Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

TORSE-FORMING VECTOR FIELDS IN T-SEMISYMMETRIC RIEMANNIAN SPACES

JOSEF MIKEŠ, LUKÁŠ RACHŮNEK

Abstract. In this paper we consider torse-forming vector fields in T-semisymmetric Riemannian spaces. We prove that if T_i- and T_{ij}-semisymmetric spaces admit a non-isotropic torse-forming vector field, then it is convergent; non-Einsteinian Ricci semisymmetric spaces with a harmonic Riemannian tensor do not admit non-recurrent torse-forming vector fields. Our paper generalizes earlier results by J. Kowolik and also results concerning almost Kenmotsu manifolds.

1. Introduction

This paper is concerned about certain questions of torse-forming vector fields in T-semisymmetric Riemannian spaces. The analysis is carried out in tensor form, locally in a class of sufficiently smooth real functions.

One of the most studied classes of special (pseudo-) Riemannian spaces V_n are semisymmetric spaces, which were introduced by N.S. Sinyukov in 1954 (see [4], [13], [17]) and which generalize symmetric spaces. Semisymmetric spaces are investigated in detail by E. Boeckx, O. Kowalski and L. Vanhecke [4].

A generalization of semisymmetric spaces is Ricci semisymmetric spaces, and these are further generalized and *T-semisymmetric* spaces are intruduced.

A Riemannian space V_n is called *T-semisymmetric* ([12], [13]), if for a tensor T the condition $R(X,Y)\circ T = 0$ holds for arbitrary vector fields X, Y, where $R(X,Y)$ denotes the corresponding curvature transformation and the symbol $\circ$ indicates the corresponding derivation on the algebra of all tensor fields. We can write this condition in the local transcription as

$$T^{\cdots}_{\cdots,[lm]} = 0 \tag{1}$$

where "," denotes the covariant derivative with respect to a (possibly *indefinite*) metric tensor g_{ij} of a Riemannian space V_n and $[jk]$ denotes the alternation with respect to j and k.

1991 *Mathematics Subject Classification.* 53B20, 53B30.

Key words and phrases. torse-forming vector field, concircular vector field, convergent vector field, recurrent vector field, T-semisymmetric Riemannian spaces, Kenmotsu manifolds.

Supported by grant No. 201/99/0265 of The Grant Agency of Czech Republic.

Evidently, a T-semisymmetric space is *semisymmetric*, or *Ricci semisymmetric*, respectively, if T is the Riemannian curvature tensor R, or Ricci tensor Ric, respectively (see [2], [3], [4], [12], [13], [17]).

The study of recurrent, convergent, concircular and torse-forming vector fields has a long history starting in 1925 by the works of H.W. Brinkmann [5], P.A. Shirokov [19] and K. Yano [22], [23]. In Riemannian spaces V_n with the above vector fields there exists a metric of a special form; these spaces are now called (*almost*) *warped products* [6]. These vector fields have been used in many areas of differential geometry, for example in conformal mappings and transformations [5], [8], [22], geodesic, almost geodesic and holomorphically projective mappings and transformations (see [1], [10] – [15], [17], [18], [20], [21]), and others [1], [2], [3], [6], [7], [9], [11], [13], [16], [17], ...

In the papers [2], [3], [7], [9], [16] there were studied semisymmetric and Ricci semisymmetric spaces which contain concircular and torse-forming vector fields satisfying some other assumptions. Our work is devoted to a generalization and extension of these results.

Particulary, we extend the following

Theorem [J. Kowolik, Th. 1, [9]]. *Let a Riemannian manifold V_n $(n \geq 4)$ be a Ricci-semisymmetric space whose Ricci tensor is a Codazzi tensor (i.e. $R_{ij,k} = R_{ik,j}$). If V_n admits a torse-forming vector field ξ then either ξ is a concircular vector field or it reduces to a recurrent one.*

Here, in Theorem 5, we generalize Kowolik's result [Th. 1] for Ricci-semisymmetric spaces V_n $(n > 2)$ where the Ricci tensor need not be a Codazzi tensor.

