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RECENT RESULTS IN VARIATIONAL SEQUENCE THEORY

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Abstract. In this paper, foundations of the higher order variational sequence theory are explained. Relations of the classes in the sequence to basic concepts of the variational calculus on fibered spaces, such as the lagrangians, Lepage forms, Euler–Lagrange forms, and the Helmholtz–Sonin forms, are discussed. Recent global results, including interpretation of the classes in the variational sequence as differential forms, are discussed.

1. Introduction

During the few last decades, there has been a growing interest in the study of global aspects of the calculus of variations. The arising theory, the calculus of variations on smooth manifolds and fibered spaces, includes the coordinate–free calculus of vector fields and differential forms, differential geometry, topology and global analysis. The most intensively studied general questions were those connected with the structure of Euler–Lagrange mapping, i.e., with variationally trivial lagrangians, the inverse problem of the calculus of variations, and the order reducibility problem.

Let Y be a fibered manifold over a base manifold X, where $n = \dim X$, and let J^rY denote the $r-$jet prolongation of Y. The need of global concepts led to the introduction of the so called Lepage $n-$forms, and Lepage equivalents of lagrangians, based on the idea of Lepage and Dedecker that there should exist a close connection between the Euler–Lagrange mapping and the exterior derivative of forms (Krupka [32], [35], [36]). Later, this concept was extended to $(n{+}1)-$forms in field theory by Krupková [57], [62] and Klapka [27]. Krupková [57], [58], [61] applied Lepage $2-$forms in higher order mechanics to the inverse problem, and to the order reducibility problem, and obtained their complete solutions.

The relationship between the Euler–Lagrange mapping and the exterior derivative operator has given rise to the theory of variational bicomplexes, and variational sequences. The idea was to discover a proper (cohomological) sequence in which

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the Euler–Lagrange mapping would be included as one "arrow"; indeed, this would give us a tool for a global characteristic of the Euler–Lagrange mapping.

A theoretical background of theory of variational bicomplexes, which is based on infinite jet constructions, was formulated at the break of seventieth and eightieth by Anderson and Duchamp [2], Dedecker and Tulczyjew [12], Takens [73], Tulczyjew [75], Vinogradov et all [76] (see also e.g. Anderson [3], Vinogradov, Krasilschik and Lychagin [77]).

Finite order variational sequences were introduced by Krupka [43] in 1989 (see also [50], [56]), and was further developed by his co-workers (Kašparová, Krbek, Musilová, Štefánek [24], [25], [29], [30], [31], [63], [72]), and others (Grigore [20], Vitolo [78], [79], Francaviglia, Palese and Vitolo [13], [14]).

A comparison of both theories can be found in Krupka [53], Pommaret [67], and Vitolo [79].

Let us discuss some most important features of the theory of finite order variational sequences.

(1) The variational sequence is defined as the quotient sequence of the De Rham sequence over J^rY by its subsequence of *contact* forms, and its morphisms keep the order r fixed. The sequence is exact, and one of its morphisms is exactly the Euler–Lagrange operator. This demonstrates the relationship of d with the Euler–Lagrange mapping.

(2) Each term of the variational sequence is, as a quotient group, determined up to an isomorphism. This means that the variational sequence can be represented by various spaces. Important representations arise when the classes of forms are represented as globally defined forms (on some J^sY, where $r \leq s$). It has already been proved that such representations do exist for spaces involving the domain and the range of the Euler–Lagrange mapping, and the next arrow in the sequence, the Helmholtz–Sonin mapping (see futher discussion).

(3) By the abstract De Rham theorem, the complex of global sections of the variational sequence has the same cohomology as the manifold Y. On the other hand, the classes in the variational sequence have a certain algebraic structure; therefore, the meaning of the cohomology conditions in the sequence differs from their meaning in the context of the variational bicomplex theory. In particular, the global variationality condition ($H^{n+1}Y = 0$) includes existence of global lagrangians of a certain analytic structure, defined by the sequence.

