ON GENERALIZED CATENOIDS *

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In [1] the author proved that a conformally flat, minimal hypersurface M^n, $n \geq 4$, of Euclidean space E^{n+1} is either totally geodesic or a hypersurface of revolution with a well defined profile curve; such a hypersurface is called a generalized catenoid. In the present paper we extend this result to higher codimension proving that if $p \leq \min\{4, n-3\}$, a conformally flat, minimal submanifold M^n of Euclidean space E^{n+p} whose Schouten tensor has at most two eigenvalues, is either totally geodesic or a generalized catenoid lying in some $(n+1)$-dimensional Euclidean space.

1. Introduction

For hypersurfaces of dimension $n \geq 4$ in Euclidean space, Cartan [3] showed that a conformally flat hypersurface is quasi-umbilical (i.e. the Weingarten map has an eigenvalue of multiplicity $\geq n-1$). Common examples, also due to Cartan, are the canal hypersurfaces, i.e. envelopes of one-parameter families of hyperspheres. Thus conformal flatness can often be viewed as a natural generalization of a surface of revolution.

In [1] the author proved that a conformally flat, minimal hypersurface M^n, $n \geq 4$, of Euclidean space E^{n+1} is either totally geodesic or a hypersurface of revolution $S^{n-1} \times \gamma(s)$ where the profile curve is a plane curve γ determined by its curvature κ as a function of arc length by $\kappa = (1-n)/u^n$ and

$$s = \int \frac{u^{n-1}du}{\sqrt{Cu^{2n-2}-1}} \ ,$$

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C being a constant. If $n = 3$, replacing conformal flatness by quasi-umbilicity gives the same result with the same proof. For $n = 2$, the profile curve is a catenary and hence these hypersurfaces are called *generalized catenoids*. In 1991 Jagy [8] gave an independent study of this question by assuming that the minimal hypersurface is foliated by spheres from the outset.

Recently Castro and Urbano [4] introduced a Lagrangian catenoid. Their result is that a minimal, Lagrangian submanifold of $\mathbb{C}^n$ which is foliated by round $(n-1)$-spheres is homothetic to the Lagrangian catenoid. In [2] the author showed that for a non-flat, conformally flat, minimal, Lagrangian submanifold of $\mathbb{C}^n$ its Schouten tensor admits an eigenvalue of multiplicity 1. Also if the Schouten tensor of a conformally flat, minimal, Lagrangian submanifold of $\mathbb{C}^n$ admits at most two eigenvalues, then either it is flat and totally geodesic or it is locally homothetic to a Lagrangian catenoid. In view of these results it seems natural to return to the question of the previous paragraph and consider conformally flat, minimal submanifolds of Euclidean space with higher codimension.

We first recall the notion of quasi-umbilicity. For an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold a (local) normal vector field is a *quasi-umbilical section* of the normal bundle if the corresponding Weingarten map has at least $n-1$ eigenvalues equal. The submanifold is said to be *quasi-umbilical* if there exist p mutually orthogonal quasi-umbilical normal sections. It is known that a quasi-umbilical submanifold of dimension ≥ 4 of a conformally flat space is conformally flat [5]. In general the converse is not true; e.g. the Lagrangian catenoid of Castro and Urbano is conformally flat but not quasi-umbilical [2].

The above result of Cartan was generalized by Moore and Morvan [10]; they showed that if $p \leq \min\{4, n-3\}$, a conformally flat submanifold M^n of Euclidean space E^{n+p} is quasi-umbilical.

Here we show that if $p \leq \min\{4, n-3\}$, a conformally flat, minimal submanifold M^n of Euclidean space E^{n+p} whose Schouten tensor has at most two eigenvalues, is either flat and totally geodesic or a generalized catenoid lying in some $(n+1)$-dimensional Euclidean space.