Moreover, our Theorem 9 shows that under the assumptions of Kowolik's theorem [Th. 1] we get the stronger assertion that a torse forming vector field is recurrent. This implies that the second theorem in Kowolik's paper is contained in our Theorem 9.

2. On the theory of torse-forming vector fields

Now we will recall results concerning torse-forming vector fields and their special cases: recurrent, convergent and concircular vector fields, which have been obtained in [1], [5]–[9], [10]–[14], [16]–[23].

A vector field ξ in a Riemannian space V_n is called *torse-forming* if it satisfies $\nabla_X \xi = \varrho X + a(X)\xi$ where $X \in TM$, $a(X)$ is a linear form and ϱ is a function. In the local transcription this reads

$$\xi^h_{,i} = \varrho\, \delta^h_i + \xi^h a_i \tag{2}$$

where ξ^h and a_i are the components of ξ and a, and δ^h_i is the Kronecker symbol. Throughout this paper we assume $\xi^h \neq 0$.

A torse-forming vector field ξ is called

a) *recurrent*, if $\varrho = 0$,

b) *concircular*, if a_i is a gradient covector (i.e. $a_i = a_{,i}$),

c) *convergent*, if it is concircular, and $\varrho = \mathrm{const} \cdot \exp(a)$.

After a suitable normalization we can characterize concircular and convergent vector fields ξ in the following form

$$\text{b) } \xi^h_{,i} = \varrho\, \delta^h_i \qquad \text{and} \qquad \text{c) } \xi^h_{,i} = \text{const}\, \delta^h_i \tag{3}$$

respectively.

A vector field ξ is called *isotropic* if $g(\xi, \xi) = 0$, where g is a metric on V_n.

Lemma 1. *A non-recurrent torse-forming vector field is non-isotropic.*

Proof. Let us suppose that $\varrho \neq 0$ and that ξ is an isotropic torse-forming vector field, i.e. $\xi^\alpha \xi^\beta g_{\alpha\beta} = 0$, where g_{ij} are components of metric g. By covariant differentiation of the last equation we get $\xi^\alpha \xi^\beta_{,i}\, g_{\alpha\beta} = 0$ and using (2) we obtain $\varrho\, \xi^\alpha g_{\alpha i} = 0$. Therefore $\varrho = 0$, which contradicts the assumption.

A non-isotropic torse-forming vector field ξ can be normalized so that $\xi^\alpha \xi^\beta g_{\alpha\beta} = e = \pm 1$ and we can write the equation (2) for a torse-forming vector field in the following form [17]:

$$\xi_{i,j} = \varrho\, (g_{ij} - e\, \xi_i \xi_j), \tag{4}$$

where $\xi_i \equiv \xi^\alpha g_{\alpha i}$ is a locally gradient covector, i.e. $\xi_i = f_{,i}$ where f is a function. Evidently, we have in this case:

a) if $\varrho = 0$, then ξ is recurrent and convergent,
b) if $\varrho = \frac{e}{f + \text{const}}$, then ξ is convergent,
c) if ϱ is a function of f, i.e. $\varrho = \varrho(f)$, then ξ is concircular,
d) if $\varrho \neq \varrho(f)$, then ξ is neither concircular nor recurrent.

Because we are studying vector fields ξ in Riemannian spaces, in what follows, we shall not distinguish *contravariant* (ξ^h) from *covariant* ($\xi_i \equiv \xi^\alpha g_{\alpha i}$) vectors.

¿From the above results it follows

Lemma 2. *Any non-isotropic recurrent torse-forming vector field is convergent.*

It is well known (see [17]) that, if a Riemannian space V_n admits a non-isotropic torse-forming vector field ξ, then in V_n there exists a coordinate system x, in which the metric takes the form

$$ds^2 = e\, (dx^1)^2 + F(x^1, x^2, \ldots, x^n)\, d\tilde{s}^2,$$

where $e = \pm 1$, $F(\neq 0)$ is a function, and $d\tilde{s}^2(x^2, \ldots, x^n)$ is the metric form of the associated Riemannian space $\tilde{V}_{n-1}$. In this coordinate system the vector ξ has the following form: $\xi^h = \delta^h_1$. Evidently, the following holds

a) if $F = \text{const}$, then ξ is recurrent and convergent,
b) if $F = c\, {x^1}^2$, where c is a constant, then ξ is convergent,
c) if $F = F(x^1)$, then ξ is concircular,
d) if $F \neq F(x^1)$, then ξ is neither concircular nor recurrent.