(4) An interesting question is the meaning of the Lepage forms and their generalizations, which play fundamental role in the global variational theory. It should be pointed out that within the context of the variational sequence, the Lepage forms are just proper representations of elements of the sequence (classes), defined by some specific properties.

Štefánek [72] found a complete (local) representation of the $r-$th order variational sequence in mechanics. Musilová [63] and Krbek and Musilová [30] described the representation by forms of the variational terms in the sequence, i.e. the terms relevant to the Euler–Lagrange, and Helmholtz–Sonin mappings. Moreover, they described a reconstruction procedure of the classes. Kašparová [24], [25], [26] has found global representations of the variational terms in the first order field theory. Her results have been extended by Krbek, Musilová and Kašparová [31] to arbitrary order field theory.

The aim of the presented review paper is to give a consistent exposition of the present situation in the variational sequence theory. We define all concepts and present all basic theorems together with ideas of their proofs. For more details, the reader should consult the references.

2. The concept of the variational sequence

The main purpose of this part of the paper is to give a brief and consistent presentation of the theory of finite-order variational sequence on the adequately abstract level.

2.1. **Differential forms on fibered manifolds.** In this section we introduce the basic geometrical structures for the formulation of variational theories, especially for the concept of global higher order variational functionals as well as for the variational sequences. Modern global variational theories are formulated by means of differential forms defined on fibered manifolds and their jet prolongations. An important role is played by some special classes of forms: horizontal and contact forms. For the theory of differential forms the reader is referred e.g. to [1], [32], [35], [43], the structure of contact forms is discussed in detail in [47], [49]. The concept of a fibered manifold and its jet prolongations is based on the general theory of jets, presented in [28], [44] and [70], and can also be found in [55].

Throughout, we use the standard notation given e.g. in [32], [43], [49], [50]. The definitions of fundamental structures and objects are presented in the form adapted to practical purposes and emphasizing their coordinate expressions. All manifolds and mappings are of class C^∞.

Y is an $(n+m)-$dimensional *fibered manifold* with an $n-$dimensional *base* X and *projection* $\pi : Y \to X$ (surjective submersion). For an arbitrary integer $r \geq 0$, J^rY is the $r-$*jet prolongation* of Y, $\pi^r : J^rY \to X$, $\pi^{r,s} : J^rY \to J^sY$, $r \geq s \geq 0$, are canonical jet projections of J^rY on X and J^sY, respectively. We denote $N_r = \dim J^rY$. It holds $N_r = n + \sum_{k=0}^{r} M_k = n + m\binom{n+r}{n}$, where $M_k = m\binom{n+k-1}{k}$. We denote by γ and $J^r_x\gamma$ a section of the fibered manifold Y (a smooth mapping $\gamma : X \to Y$ for which $\pi \circ \gamma = \mathrm{id}_X$) and its $r-$jet at the point x, respectively. The mapping $J^r\gamma : x \to J^r\gamma(x) = J^r_x\gamma$ is the $r-$*jet prolongation* of γ. $\Gamma_\Omega(\pi)$ is the set of all sections of Y defined on $\Omega \subset X$. Let (V, ψ), $\psi = (x^i, y^\sigma)$, $1 \leq i \leq n$, $1 \leq \sigma \leq m$, be a fibered chart on Y. Then we denote (U, φ) and (V^r, ψ^r) the *associated chart* on X and the *associated fibered chart* on J^rY, respectively. These charts are induced by (V, ψ) by such a way that $U = \pi(V)$, $\varphi = (x^i)$ and $V^r = (\pi^{r,0})^{-1}(V)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, \dots y^\sigma_{j_1 \dots j_k})$, where $y^\sigma_{j_1 \dots j_k}\gamma(x) = \frac{\partial y^\sigma \gamma(x)}{\partial x^{j_1} \dots \partial x^{j_k}}$ for $1 \leq k \leq r$, $x \in U$ and every $\gamma \in \Gamma_U(\pi)$. Thus, the variables $y^\sigma_{j_1 \dots j_k}$ are completely symmetric in all indices contained in each multiindex $J = (j_1 \dots j_k)$. The integer $k = |J|$ is the *length* of the multiindex J. For y^σ we put $|J| = 0$.