2. Preliminaries

For a Riemannian manifold (M^n, g), let Q denote its Ricci operator and τ its scalar curvature. The Schouten tensor of (M^n, g) is defined by

$$L = -\frac{Q}{n-2} + \frac{\tau}{2(n-1)(n-2)} I$$

and the Weyl conformal curvature tensor is given by the following decomposition of the curvature tensor.

$$\begin{aligned} g(R_{XY}Z, V) = g(W_{XY}Z, V) &- g(LX, V)g(Y, Z) + g(LX, Z)g(Y, V) \\ &- g(LY, Z)g(X, V) + g(LY, V)g(X, Z). \end{aligned} \tag{2.1}$$

It is well known that for $n \geq 4$, M^n is conformally flat if and only if the Weyl conformal curvature tensor vanishes and that this implies that L is a Codazzi tensor, i.e. a symmetric tensor field L of type $(1, 1)$ satisfying

$$(\nabla_X L)Y - (\nabla_Y L)X = 0.$$

For $n = 3$, the Weyl conformal curvature tensor vanishes identically and the manifold is conformally flat if and only if its Schouten tensor is a Codazzi tensor.

Basic properties of Codazzi tensors in general were given by Derdziński in [6]. In particular we have the following Lemma.

Lemma 2.1 ***(Derdziński)*** *If a Codazzi tensor has more than one eigenvalue, then the eigenspaces for each eigenvalue form an integrable subbundle on open sets of constant multiplicity. If an eigenvalue has multiplicity greater than 1, then the eigenvalue is constant on its integral submanifolds. Moreover the integral submanifolds are umbilical submanifolds and if the eigenvalue is constant on the manifold, then the integral submanifolds are totally geodesic.*

Returning to the form of the curvature tensor for a conformally flat manifold as given by equation (2.1) with $W = 0$, we see that if X is an eigenvector of L with eigenvalue ν_i and Y an eigenvector with eigenvalue ν_j, the sectional curvature $K(X, Y) = -(\nu_i + \nu_j)$.

It is well known, (Kurita [9]), that if a conformally flat manifold is a locally Riemannian product, say $M_1^p \times M_2^q$, $p, q \geq 2$, then each factor is of constant curvature of the opposite sign, i.e. $M_1^p(-c) \times M_2^q(c)$, same c.

Turning to submanifolds, for an isometrically immersed submanifold (M^n, g) of Euclidean space $(E^{n+p}, \langle\, ,\rangle)$ the Levi-Civita connection ∇ of g

and the second fundamental form σ are related to the ambient Levi-Civita connection $\bar{\nabla}$ by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y).$$

For a normal vector field ζ let A_ζ denote the corresponding Weingarten map and let D denote the connection in the normal bundle; in particular A_ζ and D are defined by

$$\bar{\nabla}_X \zeta = -A_\zeta X + D_X \zeta.$$

The Gauss equation is

$$R(X, Y, Z, W) = \langle \sigma(Y, Z), \sigma(X, W) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle.$$

Defining the covariant derivative of σ by

$$(\nabla' \sigma)(X, Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

the Codazzi equation is

$$(R_{X\,Y} Z)^\perp = (\nabla' \sigma)(X, Y, Z) - (\nabla' \sigma)(Y, X, Z).$$

The equation of Ricci-Kühn is

$$R^\perp(X, Y, \eta, \zeta) = g([A_\eta, A_\zeta]X, Y)$$

for normal vectors η and ζ.

We end these preliminaries with the following reduction theorem of Erbacher [7].

Theorem 2.1 ***(Erbacher)*** *Let M^n be a submanifold of a complete, simply-connected real space form $\tilde{M}^{n+p}(c)$. If there exists a normal subbundle E of rank l which is parallel in the normal bundle of the submanifold and if the first normal space (span of the second fundamental form) at each point $p \in M^n$ is contained in the fibre E_p, then M^n is contained in an $(n+l)$-dimensional totally geodesic submanifold of $\tilde{M}^{n+p}(c)$.*

3. Conformally flat, minimal submanifolds

In proving that if $p \leq \min\{4, n-3\}$, a conformally flat submanifold M^n of Euclidean space E^{n+p} is quasi-umbilical, Moore and Morvan showed that there exists an orthonormal basis $e_1, \ldots, e_n$ of the tangent space of M^n

with respect to which the second fundamental form takes the form

$$\sigma(e_i, e_j) = \begin{bmatrix} \zeta_{ab} & & & \\ & f\zeta & & \\ & & \ddots & \\ & & & f\zeta \end{bmatrix}$$

where (ζ_{ab}) is a $p \times p$ matrix of normal vectors and ζ is a unit normal vector. We now prove the following result.