In the following we shall study non-isotropic torse-forming vector fields, characterized by (4). The integrability condition arising from (4) can be written in the form

$$\xi_\alpha R^\alpha_{ijk} = g_{ij} c_k - g_{ik} c_j + \xi_i a_{jk} \tag{5}$$

where R^h_{ijk} is the Riemannian tensor of V_n, $a_{jk} \equiv -e\xi_{[j}\varrho_{k]}$ and

$$c_k \equiv \varrho_{,k} + e\varrho^2\xi_k\,. \tag{6}$$

Lemma 3. *Let ξ be a non-isotropic torse-forming vector field. If $c_i = 0$, then ξ is convergent.*

Proof. Let us suppose that $c_i = 0$. In view of (6), this assumption implies $\varrho = \varrho(f)$, which gives $\varrho' + e\varrho^2{=}0$. Therefore $\varrho = \frac{e}{f+\mathrm{const}}$ or $\varrho = 0$, which means that ξ^h is convergent.

Taking the converse to Lemma 3 we get

Lemma 4. *If a non-isotropic torse-forming vector field ξ is not convergent, then $c_i \neq 0$.*

3. Torse-forming vector fields in T-semisymmetric Riemannian spaces where T is a covector

In this section we shall be interested in T-semisymmetric Riemannian spaces where T is a covector. In accordance with the general definition in section 1, by a T_i-semisymmetric space we understand a Riemannian space V_n with a covector field T_i satisfying

$$T_{i,[lm]} = 0. \tag{7}$$

Using the Ricci identity we can write (7) in the form

$$R(X,Y)\circ T = 0 \qquad \text{or} \qquad T_\alpha R^\alpha_{ijk} = 0. \tag{8}$$

Theorem 1. *Let T $(\neq 0)$ be a covector field. A non-isotropic torse-forming vector field ξ in a T-semisymmetric space V_n $(n > 2)$ is convergent.*

This theorem is an obvious consequence of the following more general assertion.

Lemma 5. *Let T $(\neq 0)$ be a covector field. A non-isotropic torse-forming vector field ξ in a space V_n $(n > 2)$ is convergent, if $R(X,\xi)\circ T = 0$ for any X.*

Proof. Suppose there exists a non-isotropic torse-forming vector field ξ in V_n $(n > 2)$ such that $R(X,\xi)\circ T = 0$ for any X.

With help of the conditions (5) and using properties of the Riemannian tensor these conditions can be expressed in the following form

$$g_{ij}c_kT^k - T_ic_j + \xi_i a_{jk}T^k = 0 \tag{9}$$

where $T^k \equiv g^{k\alpha}T_\alpha$.

If $c_kT^k \neq 0$, then it follows from (9) that $\mathrm{rank}\|g_{ij}\| \leq 2$. Since $n > 2$ ($\Leftrightarrow$ $\mathrm{rank}\|g_{ij}\| > 2$), the formula (9) implies that $c_kT^k = 0$, and thus

$$T_ic_j = \xi_i a_{jk}T^k.$$

Let us suppose that ξ^h is not convergent, then Lemma 4 implies that $c_i \neq 0$, and, by the latter equality, the vectors T_i and ξ_i are collinear, i.e.,

$$T_i = a\xi_i, \tag{10}$$

where a is a non-zero function.

