hbar(\mathfrak{g}) \text{ such that} \\
&\qquad\qquad \mathfrak{I}_\hbar\big/\hbar\,\mathfrak{I}_\hbar \,\cong\, \pi_{U_\hbar}(\mathfrak{I}_\hbar) \,=\, \mathfrak{I} \\
&(d)\quad \text{a subalgebra, left coideal } \mathfrak{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell\, U_\hbar(\mathfrak{g}) \text{ such that} \\
&\qquad\qquad \mathfrak{C}_\hbar\big/\hbar\,\mathfrak{C}_\hbar \,\cong\, \pi_{U_\hbar}(\mathfrak{C}_\hbar) \,=\, \mathfrak{C}
\end{aligned}
\tag{3.1}
$$

In (3.1) above, the constraint $\mathcal{I}_\hbar\big/\hbar\mathcal{I}_\hbar \cong \pi_{F_\hbar}(\mathcal{I}_\hbar) = \mathcal{I}$ means the following. By construction $\mathcal{I}_\hbar \longrightarrow F_\hbar[[G]] \xrightarrow{\pi_{F_\hbar}} F_\hbar[[G]]\Big/\hbar\, F_\hbar[[G]] \cong F[[G]]$, and the composed map $\mathcal{I}_\hbar \longrightarrow F[[G]]$ factors through $\mathcal{I}_\hbar\big/\hbar\,\mathcal{I}_\hbar$; then we ask that the induced map $\mathcal{I}_\hbar\big/\hbar\,\mathcal{I}_\hbar \longrightarrow F[[G]]$ be a bijection onto $\pi_{F_\hbar}(\mathcal{I}_\hbar)$, and that the latter do coincide with $\mathcal{I}$; of course this bijection will also respects all Hopf operations, because $\pi_{F_\hbar}$ does. Similarly for the other conditions.
Moreover, let $X \in \{\mathcal{I}, \mathcal{C}, \mathfrak{I}, \mathfrak{C}\}$, $S_\hbar \in \big\{F_\hbar[[G]], U_\hbar(\mathfrak{g})\big\}$. Since $\pi_{S_\hbar}(X_\hbar) =$

$X_\hbar \Big/ \big(X_\hbar \cap \hbar \, S_\hbar \big)$, the property $X_\hbar \Big/ \hbar \, X_\hbar \cong \pi_{S_\hbar}(X_\hbar) = X$ is equivalent to $X_\hbar \cap \hbar \, S_\hbar = \hbar \, X_\hbar$. So our quantum objects can also be characterized by

$$\begin{array}{llll} (a) & \mathcal{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq} F_\hbar[[G]] & \mathcal{I}_\hbar \cap \hbar \, F_\hbar[[G]] \, = \, \hbar \, \mathcal{I}_\hbar & \mathcal{I}_\hbar \big/ \hbar \, \mathcal{I}_\hbar \, = \, \mathcal{I} \\ (b) & \mathcal{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell F_\hbar[[G]] & \mathcal{C}_\hbar \cap \hbar \, F_\hbar[[G]] \, = \, \hbar \, \mathcal{C}_\hbar & \mathcal{C}_\hbar \big/ \hbar \, \mathcal{C}_\hbar \, = \, \mathcal{C} \\ (c) & \mathfrak{I}_\hbar \trianglelefteq_\ell \dot{\trianglelefteq} \, U_\hbar(\mathfrak{g}) & \mathfrak{I}_\hbar \cap \hbar \, U_\hbar(\mathfrak{g}) \, = \, \hbar \, \mathfrak{I}_\hbar & \mathfrak{I}_\hbar \big/ \hbar \, \mathfrak{I}_\hbar \, = \, \mathfrak{I} \\ (d) & \mathfrak{C}_\hbar \leq^1 \dot{\trianglelefteq}_\ell \, U_\hbar(\mathfrak{g}) & \mathfrak{C}_\hbar \cap \hbar \, U_\hbar(\mathfrak{g}) \, = \, \hbar \, \mathfrak{C}_\hbar & \mathfrak{C}_\hbar \big/ \hbar \, \mathfrak{C}_\hbar \, = \, \mathfrak{C} \end{array} \qquad (3.1)'$$

instead of (3.1). Note that $\mathcal{I} = \mathcal{I}(K)$ and $\mathfrak{C} = \mathfrak{C}(K)$ provide an "algebraization" (in global and local terms respectively) of the subgroup K, more than of the homogeneous space $G \big/ K$; conversely, $\mathcal{C} = \mathcal{C}(K) \big(\cong F\big[\big[G/K\big]\big] \big)$ and $\mathfrak{I} = \mathfrak{I}(K)$ provide an "algebraization" (of global and local type respectively) of $G \big/ K$, more than of K . For this reason, in the sequel we shall loosely refer to $\mathcal{I}_\hbar$ and $\mathfrak{C}_\hbar$ as to *"quantum (formal) subgroups"*, and to $\mathcal{C}_\hbar$ and $\mathfrak{I}_\hbar$ instead as to *"quantum (formal) homogeneous spaces"*.

One could ask whether quantum subgroups and quantum homogeneous spaces do exist. Actually, from the very definitions one immediately finds a square *necessary condition*. In fact, next Lemma proves that the (formal) subgroup of G obtained as specialization of a quantum (formal) subgroup is *coisotropic*, and a (formal) homogeneous G–space obtained as specialization of a quantum (formal) homogeneous space is a *Poisson quotient*. This is quite a direct generalization of the situation for quantum groups, where one has that specializing a quantum group always gives a *Poisson* group.