Theorem 3.1 *Let M^n, be a conformally flat, minimal submanifold of E^{n+p} with $p \leq \min\{4, n-3\}$. If the Schouten tensor has at most two eigenvalues, then either M^n is flat and totally geodesic or a generalized catenoid lying in some $(n+1)$-dimensional Euclidean space.*

Proof. We give the proof for $p = 4$; the proofs for $p = 2, 3$ are essentially the same, only easier. Note also that for $p = 4$, $n \geq 7$ and in any case we have $n \geq 5$.

Since M^n is quasi-umbilical by the theorem of Moore and Morvan and the above remark there exist local orthonormal normal fields $\zeta_1, \ldots, \zeta_4$, giving the quasi-umbilicity, whose Weingarten maps take the following forms.

$$A_1 = \begin{bmatrix} a_{ij} & & & \\ & \Lambda_1 & & \\ & & \ddots & \\ & & & \Lambda_1, \end{bmatrix}, \qquad A_2 = \begin{bmatrix} b_{ij} & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} c_{ij} & & & \\ & \Lambda_3 & & \\ & & \ddots & \\ & & & \Lambda_3, \end{bmatrix}, \qquad A_4 = \begin{bmatrix} d_{ij} & & & \\ & \Lambda_4 & & \\ & & \ddots & \\ & & & \Lambda_4 \end{bmatrix},$$

where (a_{ij}), etc. are symmetric 4×4 matrices.

Contracting the Gauss equation and using the minimality, we see that the Ricci operator and the scalar curvature are given by

$$Q = -\sum_{i=1}^{4} A_i^2, \qquad \tau = -|\sigma|^2$$

and hence the Schouten tensor becomes

$$L = \frac{1}{n-2}\sum_{i=1}^{4} A_i^2 - \frac{|\sigma|^2}{2(n-1)(n-2)} I = \frac{1}{n-2}\Big(\sum_{i=1}^{4} A_i^2 - (\mathrm{tr}L)I\Big). \quad (3.1)$$

Thus we see that $e_5, \ldots, e_n$ are eigenvectors of L corresponding to the same eigenvalue, say ν_5. Moreover given the general form of the upper left 4×4 blocks of the Weingarten maps, we may take $e_1, \ldots, e_4$ as eigenvectors of L corresponding to eigenvalues $\nu_1, \ldots, \nu_4$ respectively.

If $A_1 \neq 0$, as the submanifold is quasi-umbilical, the characteristic polynomial, $P(\lambda)$, of (a_{ij}) has Λ_1 as an eigenvalue of multiplicity 3 and by the minimality the remaining eigenvalue is $-(n-1)\Lambda_1$. Thus we can expand the characteristic polynomial in two ways. The minimality or the coefficient of λ^3 yields

$$a_{11} + a_{22} + a_{33} + a_{44} = -(n-4)\Lambda_1.$$

The coefficient of λ^2 yields

$$\begin{aligned} a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2 - a_{11}a_{22} - a_{11}a_{33} - a_{11}a_{44} - a_{22}a_{33} \\ - a_{22}a_{44} - a_{33}a_{44} = (3n-6)\Lambda_1^2. \end{aligned}$$

Now consider the diagonal entries of A_1^2. Using the coefficient of λ^2 above we have for the (1,1) component of A_1^2

$$\begin{aligned} a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 = (3n-6)\Lambda_1^2 - (n-4)a_{11}\Lambda_1 + a_{22}a_{33} \\ - a_{23}^2 + a_{22}a_{44} - a_{24}^2 + a_{33}a_{44} - a_{34}^2. \end{aligned}$$

Similarly one finds the (2,2), (3,3) and (4,4) components. From equation (3.1) we have

$$(n-2)L + (\mathrm{tr}L)I = A_1^2 + A_2^2 + A_3^2 + A_4^2.$$

We give the (1,1) component of this diagonal matrix and the (j,j) component, $j \geq 5$, the other components being found similarly.