Next, differentiating (10) covariantly with respect to x^j and x^k, and alternating in j and k, we have

$$T_{i,[jk]} = a\xi_{i,[jk]}.$$

According to (7) and the Ricci identity we can write this equality in the form $\xi_\alpha R^\alpha_{ijk} = 0$ and in view of (5) we obtain

$$\delta^h_j c_k - \delta^h_k c_j + \xi^h a_{jk} = 0. \tag{11}$$

Since we suppose $c_k \neq 0$, there exists a vector ε^k such that $c_\alpha \varepsilon^\alpha = 1$. Contracting (11) with ε^k, we find

$$\delta^h_j - \varepsilon^h c_j + \xi^h a_{j\alpha}\varepsilon^\alpha = 0.$$

This together with $n > 2$ (rank$\|\delta^h_j\| > 2$) leads to a contradiction. Therefore $c_k = 0$ holds and we get from Lemma 3 that ξ is convergent.

4. Torse-forming vector fields in T-semisymmetric Riemannian spaces where T is a 2-tensor field

According to (1) by a 2-tensor T-semisymmetric (or simply T_{ij}-semisymmetric) space we mean a Riemannian space V_n with a tensor field T_{ij} satisfying

$$R(X,Y) \circ T = 0 \quad \text{or} \quad T_{ij,[lm]} = 0. \tag{12}$$

First, let us prove the following lemmas for symmetric and skew-symmetric tensors.

Theorem 2. *Let T $(\neq \alpha g)$ be a 2-covariant symmetric tensor field. A non-isotropic torse-forming vector field ξ in a T-semisymmetric space V_n $(n > 2)$ is convergent.*

This theorem follows from the more general lemma

Lemma 6. *Let T $(\neq \alpha g)$ be a 2-covariant symmetric tensor field. A non-isotropic torse-forming vector field ξ in a space V_n $(n > 2)$ is convergent, if $R(X,\xi) \circ T = 0$ for any X.*

Proof. Let there exist a non-isotropic torse-forming vector field ξ in V_n $(n > 2)$ with $R(X,\xi) \circ T = 0$ for any X.

Similarly as before, using (5), the assumption $T_{[ij]} = 0$, and the properties of the Riemannian tensor, we obtain

$$g_{li}T_{j\alpha}c^\alpha - T_{lj}c_i + g_{lj}T_{i\alpha}c^\alpha - T_{li}c_j + \xi_l\omega_{ij} = 0, \tag{14}$$

where ω_{ij} is a certain tensor, $c^i \equiv c_\alpha g^{i\alpha}$ and $\|g^{ij}\| = \|g_{ij}\|^{-1}$.

Let us prove that there exists a function μ such that

$$T_{i\alpha}c^\alpha = \mu c_i. \tag{15}$$

Suppose that (15) does not hold. Then we can find ε^i such that $c_i\varepsilon^i = 0$ and $T_{\alpha\beta}\varepsilon^\alpha c^\beta = 1$. Contracting (14) with such an ε^j and subsequently with ε^i, we obtain the following formulas

$$g_{li} - T_{l\alpha}\varepsilon^\alpha c_i + \varepsilon_l T_{i\alpha}c^\alpha + \xi_l\omega_{i\alpha}\varepsilon^\alpha = 0 \quad \text{and} \quad 2\varepsilon_l + \xi_l\omega_{\alpha\beta}\varepsilon^\alpha\varepsilon^\beta = 0$$

where $\varepsilon_i \equiv \varepsilon^\alpha g_{\alpha i}$. We can deduce that $\operatorname{rank}\|g_{li}\| \le 2$. But from the assumption $n > 2$ it follows that $\operatorname{rank}\|g_{li}\| > 3$, a contradiction.

Substituting (15) in (14) we get

$$F_{li}c_j + F_{lj}c_i - \xi_l\omega_{ij} = 0, \tag{16}$$

where

$$F_{ij} \equiv T_{ij} - \mu g_{ij}. \tag{17}$$

Let us choose φ^j such that $\varphi^j c_j = 1$. When contracting (16) with such a φ^j and then with φ^i, we arrive at the following formulas

$$F_{li} + F_{lj}\varphi^j c_i - \xi_l\omega_{ij}\varphi^j = 0 \tag{18}$$

and

$$F_{li}\varphi^i = \nu\xi_l,$$

where $\nu = \frac{1}{2}\omega_{ij}\varphi^j\varphi^i$. This together with (18) leads to $F_{li} = \xi_l\chi_i$ where $\chi_i = -\nu c_i + \omega_{ij}\varphi^j$.