Lemma 3.1. *Let K be a formal subgroup of G, and assume a quantization $\mathcal{I}_\hbar$, $\mathcal{C}_\hbar$, $\mathfrak{I}_\hbar$ or $\mathfrak{C}_\hbar$ of $\mathcal{I}$, $\mathcal{C}$, $\mathfrak{I}$ or $\mathfrak{C}$ respectively be given as above. Then the subgroup K is coisotropic, and the G–space $G \big/ K$ is a Poisson quotient.*

Note that, at the quantum level, one looses either commutativity or cocommutativity; then, one-sided ideals (or coideals) are not automatically two-sided! This enters in the definitions above, in that we require some objects to be *one*-sided ideals (coideals) — taking *left* rather than *right* ones is just a matter of choice. If one takes *two*-sided ones instead, the like of Lemma 3.1 is that K be a *Poisson subgroup* (for $\mathcal{I}(K)$ is a Poisson ideal).

3.5. *The existence problem*

The *existence* of any of the four possible objects providing a quantization of a coisotropic subgroup (or of the associated Poisson quotient) is an open problem. Etingof and Kahzdan [9] gave a positive answer for the subclass of those K which are also *Poisson subgroups* (which amounts to $\mathfrak{k} := Lie(K)$ being a Lie subbialgebra). Many other examples of quantizations exist too. Yet, the four existence problems are equivalent: i.e., as one solves any one of them, a solution follows for the remaining ones. Indeed, one has:

— *(a) $\Longleftrightarrow$ (d) and (b) $\Longleftrightarrow$ (c):* if $\mathcal{I}_\hbar$ exists as in *(a)*, then $\mathfrak{C}_\hbar := {\mathcal{I}_\hbar}^\perp$ (hereafter orthogonality is meant w.r.t. the fixed Hopf pairing between $F_\hbar[[G]]$ and $U_\hbar(\mathfrak{g})$) enjoys the properties in *(d)*; conversely, if $\mathfrak{C}_\hbar$ exists as in *(d)*, then $\mathcal{I}_\hbar := {\mathfrak{C}_\hbar}^\perp$ enjoys the properties in *(a)*. Similarly, the equivalence *(b) $\Longleftrightarrow$ (c)* follows from a like orthogonality argument.

— *(a) $\Longleftrightarrow$ (b) and (c) $\Longleftrightarrow$ (d):* if $\mathcal{I}_\hbar$ exists as in *(a)*, then $\mathcal{C}_\hbar := {\mathcal{I}_\hbar}^{co\mathcal{I}_\hbar}$ is an object like in *(b)*; on the other hand, if $\mathcal{C}_\hbar$ as in *(b)* is given, then $\mathcal{I}_\hbar := F_\hbar[[G]] \cdot \mathcal{C}_\hbar^+$ enjoys all properties in *(a)* (notation of Subsection 2.2). The equivalence *(c) $\Longleftrightarrow$ (d)* stems from a like argument.

3.6. *Basic assumptions*

Hereafter *we assume that quantizations $\mathcal{I}_\hbar$, $\mathcal{C}_\hbar$, $\mathfrak{I}_\hbar$, $\mathfrak{I}_\hbar$ as in (3.1) are given and that they be linked by relations like (1)–(2) in Subsection 2.2, i.e.*

$$\begin{array}{llll} (i) & \mathcal{I}_\hbar = {\mathfrak{C}_\hbar}^\perp , \quad \mathfrak{C}_\hbar = {\mathcal{I}_\hbar}^\perp & (ii) & \mathfrak{I}_\hbar = {\mathcal{C}_\hbar}^\perp , \quad \mathcal{C}_\hbar = {\mathfrak{I}_\hbar}^\perp \\ (iii) & \mathcal{I}_\hbar = F_\hbar \cdot \mathcal{C}_\hbar^+ , \ \ \mathcal{C}_\hbar = {F_\hbar}^{co\mathcal{I}_\hbar} & (iv) & \mathfrak{I}_\hbar = U_\hbar \cdot \mathfrak{C}_\hbar^+ , \ \ \mathfrak{C}_\hbar = {U_\hbar}^{co\mathfrak{I}_\hbar} \end{array} \tag{3.2}$$

In fact, one of the objects is enough to have all others, in such a way that the previous assumption holds. Indeed, if $co\mathcal{S}$ is the set of coisotropic subgroup of G, let $Y_\hbar\big(co\mathcal{S}\big) := \big\{Y_\hbar(\mathfrak{k})\big\}_{\mathfrak{k}\in co\mathcal{S}}$ for all $Y \in \big\{\mathcal{I}, \mathcal{C}, \mathfrak{I}, \mathfrak{C}\big\}$. The four equivalences *(a) $\Longleftrightarrow$ (d)*, *(b) $\Longleftrightarrow$ (c)*, *(a) $\Longleftrightarrow$ (b)* and *(c) $\Longleftrightarrow$ (d)* above are given by bijections $\mathcal{I}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathfrak{C}_\hbar\big(co\mathcal{S}\big)$, $\mathcal{C}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathfrak{I}_\hbar\big(co\mathcal{S}\big)$, $\mathcal{I}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathcal{C}_\hbar\big(co\mathcal{S}\big)$ and $\mathfrak{I}_\hbar\big(co\mathcal{S}\big) \longleftrightarrow \mathfrak{C}_\hbar\big(co\mathcal{S}\big)$ which form a *commutative* square. In fact, each of these maps, or their inverse, is of type $X_\hbar \mapsto X_\hbar^\perp$, $A_\hbar \mapsto H_\hbar A_\hbar^+$ or $K_\hbar \mapsto H_\hbar^{coK_\hbar}$ (see Subsection 2.2): since $X_\hbar \subseteq \big(X_\hbar^\perp\big)^\perp$ and $A_\hbar \subseteq H_\hbar^{co(H_\hbar A_\hbar^+)}$ in general, and these inclusions are identities at $\hbar = 0$, one gets $X_\hbar = \big(X_\hbar^\perp\big)^\perp$ and $A_\hbar = H_\hbar^{co(H_\hbar A_\hbar^+)}$.
Note that $\mathcal{I}_\hbar\big(co\mathcal{S}\big)$, $\mathfrak{C}_\hbar\big(co\mathcal{S}\big)$, $\mathcal{C}_\hbar\big(co\mathcal{S}\big)$ and $\mathfrak{I}_\hbar\big(co\mathcal{S}\big)$ are again lattices with respect to inclusion, hence they will be thought of as categories too.