$$\begin{aligned} (n-1)\nu_1 + \nu_2 + \nu_3 + \nu_4 + (n-4)\nu_5 = {} & (3n-6)(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2) \\ & - (n-4)(a_{11}\Lambda_1 + b_{11}\Lambda_2 + c_{11}\Lambda_3 + d_{11}\Lambda_4) \\ & + a_{22}a_{33} - a_{23}^2 + a_{22}a_{44} - a_{24}^2 + a_{33}a_{44} - a_{34}^2 \\ & + b_{22}b_{33} - b_{23}^2 + b_{22}b_{44} - b_{24}^2 + b_{33}b_{44} - b_{34}^2 \\ & + c_{22}c_{33} - c_{23}^2 + c_{22}c_{44} - c_{24}^2 + c_{33}c_{44} - c_{34}^2 \\ & + d_{22}d_{33} - d_{23}^2 + d_{22}d_{44} - d_{24}^2 + d_{33}d_{44} - d_{34}^2, \qquad (3.2) \end{aligned}$$

$$\nu_1 + \nu_2 + \nu_3 + \nu_4 + (2n-6)\nu_5 = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2.$$

Adding the equations corresponding to the (1,1) through (4,4) components and using the equation corresponding to the (j, j) component, we have the following two equations.

$$\nu_5 = -\frac{\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2}{2}, \tag{3.3}$$

$$\nu_1 + \nu_2 + \nu_3 + \nu_4 = (n-2)(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2) \tag{3.4}$$

Notice also that $\nu_5 < 0$ (equality with zero is the totally geodesic case) and hence the integral submanifolds of the corresponding subbundle have positive constant curvature.

The proof now proceeds by cases. Since we assume that the Schouten tensor has at most two eigenvalues, it is enough to consider the following five cases.

(1) $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu_5$

(2) $\nu_1 = \nu_2 = \nu_3 = \nu_4 \neq \nu_5$

(3) $\nu_1 = \nu_2 = \nu_3 \neq \nu_4 = \nu_5$

(4) $\nu_1 = \nu_2 \neq \nu_3 = \nu_4 = \nu_5$

(5) $\nu_1 \neq \nu_2 = \nu_3 = \nu_4 = \nu_5$

Case 1) Equations (3.3) and (3.4) immediately yield $\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 = 0$ which together with the minimality gives $A_1 = A_2 = A_3 = A_4 = 0$, the totally geodesic case of the theorem.

Case 2) Since L is a Codazzi tensor, by the lemma of Derdziński the subbundles of the tangent bundle of M^n spanned by $\{e_1, \ldots, e_4\}$ and $\{e_5, \ldots, e_n\}$ are integrable and the eigenvalues of L are constant along the respective integral submanifolds. Equations (3.3) and (3.4) now yield that $\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2$ is constant on M^n. Consequently the integral submanifolds of both subbundles are totally geodesic and therefore M^n is locally a Riemannian product, $M_1^4(-c) \times M_2^{n-4}(c)$, with $c = -2\nu_5 = 2\nu_1$. Then using equations (3.3) and (3.4) again,

$$\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2 = \frac{n-2}{2}(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2)$$

from which $\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2 = 0$. Therefore all ν_i vanish, contradicting $\nu_1 \neq \nu_5$ and hence case 2) cannot occur.

Case 3) Here the spans of $\{e_1, \ldots, e_3\}$ and $\{e_4, \ldots, e_n\}$ are the integrable subbundles and equations (3.3) and (3.4) yield

$$3\nu_1 = (n - \frac{3}{2})(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2).$$

Again we have $\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2$ constant on M^n and M^n is locally $M_1^3(-c) \times M_2^{n-3}(c)$, $c = -2\nu_5 = 2\nu_1$. Therefore

$$\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2 = \frac{2}{3}\Big(n - \frac{3}{2}\Big)(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2)$$

giving $\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2 = 0$ and again all ν_i vanish, a contradiction. Thus case 3) cannot occur.

Case 4) This case is like the previous two with $\{e_1, e_2\}$ and $\{e_3, \ldots, e_n\}$ giving the integrable subbundles. M^n is locally $M_1^2(-c) \times M_2^{n-2}(c)$ with $c = -2\nu_5 = 2\nu_1$ and

$$2\nu_1 = (n-1)(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2) = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2$$

which gives the same contradiction. Therefore case 4) cannot occur.