Then, according to the symmetry of the tensor F_{ij}, we can write

$$F_{ij} = \lambda\xi_i\xi_j, \tag{19}$$

where λ is a function, this implies that $\lambda \neq 0$.

Next, differentiating (19) covariantly with respect to x^l and x^m, and alternating in l and m, we have

$$F_{ij,[lm]} = \lambda(\xi_{i,[lm]}\xi_j + \xi_i\xi_{j,[lm]})\,.$$

From (12) and (17) it follows $F_{ij,[lm]} = 0$, which, in view of $\xi_i \neq 0$, implies $\xi_{i,[lm]} = 0$. It means that V_n is ξ_i-semisymmetric and we get from Theorem 1 that ξ^h is convergent.

Theorem 3. *Let T $(\neq 0)$ be a 2-covariant skew-symmetric tensor field. A non-isotropic torse-forming vector field ξ in a T-semisymmetric space V_n $(n > 3)$ is convergent.*

Similarly as above, this theorem follows from the following

Lemma 7. *Let T $(\neq 0)$ be a 2-covariant skew-symmetric tensor field. A non-isotropic torse-forming vector field ξ in a space V_n $(n > 3)$ is convergent, if $R(X,\xi)\circ T = 0$ for any X.*

Proof. Let there exist a non-isotropic torse-forming vector field ξ^h in V_n $(n > 3)$ with $R(X,\xi)\circ T = 0$.

Again, using (5) and the properties of the Riemannian tensor and, in addition, the assumption $T_{ij} + T_{ji} = 0$, we obtain

$$g_{li}T_{\alpha j}c^{\alpha} - T_{lj}c_i - g_{lj}T_{\alpha i}c^{\alpha} + T_{li}c_j - \xi_l\omega_{ij} = 0, \tag{20}$$

where ω_{ij} is a certain tensor and $c^i \equiv c_\alpha g^{\alpha i}$.

Let us prove that there exists a function μ such that (15) is true. Suppose, on the contrary, that (15) does not hold. Then we can find ε^i such that $T_{\alpha\beta}c^{\alpha}\varepsilon^{\beta} = 1$ and $c_i\varepsilon^i = 0$. Contracting (20) with such an ε^j, we can deduce that rank$\|g_{li}\| \leq 3$. But from the assumption $n > 3$ it follows that rank$\|g_{li}\| > 3$, a contradiction.

Substituting (15) in (20) we get

$$(T_{li} - \mu g_{li})c_j - (T_{lj} - \mu g_{lj})c_i - \xi_l\omega_{ij} = 0. \tag{21}$$

Let us suppose that $c_j \neq 0$. Then there exist φ^i such that $\varphi^i c_i = 1$. Contracting (21) with φ^j we get

$$T_{li} - \mu g_{li} = \xi_l\eta_i + \chi_l c_i \tag{22}$$

where η_i and χ_i are suitable vectors. Symmetrizing (22) we obtain

$$-2\mu g_{li} = \xi_l\eta_i + \chi_l c_i + \xi_i\eta_l + \chi_i c_l. \tag{23}$$

Provided the vectors $\xi_i, c_i, \eta_i, \chi_i$ were linearly independent, we could use (23) and verify that all they are isotropic. Since, however, ξ_i is non-isotropic, these vectors have to be linearly dependent. If $\mu \neq 0$, then the equality (23) implies rank$\|g_{li}\| \leq 3$ which contradicts the assumption $n > 3$. Therefore $\mu = 0$ and (22) has the form

$$T_{li} = \xi_l\eta_i + \chi_l c_i. \tag{24}$$

Using the fact that T_{ij} is skew-symmetric, we get from (24) that there is a vector ν_i such that

$$T_{ij} = \xi_i\nu_j - \xi_j\nu_i. \tag{25}$$

Having in mind that (12) and (25) are valid, we obtain

$$\xi_{i,[lm]}\nu_j + \xi_i\nu_{j,[lm]} - \xi_{j,[lm]}\nu_i - \xi_j\nu_{i,[lm]} = 0.$$