3.7. *Remark*

If a quadruple $\big(\mathcal{I}_\hbar , \mathcal{C}_\hbar , \mathfrak{I}_\hbar , \mathfrak{C}_\hbar\big)$ is given which enjoys all properties in the first and the second column of $(3.1)'$, along with relations (3.2), then one easily checks that the four specialized objects $\mathcal{I} := \mathcal{I}_\hbar\big|_{\hbar=0}$, $\mathcal{C} := \mathcal{C}_\hbar\big|_{\hbar=0}$,

$\mathfrak{I} := \mathfrak{I}_\hbar\big|_{\hbar=0}$ and $\mathfrak{C} := \mathfrak{C}_\hbar\big|_{\hbar=0}$ verify relations *(1)* and *(2)* in Subsection 2.2. Therefore, these four objects define just *one* single pair *(coisotropic subgroup, Poisson quotient)*, and the quadruple $\big(\mathcal{I}_\hbar\,,\,\mathcal{C}_\hbar\,,\,\mathfrak{I}_\hbar\,,\,\mathfrak{C}_\hbar\big)$ yields a quantization of the latter in the sense of Subsection 3.4.

3.8. *General program*

From the setup of Subsection 2.2, we follow this scheme:

$$\begin{array}{llll}
(a) & \big(F[[G]]\supseteq\big)\ \mathcal{I} \xrightarrow{(1)} \mathcal{I}_\hbar \xrightarrow{(2)} {\mathcal{I}_\hbar}^{\curlyvee} \xrightarrow{(3)} {\mathcal{I}_0}^{\curlyvee}\ \ \big(\subseteq U\big(\mathfrak{g}^*\big)\big) \\
(b) & \big(F[[G]]\supseteq\big)\ \mathcal{C} \xrightarrow{(1)} \mathcal{C}_\hbar \xrightarrow{(2)} \mathcal{C}_\hbar^{\triangledown} \xrightarrow{(3)} \mathcal{C}_0^{\triangledown}\ \ \big(\subseteq U\big(\mathfrak{g}^*\big)\big) \\
(c) & \big(U(\mathfrak{g})\supseteq\big)\ \mathfrak{I} \xrightarrow{(1)} \mathfrak{I}_\hbar \xrightarrow{(2)} \mathfrak{I}_\hbar^{\,!} \xrightarrow{(3)} \mathfrak{I}_0^{\,!}\ \ \big(\subseteq F[[G^*]]\big) \\
(d) & \big(U(\mathfrak{g})\supseteq\big)\ \mathfrak{C} \xrightarrow{(1)} \mathfrak{C}_\hbar \xrightarrow{(2)} \mathfrak{C}_\hbar^{\,\daleth} \xrightarrow{(3)} \mathfrak{C}_0^{\,\daleth}\ \ \big(\subseteq F[[G^*]]\big)
\end{array}$$

Its meaning is the following. Starting from the first column (in left-hand side) we move a first step — arrows (1) — which is some quantization process. Instead, the last step — arrows (3) — is a specialization (at $\hbar = 0$) process. In between, the middle step — arrows (2) — demands some new idea: here we shall apply some suitable “adaptations” of Drinfeld's functors to the quantizations (of a coisotropic subgroup or a Poisson quotient) obtained from step (1). Roughly, the idea is to take the suitable Drinfeld's functor on the quantum group — $F_\hbar[[G]]$ or $U_\hbar(\mathfrak{g})$ — and to “restrict it” to the quantum sub-object given by step (1). The points to show then are

First: each one of the four objects in the third column above — that is, provided by arrows (2) — is one of the four algebraic objects which describe a quantum (closed formal) coisotropic subgroup or Poisson quotient of G^*. Namely, the correspondence of “types” is

$$(a) \Longrightarrow (c)\ ,\qquad (b) \Longrightarrow (d)\ ,\qquad (c) \Longrightarrow (a)\ ,\qquad (d) \Longrightarrow (b)$$

where notation *(x)* $\Longrightarrow$ *(y)* means that an object of “type *(x)*” — referring to the classification of (3.1) or of (3.1)′ — yields an object of “type *(y)*”.

Second: the four formal subgroups or homogeneous spaces of G^* obtained above provide only *one* single pair *(subgroup, homogeneous space)*.

Third: if we start from a coisotropic subgroup K, and/or a Poisson quotient $G\big/K$, of G, then the Poisson quotient and/or the subgroup of G^* obtained above are $G^*\big/K^\perp$ and/or $K^\perp$ (cf. Definition 2.1) respectively.

4. Drinfeld-like functors on quantum subgroups and quantum Poisson quotients

4.1. *Restricting Drinfeld's functors*

The main idea in the program sketched in Subsection 3.8 is the intermediate step — provided by arrows (2) — namely that of "restricting" Drinfeld's functors, originally defined for quantum groups, to the quantum sub-objects we are interested in. To this end, the "right" definition is the following:

Definition 4.1. Let $J := Ker\big(\epsilon_{F_\hbar[[G]]}\big)\,,\ I := J + \hbar\, F_\hbar[[G]]\,.$

(a) $\mathcal{I}_\hbar{}^{\curlyvee} \;:=\; \sum\limits_{n=1}^{\infty} \hbar^{-n} \cdot I^{n-1} \cdot \mathcal{I}_\hbar \;=\; \sum\limits_{n=1}^{\infty} \hbar^{-n} \cdot J^{n-1} \cdot \mathcal{I}_\hbar$