Case 5) This case is the most involved. By the lemma of Derdziński the eigenspaces of ν_2 $(= \nu_3 = \nu_4 = \nu_5)$ are integrable and the integral submanifolds are $(n-1)$-dimensional umbilical submanifolds in M^n. Also recall that $n \geq 7$ $(\geq 6, 5$ for lower codimension) and hence $n - 1$ is certainly ≥ 3. Thus the integral submanifolds are of positive constant curvature and we can write the metric in the form

$$ds^2 = e^{2f(u_1)}\Big(du_1^2 + \frac{du_2^2 + \cdots + du_n^2}{\big(1 + \frac{1}{4}\sum_{i=2}^n u_i^2\big)^2}\Big).$$

With respect to the orthonormal basis

$$e_1 = e^{-f}\frac{\partial}{\partial u_1} \text{ and } e_j = e^{-f}(1 + \frac{1}{4}\sum_{i=2}^n u_i^2)\frac{\partial}{\partial u_j},\ j > 1$$

the Levi-Civita connection is given as follows where $i, j > 1$, $i \neq j$:

$$\nabla_{e_1} e_1 = 0, \qquad \nabla_{e_1} e_j = 0, \qquad \nabla_{e_i} e_1 = (e_1 f)e_i,$$

$$\nabla_{e_i} e_i = -(e_1 f)e_1 + \frac{e^{-f}}{2}\sum_{l \neq 1, i} u_l e_l, \qquad \nabla_{e_i} e_j = -\frac{e^{-f}}{2} u_j e_i.$$

The computation of the curvature is now straightforward and $\{e_1, \ldots, e_n\}$ is an eigenvector basis of L.

For local orthonormal normal fields, ζ_a, $a = 1, \ldots, 4$, set $\omega_{ba}(X) = \langle D_X \zeta_b, \zeta_a \rangle$. The Codazzi equation now takes the form

$$g((\nabla_X A_a)Y, Z) + \sum_{b \neq a} g(A_b Y, Z)\omega_{ba}(X) = \\ = g((\nabla_Y A_a)X, Z) + \sum_{b \neq a} g(A_b X, Z)\omega_{ba}(Y). \quad (3.5)$$

Consider the Weingarten map A_1 and set $X = e_1$, $Y = e_2$ and $Z = e_5$ in the Codazzi equation. Then

$$g((\nabla_{e_1}(a_{12}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4) - A_1 \nabla_{e_1} e_2, e_5) + \sum_{b \neq 1} g(A_b e_2, e_5)\omega_{b1}(e_1)$$

$$= g((\nabla_{e_2}(a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4) - A_1 \nabla_{e_2} e_1, e_5) + \sum_{b \neq 1} g(A_b e_1, e_5)\omega_{b1}(e_2).$$

From the form of the matrices of the A_a and the Levi-Civita connection as given above, the only surviving term in this equation is $g(a_{12}\nabla_{e_2} e_2, e_5)$ and we have

$$0 = a_{12} \frac{e^{-f}}{2} u_5$$

giving $a_{12} = 0$. Similarly setting $X = e_1$, $Y = e_3$ and $Z = e_5$, and $X = e_1$, $Y = e_4$ and $Z = e_5$, we find $a_{13} = 0$ and $a_{14} = 0$. Thus the matrix of A_1 is of one of the two following forms:

$$\begin{bmatrix} -(n-1)\Lambda_1 & & & & & & \\ & a_{22} & a_{23} & a_{24} & & & \\ & a_{23} & a_{33} & a_{34} & & & \\ & a_{24} & a_{34} & a_{44} & & & \\ & & & & \Lambda_1 & & \\ & & & & & \ddots & \\ & & & & & & \Lambda_1, \end{bmatrix}, \quad \begin{bmatrix} \Lambda_1 & & & & & & \\ & a_{22} & a_{23} & a_{24} & & & \\ & a_{23} & a_{33} & a_{34} & & & \\ & a_{24} & a_{34} & a_{44} & & & \\ & & & & \Lambda_1 & & \\ & & & & & \ddots & \\ & & & & & & \Lambda_1 \end{bmatrix}.$$

Now compute the characteristic polynomial, $P(\lambda)$, of the 3×3 block in each of these matrices in two ways. In the first case

$$P(\lambda) = \lambda^3 - 3\Lambda_1 \lambda^2 + 3\Lambda_1^2 \lambda - \Lambda_1^3.$$

Comparing with the standard expansion of the characteristic polynomial we have

$$a_{22} + a_{33} + a_{44} = 3\Lambda_1, \quad (3.6)$$

$$a_{23}^2 + a_{24}^2 + a_{34}^2 - a_{22}a_{33} - a_{22}a_{44} - a_{33}a_{44} = -3\Lambda_1^2. \quad (3.7)$$