We substitute $\xi_{i,[lm]} \equiv -\xi_\alpha R^{\alpha}_{ilm}$ and then, by (5), we have

$$\begin{aligned}(g_{il}c_m - g_{im}c_l + \xi_i a_{lm})\nu_j + \xi_i\nu_{j,[lm]} - \\ -(g_{jl}c_m - g_{jm}c_l + \xi_j a_{lm})\nu_i - \xi_j\nu_{i,[lm]} = 0.\end{aligned} \tag{26}$$

From (25) and the assumption $T_{ij} \neq 0$ it follows that the vectors ξ_i and ν_i cannot be collinear. Therefore there is ε^i such that $\varepsilon^i\nu_i = 1$ and $\varepsilon^i\xi_i = 0$ Contracting (26) with ε^j we get

$$g_{il}c_m - g_{im}c_l + \xi_i b_{lm} + \nu_i c_{lm} = 0, \tag{27}$$

where b_{lm} and c_{lm} are certain tensors. Contracting (27) with φ^m (this vector satisfies $\varphi^m c_m = 1$) we find that rank$\|g_{li}\| \leq 3$, a contradiction. This contradiction implies that $c_i = 0$, which means, by Lemma 3, that ξ^h is convergent.

For torse-forming vector fields in T_{ij}-semisymmetric Riemannian spaces an assertion which is analogous to Theorem 1 and Lemma 5 holds.

Theorem 4. *Let T ($\neq \alpha g$) be a 2-covariant tensor field. A non-isotropic torse-forming vector field ξ in a T-semisymmetric space V_n ($n > 3$) is convergent.*

Analogously we show that the following is true.

Lemma 8. *Let T ($\neq \alpha g$) be a 2-covariant tensor field. A non-isotropic torse-forming vector field ξ in a space V_n ($n > 3$) is convergent, if $R(X,\xi) \circ T = 0$ for any X.*

Proof. Let T ($\neq \alpha g$) be a 2-covariant tensor field in V_n ($n > 3$) with $R(X,\xi)\circ T = 0$ for any X. The tensor T can be expressed uniquely in the form $T = U + V$ where U is symmetric and V is skew-symmetric. Then $U(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X))$ and $V(X,Y) = \frac{1}{2}(T(X,Y) - T(Y,X))$. From $R(X,\xi)\circ T = 0$ we get $R(X,\xi)\circ U = 0$ and $R(X,\xi) \circ V = 0$.

Further, let us suppose that there exists a non-isotropic torse-forming vector field ξ in V_n which is not convergent. Therefore we can use Lemma 6 and Lemma 7 and get that $U = \alpha g$ and $V = 0$. It means that $T = \alpha g$, a contradiction. This implies that the vector field ξ has to be convergent.

5. Torse-forming vector fields in special T-semisymmetric spaces

Now, we will consider a special case of a T-semisymmetric space, namely, such that T is the Ricci tensor. A Riemannian space V_n is called *Ricci-semisymmetric* if the Ricci tensor Ric satisfies

$$R(X,Y) \circ Ric = 0.$$

For non-Einsteinian spaces we have the inequality $Ric \neq \alpha g$. The following theorem follows from Theorem 2:

Theorem 5. *A non-isotropic torse-forming vector field ξ in a non-Einsteinian Ricci-semisymmetric space V_n ($n > 2$) is convergent.*

This theorem follows from

Lemma 9. *A non-isotropic torse-forming vector field ξ in a non-Einsteinian space V_n ($n > 2$) is convergent, if $R(X,\xi) \circ Ric = 0$ for any X.*

The structure F_i^h in Kählerian spaces is covariantly constant, and evidently in this case K_n is F_i^h-semisymmetric. Therefore we have, using Theorem 4

Theorem 6. *A non-isotropic torse-forming vector field ξ in a Kählerian space K_n ($n > 3$) is convergent.*

For Einsteinian spaces we have

Theorem 7. *A non-isotropic torse-forming vector field ξ in an Einstenian space V_n ($n > 2$) is concircular.*