Let Ξ be a vector field on an open subset W of Y. It is called $\pi-$*projectable*, if there exists a vector field ξ on $\pi(V)$ such that $T\pi \cdot \Xi = \xi \circ \pi$, $T\pi$ being the tangent mapping to π. Then ξ is unique and it is called the $\pi-$*projection* of Ξ. In a fibered chart (V, ψ), $V \subset W$, $\psi = (x^i, y^\sigma)$, it holds

$$\Xi = \xi^i(x^j)\frac{\partial}{\partial x^i} + \Xi^\sigma(x^i, y^\sigma)\frac{\partial}{\partial y^\sigma}.$$

Let (V, ψ) be a chart on Y. Let $\alpha : V \to Y$ be a local isomorphism of Y and $\alpha_0 : U \to X$ its projection, i.e. $\pi \circ \alpha = \alpha_0 \circ \pi$. We define the local isomorphism $J^r\alpha : V^r \to J^rY$ of J^rY by the relation

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}\alpha\gamma\alpha_0^{-1}.$$

$J^r\alpha$ is called the $r-$*jet prolongation of* α. Using prolongations of local isomorphisms connected with the one-parameter group of a projectable vector field we can define jet prolongations of this vector field: Let Ξ be a $\pi-$projectable vector field on Y and let ξ be its $\pi-$projection. Let α_t be the local one-parameter group of Ξ. Then we define

$$J^r\Xi(J^r_x\gamma) = \left(\frac{\mathrm{d}}{\mathrm{d}t}J^r\alpha_t(J^r_x\gamma)\right)_{t=0}$$

for each $J^r_x\gamma \in \operatorname{dom} J^r\alpha_t$. This relation defines the vector field on J^rY called the $r-$*jet prolongation of* Ξ. Its chart expression is as follows:

$$J^r\Xi = \xi^i\frac{\partial}{\partial x^i} + \Xi^\sigma_J\frac{\partial}{\partial y^\sigma_J}, \quad \Xi^\sigma_{j_1\dots j_k} = \mathrm{d}_{j_k}\xi^\sigma_{j_1\dots j_{k-1}} - y^\sigma_{j_1\dots j_{k-1}i}\frac{\partial \xi^i}{\partial x^{j_k}}, \quad 0 \le |J| \le r$$

(in details see e.g. [49]), where d_i denotes the *total (formal) derivative* operator for any function $f : W \to \mathbf{R}$ in a fibered chart (V, ψ), $V \subset W$:

$$\mathrm{d}_i f = \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^\sigma_J}y^\sigma_{Ji}, \quad 0 \le |J| \le r.$$

It can be easily seen that $J^r\Xi$ is $\pi^{r,s}-$projectable for every $0 \le s \le r$ and it is also π^r-projectable. Denote $W^r = (\pi^{r,0})^{-1}W$. Let Ξ be a vector field on W^r. It is called π^r-*projectable*, if there exists a vector field ξ on $\pi(W)$ such that $T\pi^r \cdot \Xi = \xi \circ \pi^r$. In a fibered chart (V, ψ), $V \subset W$, $\psi = (x^i, y^\sigma)$ we have

$$\Xi = \xi^i(x^j) + \Xi^\sigma_J\frac{\partial}{\partial y^\sigma_J}, \quad \Xi^\sigma_J = \Xi^\sigma_J(x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1\dots j_r}).$$

A vector field Ξ on W^r is called $\pi^{r,s}-$*projectable* for $0 \le s \le r$, if there exists a vector field ξ on W^s such that

$T\pi^{r,s} \cdot \Xi = \xi \circ \pi^{r,s}$, i.e. $\Xi^\sigma_{j_1\dots j_k} = \Xi^\sigma_{j_1\dots j_k}(x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1\dots j_s})$ for $s \le k \le r$.