Squaring equation (3.6) and using equation (3.7) we have

$$\begin{aligned} a_{22}^2 + a_{33}^2 &+ a_{44}^2 + 2a_{22}a_{33} + 2a_{22}a_{44} + 2a_{33}a_{44} \\ &= 3(-a_{23}^2 - a_{24}^2 - a_{34}^2 + a_{22}a_{33} + a_{22}a_{44} + a_{33}a_{44}). \end{aligned}$$

Using this we then have

$$\begin{aligned} (a_{22} - a_{33})^2 &+ (a_{22} - a_{44})^2 + (a_{33} - a_{44})^2 \\ &= 2a_{22}^2 + 2a_{33}^2 + 2a_{44}^2 - 2a_{22}a_{33} - 2a_{22}a_{44} - 2a_{33}a_{44} \\ &= -a_{23}^2 - 6a_{24}^2 - 6a_{34}^2. \end{aligned}$$

Therefore $a_{23} = a_{24} = a_{34} = 0$ and $a_{22} = a_{33} = a_{44} = \Lambda_1$. We will return to this form of A_1 after eliminating the second possibility.

In the second form of the matrix of A_1, $-(n-1)\Lambda_1$ must be an eigenvalue of $P(\lambda)$ and we have

$$P(\lambda) = \lambda^3 + (n-3)\Lambda_1\lambda^2 - (2n-3)\Lambda_1^2\lambda + (n-1)\Lambda_1^3.$$

This time the comparison with the standard expansion of the characteristic polynomial gives

$$\begin{aligned} a_{22} + a_{33} + a_{44} &= -(n-3)\Lambda_1, \\ a_{23}^2 + a_{24}^2 + a_{34}^2 - a_{22}a_{33} - a_{22}a_{44} - a_{33}a_{44} &= (2n-3)\Lambda_1^2. \end{aligned}$$

Then from equation (3.2) we have

$$\begin{aligned} (n-1)\nu_1 + (n-1)\nu_5 = (3n-6)&(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2) \\ &- (n-4)(\Lambda_1^2 + b_{11}\Lambda_2 + c_{11}\Lambda_3 + d_{11}\Lambda_4) - (2n-3)\Lambda_1^2 \\ &+ b_{22}b_{33} - b_{23}^2 + b_{22}b_{44} - b_{24}^2 + b_{33}b_{44} - b_{34}^2 \\ &+ c_{22}c_{33} - c_{23}^2 + c_{22}c_{44} - c_{24}^2 + c_{33}c_{44} - c_{34}^2 \\ &+ d_{22}d_{33} - d_{23}^2 + d_{22}d_{44} - d_{24}^2 + d_{33}d_{44} - d_{34}^2. \end{aligned}$$

On the other hand by equations (3.3) and (3.4)

$$(n-1)\nu_1 + (n-1)\nu_5 = (n-1)^2(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2).$$

Comparing we have

$$\begin{aligned} n(n-2)\Lambda_1^2 &+ (n^2 - 5n + 7)(\Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2) = \\ &= (n-4)(b_{11}\Lambda_2 + c_{11}\Lambda_3 + d_{11}\Lambda_4) \\ &\quad + b_{22}b_{33} - b_{23}^2 + b_{22}b_{44} - b_{24}^2 + b_{33}b_{44} - b_{34}^2 \\ &\quad + c_{22}c_{33} - c_{23}^2 + c_{22}c_{44} - c_{24}^2 + c_{33}c_{44} - c_{34}^2 \\ &\quad + d_{22}d_{33} - d_{23}^2 + d_{22}d_{44} - d_{24}^2 + d_{33}d_{44} - d_{34}^2. \end{aligned} \tag{3.8}$$