Proof. Let V_n $(n > 2)$ be an Einsteinian space. The Ricci tensor of this space satisfies the following equation $R_{ij} = \frac{R}{n} g_{ij}$, where $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature. Let there exist a non-istropic torse-forming vector field ξ^h in V_n. Then the condition (5) is satisfied. By contracting of (5) with g^{ij} we obtain

$$(n-2)\varrho_{,k} = \xi_k \left(\tfrac{R}{n} + e(n-1)a^2 - e\xi^\alpha \varrho_\alpha\right). \tag{28}$$

Since ξ_k is a gradient vector, i.e., $\xi_k \equiv \xi_{,k}$, it follows from (28) that $\varrho = \varrho(\xi)$ which implies that ξ^h is concircular. The proof of Theorem 7 is complete.

Using Theorems 5 and 7 and the property of the concircular vector field in a semisymmetric space V_n $(n > 2)$ with non-constant curvature [13] we have

Theorem 8. *A non-isotropic torse-forming vector field ξ in a semisymmetric space V_n $(n > 3)$ with non-constant curvature is convergent.*

The Einsteinian, Kählerian, semisymmetric and Ricci-semisymmetric Riemannian spaces with concircular (or convergent) vector field are described in [5] – [9], [11] – [14], [16].

Spaces which generalize Einsteinian spaces are Riemannian spaces *with a harmonic curvature tensor*, they are characterized by the following formula:

$$R^\alpha_{ijk,\alpha} = 0 \qquad (\Leftrightarrow\ R_{ij,k} = R_{ik,j}). \tag{29}$$

These spaces are studied by many authors, for example [9], [15], [20].

We have

Theorem 9. *A torse-forming vector field ξ in a non-Einsteinian Ricci-semisymmetric space V_n $(n > 2)$ with harmonic curvature tensor is recurrent.*

Proof. Let there exist a non-recurrent torse-forming vector field ξ^h in a non-Einsteinian Ricci-semisymmetric space V_n $(n > 2)$ with harmonic curvature tensor. Evidently, the vector ξ^h is non-isotropic. Therefore we can use Theorem 5 and get that ξ^h is convergent.

For this vector formula (3c) applies in the following form:

$$\xi_{i,j} = \varrho\, g_{ij}, \qquad \varrho \equiv \text{const} \neq 0. \tag{30}$$

The condition of integrability of the equation (30) has the form $\xi_\alpha R^\alpha_{ijk} = 0$. Differentiating covariantly the last formula we obtain

$$\xi_\alpha R^\alpha_{ijk,l} + \varrho\, R_{lijk} = 0. \tag{31}$$

Contracting (31) with g^{kl} and using properties of the Riemannian tensor and (29) we get:

$$\varrho\, R_{ij} = 0.$$

Because of $\varrho \neq 0$ (ξ^h is not reccurent) we have $R_{ij} = 0$. This contradics to the fact that V_n is not an Einsteinian space, and we are done.

Remark. T. Q. Binh, U. C. De, L. Tamássy and M. Tarafdar [2], [3] studied Ricci-semisymmetric and semisymmetric almost Kenmotsu manifolds. In Kenmotsu manifolds there exists a unit vector field ξ satisfying the condition $\nabla_X \xi =$

$X - \eta(X)\xi$, where $\eta(X) = g(X, \xi)$. By simple observation we convince ourselves that this vector field is non-isotropic and torse-forming, is not convergent and, consequently, is not recurrent. Therefore many results of [2] and [3] follow immediately from the properties of the torse-forming fields introduced in our article.