(b) $\mathcal{C}_\hbar^{\triangledown} \;:=\; \mathcal{C}_\hbar \,+\, \sum\limits_{n=1}^{\infty} \hbar^{-n} \cdot \big(\mathcal{C}_\hbar \cap I\,\big)^n \;=\; \Bbbk[[\hbar]] \cdot 1 \,+\, \sum\limits_{n=1}^{\infty} \hbar^{-n} \cdot \big(\mathcal{C}_\hbar \cap J\,\big)^n$

(c) $\mathfrak{I}_\hbar^{\,!} \;:=\; \left\{ x \in \mathfrak{I}_\hbar \;\middle|\; \delta_n(x) \in \hbar^n \sum\limits_{s=1}^{n} U_\hbar{}^{\widehat{\otimes}\,(s-1)}\, \widehat{\otimes}\, \mathfrak{I}_\hbar \,\widehat{\otimes}\, U_\hbar{}^{\widehat{\otimes}\,(n-s)},\ \forall\, n \in \mathbb{N}_+ \right\}$

(d) $\mathfrak{C}_\hbar^{\,\Lsh} \;:=\; \left\{ x \in \mathfrak{C}_\hbar \;\middle|\; \delta_n(x) \in \hbar^n\, U_\hbar{}^{\widehat{\otimes}\,(n-1)}\, \widehat{\otimes}\, \mathfrak{C}_\hbar\,,\ \forall\, n \in \mathbb{N}_+ \right\}$

Indeed, directly by definitions one has that

$$\mathcal{I}_\hbar{}^{\curlyvee} \supseteq \mathcal{I}_\hbar \quad , \qquad \mathcal{C}_\hbar^{\triangledown} \supseteq \mathcal{C}_\hbar \quad , \qquad \mathfrak{I}_\hbar^{\,!} \subseteq \mathfrak{I}_\hbar \quad , \qquad \mathfrak{C}_\hbar^{\,\Lsh} \subseteq \mathfrak{C}_\hbar \quad .$$

But even more, a careful (yet easy) analysis of definitions and of the relationship between each quantum groups and its relevant sub-objects shows that, in force of $(3.1)'$ — in particular, the mid column there — one has

$$\mathcal{I}_\hbar \,=\, \mathcal{I}_\hbar{}^{\curlyvee} \cap F_\hbar\,, \quad \mathcal{C}_\hbar \,=\, \mathcal{C}_\hbar^{\triangledown} \cap F_\hbar\,, \quad \mathfrak{I}_\hbar^{\,!} = \mathfrak{I}_\hbar \cap U_\hbar{}'\,, \quad \mathfrak{C}_\hbar^{\,\Lsh} = \mathfrak{C}_\hbar \cap U_\hbar{}'$$

This proves that, in very precise sense, Definition 4.1 really provides a "restriction" of Drinfeld's functors from quantum groups to our quantum subgroups — in cases *(a)* and *(d)* — or quantum Poisson quotients — in cases *(b)* and *(c)*. This also motivates the notation: indeed, the symbols ${}^{\curlyvee}$ and ${}^{\triangledown}$ are (or should be) remindful of ${}^{\vee}$, while ${}^{!}$ and ${}^{\Lsh}$ are remindful of ${}'$.

We can now state the QDP for coisotropic subgroups and Poisson quotients:

Theorem 4.1. *("QDP for Coisotropic Subgroups and Poisson Quotients")*

(a) Definition 4.1 provides category equivalences

$$(\ )^{\curlyvee} :\ \mathcal{I}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\;\cong\;} \mathfrak{I}_\hbar\big(co\mathcal{S}(G^*)\big)\ , \qquad (\ )^{\triangledown} :\ \mathcal{C}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\;\cong\;} \mathfrak{C}_\hbar\big(co\mathcal{S}(G^*)\big)$$

$$(\ )^{!} :\ \mathfrak{I}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\;\cong\;} \mathcal{I}_\hbar\big(co\mathcal{S}(G^*)\big)\ , \qquad (\ )^{\Lsh} :\ \mathfrak{C}_\hbar\big(co\mathcal{S}(G)\big) \xrightarrow{\;\cong\;} \mathcal{C}_\hbar\big(co\mathcal{S}(G^*)\big)$$

along with the similar ones with G and G^* interchanged, such that $(\ )^{!}$ and $(\ )^{\curlyvee}$ are inverse to each other, and $(\ )^{\Lsh}$ and $(\ )^{\triangledown}$ are inverse to each other.

(b) (the QDP) For any $K \in co\mathcal{S}(G)$ and $\mathfrak{k} := Lie(K)$, we have

$$\mathcal{I}(K)_\hbar^{\curlyvee} \mod \hbar\, F_\hbar[[G]]^\vee = \mathfrak{I}\big(\mathfrak{k}^\perp\big)\,, \qquad \mathcal{C}(K)_\hbar^{\triangledown} \mod \hbar\, F_\hbar[[G]]^\vee = \mathfrak{C}\big(\mathfrak{k}^\perp\big)\,,$$
$$\mathfrak{I}(\mathfrak{k})_\hbar^{!} \mod \hbar\, U_\hbar(\mathfrak{g})' = \mathcal{I}\big(K^\perp\big)\,, \qquad \mathfrak{C}(\mathfrak{k})_\hbar^{\Lsh} \mod \hbar\, U_\hbar(\mathfrak{g})' = \mathcal{C}\big(K^\perp\big)\,.$$

That is, $\Big(\mathcal{I}(K)_\hbar^{\curlyvee}, \mathcal{C}(K)_\hbar^{\triangledown}, \mathfrak{I}(\mathfrak{k})_\hbar^{!}, \mathfrak{C}(\mathfrak{k})_\hbar^{\Lsh}\Big)$ is a quantization of the quadruple $\Big(\mathfrak{I}\big(\mathfrak{k}^\perp\big), \mathfrak{C}\big(\mathfrak{k}^\perp\big), \mathcal{I}\big(K^\perp\big), \mathcal{C}\big(K^\perp\big)\Big)$ w.r.t. the quantization $\Big(F_\hbar[[G]]^\vee, U_\hbar(\mathfrak{g})'\Big)$ of $\Big(U(\mathfrak{g}^*)\,,\, F[[G^*]]\Big)$, which again satisfies relations like in (3.2) .