Now the matrix of A_2 also has one of the above to forms, i.e. $b_{11} = \Lambda_2$ or $b_{11} = -(n-1)\Lambda_2$. Then respectively

$$b_{22}b_{33} - b_{23}^2 + b_{22}b_{44} - b_{24}^2 + b_{33}b_{44} - b_{34}^2 = 3\Lambda_2^2 \quad \text{or} \quad -(2n-3)\Lambda_2^2$$

and similarly for the matrices of A_3 and A_4 Therefore equation (3.8) takes the form

$$n(n-2)\Lambda_1^2 + (n^2-5n+7)(\Lambda_2^2+\Lambda_3^2+\Lambda_4^2)$$

$$= \left\{ \begin{array}{r} (n^2-5n+7)\Lambda_2^2 \\ \text{or} \\ -(3n-7)\Lambda_2^2 \end{array} \right\} + \left\{ \begin{array}{r} (n^2-5n+7)\Lambda_3^2 \\ \text{or} \\ -(3n-7)\Lambda_3^2 \end{array} \right\} + \left\{ \begin{array}{r} (n^2-5n+7)\Lambda_4^2 \\ \text{or} \\ -(3n-7)\Lambda_4^2 \end{array} \right\}.$$

The results of the various sums are that $n(n-2)\Lambda_1^2$ plus $n(n-2)$ times the sum of some or none of the remaining Λ_a^2 vanish. In any case we see that $A_1 = 0$.

Therefore we are now at the point that if M^n is not totally geodesic, not all of the A_a's vanish and the non-vanishing ones are of the form

$$A_a = \begin{bmatrix} -(n-1)\Lambda_a & & & \\ & \Lambda_a & & \\ & & \ddots & \\ & & & \Lambda_a \end{bmatrix}.$$

Using the Codazzi equation (3.5) for A_a with $X = e_1$, $Y = Z = e_2$ we have

$$g(\nabla_{e_1}\Lambda_a e_2, e_2) + \sum_{b\neq a} \Lambda_b \omega_{ba}(e_1) = g(\nabla_{e_2}(-(n-1)\Lambda_a)e_1, e_2) - g(\Lambda_a \nabla_{e_2} e_1, e_2)$$

from which we obtain

$$e_1\Lambda_a + \sum_{b\neq a} \Lambda_b \omega_{ba}(e_1) = -n\Lambda_a(e_1 f). \tag{3.9}$$

Similarly setting $X = e_2$, $Y = e_j$, $Z = e_2$ for $j \geq 3$ and respectively $X = e_3$, $Y = e_2$, $Z = e_3$ to deal with $j = 2$ we have

$$e_j\Lambda_a + \sum_{b\neq a} \Lambda_b \omega_{ba}(e_j) = 0, \quad j \geq 2. \tag{3.10}$$

To complete the proof we introduce new normal fields $\eta_a = \sum_b P_{ba}\zeta_b$, $P \in SO(4)$. Then

$$\bar{\nabla}_X \eta_a = \sum_b (XP_{ba})\zeta_b + \sum_b P_{ba}\Big(- A_b X + \sum_c \omega_{bc}(X)\zeta_c \Big).$$

Thus for the corresponding Weingarten maps, B_a, and covariant derivative in the normal bundle we have

$$B_a = \sum_b P_{ba} A_b, \quad D_X \eta_a = \sum_b \Big(X P_{ba} + \sum_c P_{ca} \omega_{cb}(X) \Big) \zeta_b.$$

Now

$$B_a = \Big(\sum_b \Lambda_b P_{ba} \Big) \begin{bmatrix} -(n-1) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

For $a = 2, 3, 4$ choose P such that $\sum_b \Lambda_b P_{ba} = 0$, then

$$P_{b1} = \frac{\Lambda_b}{\sqrt{\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2}}.$$

Using (3.10) we have for $j \geq 2$

$$e_j P_{b1} = -\frac{\sum_c \Lambda_c \omega_{cb}(e_j)}{\sqrt{\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2}}$$

and

$$\sum_c P_{c1} \omega_{cb}(e_j) = \frac{\sum_c \Lambda_c \omega_{cb}(e_j)}{\sqrt{\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2}}.$$

Therefore

$$e_j P_{b1} + \sum_c P_{c1} \omega_{cb}(e_j) = 0.$$

Making a similar calculation using (3.9) we have

$$e_1 P_{b1} + \sum_c P_{c1} \omega_{cb}(e_1) = 0$$

and hence in general

$$X P_{b1} + \sum_c P_{c1} \omega_{cb}(X) = 0.$$

Thus we see that η_1 spans the first normal space and $D_X \eta_1 = 0$. Therefore by the Erbacher reduction theorem M^n lies in some Euclidean space E^{n+1} as a hypersurface and the theorem follows from the result in [1].

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