References

[1] Aminova A.V.: *Groups of transformations of Riemannian manifolds.* J. Sov. Math.,1991, 55, No.5, 1996-2041; translation from Itogi Nauki Tekh., Ser. Probl. Geom., 1990, 22, 97–165.
[2] Binh T.Q.: *Some remarks on Kenmotsu manifolds.* Abstract in Int. Congress on Diff. Geom. in memory of Alfred Gray, Sept. 18–23, 2000, Univ. del País Vasco, Bilbao, p.12.
[3] Binh T.Q., De U.C., Tamassy L., Tarafdar M.: *Some remarks on Kenmotsu manifolds.* In poster of Int. Congress on Diff. Geom. in memory of Alfred Gray, Sept. 18–23, 2000, Univ. del País Vasco, Bilbao.
[4] Boeckx E., Kowalski O., Vanhecke L.: *Riemannian manifolds of conullity two.* World Scientific Publ. Co., Singapore, 1996.
[5] Brinkmann H.W.: *Einstein spaces which mapped conformally on each other.* Math. Ann., 94, 1925.
[6] Defever F., Deszcz R.: *On warped product manifolds satisfying a certain curvature condition.* Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis. Mat. Nat. 1991, 69, 213–236.
[7] Deszcz R.: *On some Riemannian manifolds admitting a concircular vector field.* Demonstr. math., 1976, 9, No. 3, 487–495.
[8] Fialkow A.: *Conformal geodesics.* Trans. Am. Math. Soc., 1939, 45, 443-473.
[9] Kowolik J.: *On some Riemannian manifolds admitting torse-forming vector fields.* Dem. Math., 18, N. 3, 1985, 885–891.
[10] Mikeš J.: *Geodesic Ricci mappings of two-symmetric Riemann spaces.* Math. Notes, 1981, 28, 622-624.
[11] Mikeš J.: *On Sasaki spaces and equidistant Kaehler spaces.* Sov. Math., Dokl., 1987, 34, 428–431; translation from Dokl. Akad. Nauk SSSR, 1986, 291, 33-36.
[12] Mikeš J.: *Geodesic mappings of special Riemannian spaces.* Coll. Math. Soc. J. Bolyai. 46. Top. in diff. geom. Debrecen, 1984, Vol. 2, Amsterdam etc., 1988, 793–813.
[13] Mikeš J.: *Geodesic mappings of affine-connected and Riemannian spaces.* J. Math. Sci. New York, 1996, 78, 3, 311–333.
[14] Mikeš J.: *Holomorphically projective mappings and their generalizations.* J. Math. Sci., New York, 1998, 89, No.3, 1334–1353.
[15] Mikeš J., Radulović Ž.: *On projective transformations of Riemannian spaces with harmonic curvature.* New developments in differential geometry, Budapest 1996. Proceedings of the conference, Budapest, Hungary, July 27-30, 1996. Dordrecht: Kluwer Academic Publishers., 1999, 279–283.
[16] Roter W.: *On a class of conformally recurrent manifolds.* Tensor N. S., 1982, 39, 207–217.
[17] Sinyukov N. S.: *Geodesic mappings of Riemannian spaces.* Nauka, Moscow. 1979, 256pp.
[18] Sinyukov N. S.: *Almost-geodesic mappings of affinely connected and Riemann spaces.* J. Sov. Math., 1984, 25, 1235–1249.
[19] Shirokov P.A.: *Collected works of geometry.* Kazan univ. press, 1966, 432 pp.
[20] Sobchuk V.S.: *Ricci generalized symmetric Riemannian spaces admit nontrivial geodesic mappings.* Sov. Math. Dokl., 1982, 26, 699-701; translation from Dokl. Akad. Nauk SSSR, 1982, 267, 793–795.
[21] Solodovnikov, A.S.: *Spaces with common geodesics.* Tr. Semin. Vektor. tensor. Anal.,1961, 11, 43–102.
[22] Yano K.: *Concircular geometry, I–IV.* Proc. Imp. Acad. Tokyo, 1940, 16, 195–200, 354–360, 442–448, 505–511.

[23] Yano K.: *On torse-forming directions in Riemannian spaces*. Proc. Imp. Acad. Tokyo, 1944, 20, 701–705.

Dept. of Algebra and Geometry, Fac. Sci., Palacky Univ., Tomkova 40, 779 00 Olomouc, Czech Rep.

Dept. of Mathematics TU, Masaryk Square 275, 762 72 Zlín, Czech Rep.
E-mail address: Mikes@matnw.upol.cz
E-mail address: Rachunek@zlin.vutbr.cz