Let $W \subset Y$ be again an open set. We denote by $\Omega^r_0 W$ the ring of smooth functions on W and by $\Omega^r_q W$ the $\Omega^r_0 W-$module of smooth differential $q-$forms on W^r. The fibered structure on Y leads to the concept of vertical vectors and vector fields and of horizontal forms, as follows: A vector $\Xi \in T_y J^r Y$ is called π^r-*vertical* if $T\pi^r \cdot \Xi = 0$. If the same holds for a vector field Ξ on $W^r \subset J^r Y$ at every point $y \in W^r$, we have the π^r-*vertical vector field.* (In coordinates this means that $\xi^i = 0$ on $\pi^r(W^r)$.) Let $0 \le s \le r$ be integers. A vector $\Xi \in T_y J^r Y$ (or a vector field Ξ on $J^r Y$) is called $\pi^{r,s}-$*vertical*, if $T_y \pi^{r,s} \cdot \Xi = 0$ (or $T\pi^{r,s} \cdot \Xi = 0$, respectively).

A form $\varrho \in \Omega^r_q W$ is called π^r- *horizontal* (or simply horizontal), if it takes zero value whenever some of its vector arguments are π^r-vertical vectors. It can be proved that for every form $\varrho \in \Omega^r_q W$, $q \ge 1$, there exists the uniquely defined horizontal form $h\varrho \in \Omega^{r+1}_q W$ for which $J^r\gamma^*\varrho = J^{r+1}\gamma^* h\varrho$ for all sections γ of Y, $*$ denoting the pullback mapping. Putting in addition $hf = f \circ \pi^{r+1,r}$ for a function $f : W^r \to \mathbf{R}$, we obtain a morphism $h : \Omega^r_q W \to \Omega^{r+1}_q W$ which is induced by the fibered structure on Y. This morphism is called the *horizontalization.* For the chart expressions it holds

$$(1)\quad h\,\mathrm{d}x^i = (\pi^{r+1,r})^*\mathrm{d}x^i = \mathrm{d}x^i, \quad h\,\mathrm{d}y^\sigma_{j_1\dots j_k} = y^\sigma_{j_1\dots j_k i}\,(\pi^{r+1,r})^*\mathrm{d}x^i = y^\sigma_{j_1\dots j_k i}\,\mathrm{d}x^i,$$

for $1 \le k \le r$. It holds $h(\omega \wedge \eta) = h\omega \wedge h\eta$.

A form $\varrho \in \Omega^r_q W$ is called *contact* if it holds $J^r\gamma^*\varrho = 0$ for every section γ of Y, or equivalently, if $h\varrho = 0$. Let (V, ψ) be a fibered chart on Y. We define

$$\omega^\sigma_{j_1\dots j_k} = \mathrm{d}y^\sigma_{j_1\dots j_k} - y^\sigma_{j_1\dots j_k i}\,\mathrm{d}x^i, \quad 0 \le k \le r-1. \tag{2}$$

We can see that the integers $M_k = m\binom{n+k-1}{k}$ defined previously give also the number of independent forms $\omega^\sigma_{j_1\dots j_k}$.

The forms $\omega^\sigma_{j_1\dots j_k}$ defined by (2) are contact, as can be easily verified. Then we can use the so called *contact base of* $1-$*forms on* V^r

$$(\mathrm{d}x^i, \omega^\sigma, \omega^\sigma_{j_1}, \dots, \omega^\sigma_{j_1\dots j_{r-1}}, \mathrm{d}y^\sigma_{j_1\dots j_r})$$

instead of the canonical one, $(\mathrm{d}x^i, \mathrm{d}y^\sigma, \mathrm{d}y^\sigma_{j_1}, \dots, \mathrm{d}y^\sigma_{j_1\dots j_r})$.