Sketch proof. Let us draw a quick sketch of the proof of Theorem 4.1, as it is given in the extended version [3]. The main idea is to reduce everything to the study of $\mathcal{I}(K)_\hbar^{\curlyvee}$ and $\mathcal{C}(K)_\hbar^{\triangledown}$, and to get the rest via an indirect approach. Indeed, this amounts to show that the quadruple $\Big(\mathcal{I}(K)_\hbar^{\curlyvee}, \mathcal{C}(K)_\hbar^{\triangledown}, \mathfrak{I}(\mathfrak{k})_\hbar^{!}, \mathfrak{C}(\mathfrak{k})_\hbar^{\Lsh}\Big)$ satisfies relations similar to (3.2), namely

$$\begin{gathered} \mathcal{I}_\hbar^{\curlyvee} = \big(\mathfrak{C}_\hbar^{\Lsh}\big)^\perp, \quad \mathfrak{C}_\hbar^{\Lsh} = \big(\mathcal{I}_\hbar^{\curlyvee}\big)^\perp, \qquad \mathfrak{I}_\hbar^{!} = \big(\mathcal{C}_\hbar^{\triangledown}\big)^\perp, \quad \mathcal{C}_\hbar^{\triangledown} = \big(\mathfrak{I}_\hbar^{!}\big)^\perp \\ \mathcal{I}_\hbar^{\curlyvee} = F_\hbar^\vee \big(\mathcal{C}_\hbar^{\triangledown}\big)^+,\ \mathcal{C}_\hbar^{\triangledown} = \big(F_\hbar^\vee\big)^{co\mathcal{I}_\hbar^{\curlyvee}},\ \mathfrak{I}_\hbar^{!} = U_\hbar' \big(\mathfrak{C}_\hbar^{\Lsh}\big)^+,\ \mathfrak{C}_\hbar^{\Lsh} = \big(U_\hbar'\big)^{co\mathfrak{I}_\hbar^{!}} \end{gathered} \tag{4.1}$$

The relations in the top line of (4.1) follow from the similar relations for the elements of the initial quadruple — the top line relations in (3.2) — passing through the duality-preserving property of Drinfeld's functors given by Proposition 3.1. In fact, these orthogonality relations are proved much like Proposition 3.1 itself. Similarly, the bottom line relations in (4.1) follow from the bottom line relations in (3.2) and the very definitions.
In force of (4.1), it is enough to prove that just *one* of the four objects involved — those on left-hand side of the identities in bottom line of (4.1) — has the required properties, in particular, it is a quantum subgroup for $K^\perp$ or a quantum Poisson quotient for $G^*\big/K^\perp$. In fact, the similar results for the other three objects will then follow as a consequence, due to (4.1).

Next step is to prove that our quantum sub-objects have the "right" Hopf algebraic properties, i.e. those occurring in first column of $(3.1)'$, namely

$$\begin{array}{llll} (a)\ \ \mathcal{I}_\hbar^{\curlyvee} \trianglelefteq_\ell F_\hbar^\vee & (b)\ \ \mathcal{C}_\hbar^{\triangledown} \leq^1 F_\hbar^\vee & (c)\ \ \mathfrak{I}_\hbar^{!} \trianglelefteq_\ell U_\hbar' & (d)\ \ \mathfrak{C}_\hbar^{\Lsh} \leq^1 U_\hbar' \\ (e)\ \ \mathcal{I}_\hbar^{\curlyvee} \dot{\trianglelefteq} F_\hbar^\vee & (f)\ \ \mathcal{C}_\hbar^{\triangledown} \dot{\trianglelefteq}_\ell F_\hbar^\vee & (g)\ \ \mathfrak{I}_\hbar^{!} \dot{\trianglelefteq} U_\hbar' & (h)\ \ \mathfrak{C}_\hbar^{\Lsh} \dot{\trianglelefteq}_\ell U_\hbar' \end{array}$$

This is an easy task, with some shortcuts available thanks to (4.1) again.

As a third step, we must prove that our quantum sub-objects have "the good property" with respect to specialization — see $(3.1)'$ — namely

$$\begin{array}{ll} (a)\quad \mathcal{I}_\hbar^{\curlyvee} \bigcap \hbar\, F_\hbar^\vee = \hbar\, \mathcal{I}_\hbar^{\curlyvee} & (b)\quad \mathcal{C}_\hbar^{\triangledown} \bigcap \hbar\, F_\hbar^\vee = \hbar\, \mathcal{C}_\hbar^{\triangledown} \\ (c)\quad \mathfrak{I}_\hbar^{!} \bigcap \hbar\, U_\hbar' = \hbar\, \mathfrak{I}_\hbar^{!} & (d)\quad \mathfrak{C}_\hbar^{\Lsh} \bigcap \hbar\, U_\hbar' = \hbar\, \mathfrak{C}_\hbar^{\Lsh} \end{array}$$

Proving this requires a different analysis according to whether one deals with cases *(a)* and *(b)* or cases *(c)* and *(d)*. Indeed, the latter identities are proved easily, just looking at definitions of ${\mathcal{I}_\hbar}^\curlyvee$ and ${\mathcal{C}_\hbar}^\triangledown$ and reminding that $\mathcal{I}_\hbar$ and $\mathcal{C}_\hbar$ do satisfy the "good property" for specialization, by assumption. Instead, cases *(a)* and *(b)* require a careful description of ${\mathcal{I}_\hbar}^\curlyvee$ and ${\mathcal{C}_\hbar}^\triangledown$.
Let $I := I_{F_\hbar}$ and $J := J_{F_\hbar}$ be as in Subsection 3.3, and $J^\vee := \hbar^{-1} J \subset {F_\hbar}^\vee$. Then $J \mod \hbar\, F_\hbar = J_G := Ker\big(\epsilon : F[[G]] \longrightarrow \Bbbk\big)$, and $J_G \big/ {J_G}^2 = \mathfrak{g}^*$. Let $\{y_1, \dots, y_n\}$, with $n := \dim(G)$, be a $\Bbbk$–basis of $J_G \big/ {J_G}^2$, and pull it back to a subset $\{j_1, \dots, j_n\}$ of J . Then $\big\{ \hbar^{-|\underline{e}|} j^{\,\underline{e}} \mod \hbar\, {F_\hbar}^\vee \,\big|\, \underline{e} \in \mathbb{N}^n \big\}$ (with $j^{\,\underline{e}} := \prod_{s=1}^n j_s^{\underline{e}(i)}$), is a $\Bbbk$–basis of ${F_0}^\vee$ and, if ${j_s}^\vee := \hbar^{-1} j_s$ for all s, the set $\big\{ j_1^\vee, \dots, j_n^\vee \big\}$ is a $\Bbbk$–basis of $\mathfrak{t} := J^\vee \mod \hbar\, {F_\hbar}^\vee$. Moreover, since $j_\mu\, j_\nu - j_\nu\, j_\mu \in \hbar\, J$ (for $\mu, \nu \in \{1, \dots, n\}$) we have $j_\mu\, j_\nu - j_\nu\, j_\mu = \hbar \sum_{s=1}^n c_s\, j_s + \hbar^2 \gamma_1 + \hbar\, \gamma_2$ for some $c_s \in \Bbbk[[\hbar]]$, $\gamma_1 \in J$ and $\gamma_2 \in J^2$, whence $\big[j_\mu^\vee, j_\nu^\vee \big] := j_\mu^\vee\, j_\nu^\vee - j_\nu^\vee\, j_\mu^\vee \equiv \sum_{s=1}^n c_s\, j_s^\vee \mod \hbar\, {F_\hbar}^\vee$, thus $\mathfrak{t} := J^\vee \mod \hbar\, {F_\hbar}^\vee$ is a Lie subalgebra of ${F_0}^\vee$: indeed, ${F_0}^\vee = U(\mathfrak{t})$ as Hopf algebras. Even more, this description also shows that the linear map $\mathfrak{t} \longrightarrow \mathfrak{g}^*$ given by $y_s \mapsto j_s^\vee \, \big(\mathrm{mod}\ \hbar\, {F_\hbar}^\vee\big)$, $s = 1, \dots, n$, is a Lie bialgebra isomorphism.
Let us now fix the set $\{y_1, \dots, y_n\}$ as follows. If $k := \dim(K)$, we can choose a system of parameters for G , say $\big\{ j_1, \dots, j_k, j_{k+1}, \dots, j_n \big\}$ such that $\mathcal{C}(K) := F[[G]]^K = \Bbbk[[j_{k+1}, \dots, j_n]]$, the topological *subalgebra* of $F[[G]]$ generated by $\big\{ j_{k+1}, \dots, j_n \big\}$, and $\mathcal{I}(K) = \big(j_{k+1}, \dots, j_n \big)$, the *ideal* of $F[[G]]$ topologically generated by $\big\{ j_{k+1}, \dots, j_n \big\}$. Set also $y_s := j_s \mod {J_G}^2$ $(s = 1, \dots n)$. Then $Span\big(\{y_{k+1}, \dots, y_n\}\big) = \mathfrak{k}^\perp$.
Basing on this, one finds that $\mathcal{C}(K)_\hbar^\triangledown$ is just the topological subalgebra of $U_\hbar(\mathfrak{g}) = \Bbbk\big[j_1^\vee, \dots, j_n^\vee\big][[\hbar]]$ generated by $\big\{ j_{k+1}^\vee, \dots, j_n^\vee \big\}$, that is $\Bbbk\big[j_{k+1}^\vee, \dots, j_n^\vee\big][[\hbar]]$. Similarly, ${\mathcal{I}_\hbar}^\curlyvee$ is the left ideal of $U_\hbar(\mathfrak{g})$ generated by $\big\{ j_{k+1}^\vee, \dots, j_n^\vee \big\}$, that is the set of all series (in $\hbar$) in $\Bbbk\big[j_1^\vee, \dots, j_n^\vee\big][[\hbar]]$ whose coefficients belong to the ideal of $\Bbbk\big[j_1^\vee, \dots, j_n^\vee\big]$ generated by $\big\{ j_{k+1}^\vee, \dots, j_n^\vee \big\}$. Then $(3.1)'$ implies that $\mathcal{C}(K)_\hbar^\triangledown \cap \hbar\, F_\hbar[G]^\vee \subseteq \hbar\, \mathcal{C}(K)_\hbar^\triangledown$, while the converse is obvious. This proves *(b)*, and case *(a)* is similar.

At this point, one has proved that each element of the quadruple $\Big(\mathcal{I}(K)_\hbar^\curlyvee \, , \, \mathcal{C}(K)_\hbar^\triangledown \, , \, \mathfrak{I}(\mathfrak{k})_\hbar^{\,!} \, , \, \mathfrak{C}(\mathfrak{k})_\hbar^{\,\dagger} \Big)$ is a quantum subgroup — the second and third element — or a quantum Poisson quotient — the first and fourth element — of the dual Poisson group G^* , with respect to the fixed quantization $\Big(F_\hbar[[G]]^\vee, U_\hbar(\mathfrak{g})' \Big)$ of $\Big(U(\mathfrak{g}^*) \, , \, F[[G^*]] \Big)$. In addition, relations (4.1) induce similar relations when specializing at $\hbar = 0$, so these quantum subgroups and quantum Poisson quotients all provide a quantization of *one* single pair $\big(T \, , \, G^*\!/T \, \big)$, for some coisotropic (formal) subgroup T of G^* .

The fourth step concerns last part of claim *(a)*: it amounts to prove that

$$\big(\mathcal{I}_\hbar^{\curlyvee}\big)^! = \mathcal{I}_\hbar \quad , \quad \big(\mathcal{C}_\hbar^{\triangledown}\big)^{\Lsh} = \mathcal{C}_\hbar \quad , \quad \big(\mathfrak{I}_\hbar^{!}\big)^{\curlyvee} = \mathfrak{I}_\hbar \quad , \quad \big(\mathfrak{C}_\hbar^{\Lsh}\big)^{\triangledown} = \mathfrak{C}_\hbar$$

Now, the very definitions imply at once one-way inclusions

$$\big(\mathcal{I}_\hbar^{\curlyvee}\big)^! \supseteq \mathcal{I}_\hbar \quad , \quad \big(\mathcal{C}_\hbar^{\triangledown}\big)^{\Lsh} \supseteq \mathcal{C}_\hbar \quad , \quad \big(\mathfrak{I}_\hbar^{!}\big)^{\curlyvee} \subseteq \mathfrak{I}_\hbar \quad , \quad \big(\mathfrak{C}_\hbar^{\Lsh}\big)^{\triangledown} \subseteq \mathfrak{C}_\hbar$$

For the converse inclusions, in the first or the second case they follow again from the description of $\mathcal{I}_\hbar^{\curlyvee}$ or $\mathcal{C}_\hbar^{\triangledown}$, respectively. Then one uses the identities just proved, along with the orthogonality-preserving properties of Drinfeld-like functors — the top line in (4.1) — applied twice, to obtain the full identities in the other cases too. Note that (again) one could simply prove only one of the identities involved, and then get the others via (4.1).

The last step is to show that the coisotropic (formal) subgroup T of G^* found above is $K^\perp$, i.e. $\mathit{Lie}\,(T) = \mathfrak{k}^\perp$. This means that one has to specialize our quantum subobjects at $\hbar = 0$. By the third step, it is enough to do it for any one of them. The best choice is again $\mathcal{C}_\hbar^{\triangledown}$ or $\mathcal{I}_\hbar^{\curlyvee}$, whose (almost) explicit description yields an explicit description of its specialization too.

4.2. *Remark*

We point out that quantum coisotropic subgroups such as $\mathcal{I}(K)_\hbar$ and $\mathfrak{C}(\mathfrak{k})_\hbar$ provide quantum Poisson quotients $\mathcal{I}(K)_\hbar^{\curlyvee} = \mathfrak{I}\big(\mathfrak{k}^\perp\big)_\hbar$ and $\mathfrak{C}(\mathfrak{k})_\hbar^{\Lsh} = \mathcal{C}\big(K^\perp\big)_\hbar$ respectively, while the quantum Poisson quotients $\mathcal{C}(K)_\hbar$ and $\mathfrak{I}(\mathfrak{k})_\hbar$ yield quantum coisotropic subgroups $\mathcal{C}(K)_\hbar^{\triangledown} = \mathfrak{C}\big(\mathfrak{k}^\perp\big)_\hbar$ and $\mathfrak{I}(\mathfrak{k})_\hbar^{!} = \mathcal{I}\big(K^\perp\big)_\hbar$ respectively. Thus, Drinfeld-like functors map quantum coisotropic subgroups to quantum Poisson quotients, and viceversa.

5. Generalizations and effectiveness

5.1. *QDP with half quantizations*

In this work we start from a pair of mutually dual quantum groups, i.e. $\big(F_\hbar[[G]]\,,\,U_\hbar(\mathfrak{g})\big)$. This is used in the proofs to apply orthogonality arguments. However, this is only a matter of choice. Indeed, our QDP deals with quantum subgroups or quantum homogeneous spaces which are contained either in $F_\hbar[[G]]$ or in $U_\hbar(\mathfrak{g})$. In fact *we might prove every step in our discussion using only the single quantum group which is concerned*, and only one quantum subgroup (such as $\mathcal{I}_\hbar$, or $\mathcal{C}_\hbar$, etc.) at the time, by a direct method with no orthogonality arguments.

5.2. *QDP with global quantizations*

In this paper we use quantum groups in the sense of Definition 3.1; these are sometimes called *local* quantizations. Instead, one can consider *global quantizations*: quantum groups like Jimbo's, Lusztig's, etc. The latter differ from the former in that

–1) they are standard (rather than topological) Hopf algebras;

–2) they may be defined over any ring R, the rôle of $\hbar$ being played by a suitable element of that ring (for example, $R = \Bbbk\big[q, q^{-1}\big]$ and $\hbar = q - 1$).

Now, our analysis may be done also in terms of *global quantum groups* and their specializations. First, one starts with *algebraic* (instead of formal) Poisson groups and Poisson homogeneous spaces. Then one defines Drinfeld-like functors in a similar manner; the key fact is that the QDP for quantum groups has a *global version [12]*, and the recipes of Section 4 to define Drinfeld-like functors still do make sense. Moreover, one can extend our QDP for coisotropic subgroups (and Poisson quotients) to *all* closed subgroups and homogeneous spaces: all this will be treated separately.

5.3. *$*$–structures and QDP in the real case*

If one looks for quantizations of *real* subgroups and homogeneous spaces, then one must consider $*$–structures on the quantum group Hopf algebras. Then one can perform all our construction in this setting, and state and prove a version of the QDP for *real* quantum subgroups and quantum homogeneous spaces too, both in the formal and in the global setting.

5.4. *QDP for pointed Poisson varieties*

Let *pointed Poisson variety (=p.P.v.)* be any pair $(M, \bar{m})$ where M is a Poisson variety and $\bar{m} \in M$ is such that $\{\bar{m}\}$ is a symplectic leaf of M. A *morphism of p.P.v.'s* $(M, \bar{m})$, $(N, \bar{n})$ is any Poisson map $\varphi : M \longrightarrow N$ such that $\varphi(\bar{m}) = \bar{n}$. This defines a subcategory of the category of all Poisson varieties, whose morphisms are those which map distinguished points into distinguished points. In terms of affine algebraic geometry, a p.P.v. $(M, \bar{m})$ is given by the pair $\big(F[M], \mathfrak{m}_{\bar{m}}\big)$ where $F[M]$ is the function algebra of M and $\mathfrak{m}_{\bar{m}}$ is the defining ideal of $\bar{m} \in M$ in $F[M]$.

By assumptions, the Poisson bracket of $F[M]$ restricts to a Lie bracket onto $\mathfrak{m}_{\bar{m}}$: then [16] $\mathcal{L}_M := \mathfrak{m}_{\bar{m}} \big/ \mathfrak{m}_{\bar{m}}^2$ (the cotangent space to M at $\bar{m}$) inherits a Lie algebra structure too, the so-called "linear approximation of M at $\bar{m}$". Poisson quotients are natural examples of p.P.v.'s. Other examples are Poisson monoids (= unital Poisson semigroups), each one being pointed by

its unit element. If $(M,\bar{m}) = (\varLambda,e)$ is a Poisson monoid, then $F[\varLambda]$ is a bialgebra (and conversely), and $\mathcal{L}_\varLambda$ has a natural structure of *Lie bialgebra*, hence $U(\mathcal{L}_\varLambda)$ is a co-Poisson Hopf algebra. The Lie cobracket is induced by the coproduct of $F[\varLambda]$, hence (dually) by the multiplication in $\varLambda$. In particular, when the monoid $\varLambda$ is a Poisson group G we have $\varLambda = \mathfrak{g}^*$.
We call *quantization of a p.P.v.* $(M,\bar{m})$ any unital algebra A in $\mathcal{T}_{\widehat{\otimes}}$ or $\mathcal{P}_{\widetilde{\otimes}}$ (see Subsection 3.1) along with a morphism of topological unital algebras $\underline{\epsilon}_A : A \longrightarrow \Bbbk[[\hbar]]$, such that $A\big|_{\hbar=0} \cong F[M]$ as Poisson $\Bbbk$–algebras and $\pi_{\Bbbk[[\hbar]]} \circ \underline{\epsilon}_A$, with $\pi_A : A \twoheadrightarrow A\big|_{\hbar=0} \cong F[M]$, $\pi_{\Bbbk[[\hbar]]} :\ \Bbbk[[\hbar]] \twoheadrightarrow \Bbbk[[\hbar]]$ the specialisation maps ($\hbar \mapsto 0$); in this case we write $A = F_\hbar[M]$. For any such object we set $J_{\hbar,M} := Ker\big(\underline{\epsilon}_{F_\hbar[M]}\big)$ and $I_{\hbar,M} := J_{\hbar,M} + \hbar\, F_\hbar[M]$. A *morphism of quantizations of p.P.v.'s* is any morphism $\phi : F_\hbar[M] \longrightarrow F_\hbar[N]$ in $\mathcal{A}^+$ such that $\underline{\epsilon}_{F_\hbar[N]} \circ \phi = \underline{\epsilon}_{F_\hbar[M]}$. Quantizations of p.P.v.'s and their morphisms form a subcategory of $\mathcal{T}_{\widehat{\otimes}}$, or $\mathcal{P}_{\widetilde{\otimes}}$, respectively. Also, we might repeat the same construction in the setting of formal geometry, or we might use global quantizations, as in Subsection 5.2.
Now define $F_\hbar[M]^\vee$ like in Subsection 3.3, replacing H with $A = F_\hbar[M]$ and J_H with $J_{\hbar,M}$. Then the same analysis made to prove the parts of Theorem 4.1 concerning $F_\hbar[[G]]^\vee$ proves also the following:

Theorem 5.1. *Let $F_\hbar[M] \in \mathcal{A}^+$ be a quantization of a pointed Poisson manifold $(M,\bar{m})$ as above. Then $F_\hbar[M]^\vee$ is a quantization of $U(\mathcal{L}_M)$, i.e.*

$$F_\hbar[M]^\vee\Big|_{\hbar=0} := F_\hbar[M]^\vee \Big/ \hbar\, F_\hbar[M]^\vee = U(\mathcal{L}_M)$$

If in addition M is a Poisson monoid and $F_\hbar[M]$ is a quantization of $F[M]$ as a bialgebra, then the last identification above is one of Hopf algebras. Moreover, the construction $F_\hbar[M] \mapsto F_\hbar[M]^\vee$ is functorial.

5.5. *Computations*

In the extended version [3] of this work a nontrivial example is treated in detail. Here instead, let us consider the 1–parameter family of *quantum spheres* $\mathbb{S}^2_{q,c}$ described by Dijkhuizen and Koornwinder [4]. These are quantum homogeneous spaces for the standard $SU_q(2)$, described by objects of type *(c)*. Such objects, corresponding to 1–dimensional subgroups conjugated to the diagonally embedded $\mathbb{S}^1$, are right ideals and two sided coideals in $U_q\big(\mathfrak{su}(2)\big)$ generated by a single twisted primitive element X_ρ, i.e. $X_\rho = \big(q^2+1\big)^{-1/2} L^{-1} E + \big(q^2+1\big)^{-1/2} F\, L - \rho\, \frac{(q+q^{-1})^{1/2}}{q-q^{-1}} \big(L - L^{-1}\big)$, using notations of the work of Gavarini [12]. Applying the suitable duality functor, these elements correspond to generators of dual right ideals and two sided coideals in $F_q\big[SU(2)^*\big]$, say $\xi_\rho = x + y - \rho\left(z - z^{-1}\right)$ (in Gavarini's [12] notation again). Modding out these elements provides the corresponding quantum coisotropic subgroups. Here much of the Poisson aspects of the theory trivializes. It would be interesting to carry out similar computations for the 1–parameter families of quantum projective spaces described by Dijkhuizen and Noumi [5].

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