Recall that a form $\varrho \in \Omega^r_q W$ is called π^r-*projectable* if there exists a form η on $\pi^r(W)$ for which $\varrho = (\pi^r)^*\eta$. A form $\varrho \in \Omega^r_q W$ is called $\pi^{r,s}-$*projectable* for $r \ge s \ge 0$ if there exists a form $\eta \in \Omega^s_q W$ for which $\varrho = (\pi^{r,s})^*\eta$. Let $\varrho \in \Omega^r_q W$ be a form. We denote $p\varrho = (\pi^{r+1,r})^*\varrho - h\varrho$ its *contact part* ($p\varrho$ is of course contact, as can be immediately proved with the use of definition of $h\varrho$). There exists the unique decomposition

$$(\pi^{r+1,r})^*\varrho = h\varrho + p_1\varrho + \cdots + p_q\varrho \tag{3}$$

of the form $(\pi^{r+1,r})^*\varrho$ in which $p_k\varrho$, for every $1 \le k \le q$, is the contact form, called the $k-$*contact component* of ϱ. In an arbitrarily chosen fibered chart (V, ψ) on Y the chart expression of $p_k\varrho$ is a linear combination of exterior products

$$\omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_k}_{I_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \dots \wedge \mathrm{d}x^q$$

with coefficients from $\Omega^{r+1}_0 V$, where $I_p = (j_1 \dots j_p)$, $0 \le p \le r$, are multiindices. Every such product contains exactly k exterior factors of the type $\omega^\sigma_{j_1\dots j_p}$, $0 \le p \le r$. The form $h\varrho$ is the *horizontal* or $0-$*contact component* of the form ϱ. The lowest integer k for which $p_k \ne 0$ is called the *degree of contactness* of the form ϱ. We denote the submodule of horizontal $q-$forms on W^r by $\Omega^r_{q,X}W$. A $q-$form $\varrho \in \Omega^r_q W$ is called $\pi^{r,s}-$*horizontal* if for every $\pi^{r,s}-$vertical vector field Ξ on J^rY it holds $i_\Xi \varrho = 0$. The decomposition (3) is, of course, coordinate invariant. In a fibered chart (V,ψ) it can be expressed as follows: Let $\varrho \in \Omega^r_q W$ have, in a fibered chart (V,ψ), $V \subset W$, the chart expression

$$\varrho = \sum_{s=0}^{q} A^{I_1}_{\sigma_1} \cdots {}^{I_s}_{\sigma_s, i_{s+1},\dots i_q}\, \mathrm{d}y^{\sigma_1}_{I_1} \wedge \ldots \wedge \mathrm{d}y^{\sigma_s}_{I_s} \wedge \mathrm{d}x^{i_{s+1}} \wedge \ldots \wedge \mathrm{d}x^{i_q} \tag{4}$$

in which coefficients $A^{I_1}_{\sigma_1} \cdots {}^{I_s}_{\sigma_s, i_{s+1},\dots i_q} \in \Omega^r_0 V$ are antisymmetric in all multiindices $\left(\binom{I_1}{\sigma_1}, \dots, \binom{I_s}{\sigma_s}\right)$, $0 \le |I_p| \le r$, antisymmetric in all indices $(i_{s+1}, \dots, i_q)$ and symmetric in all indices contained in each multiindex I_p. Then for every $0 \le k \le q$ it holds

$$p_k \varrho = C^{I_1}_{\sigma_1} \cdots {}^{I_k}_{\sigma_k, i_{k+1},\dots i_q}\, \omega^{\sigma_1}_{I_1} \wedge \ldots \wedge \omega^{\sigma_k}_{I_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \mathrm{d}x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}x^{i_q}, \tag{5}$$

$$C^{I_1}_{\sigma_1} \cdots {}^{I_k}_{\sigma_k, i_{k+1},\dots i_q} = \\ = \sum_{s=k}^{q} \binom{s}{k} A^{I_1}_{\sigma_1} \cdots {}^{I_k}_{\sigma_k} \cdots {}^{I_s}_{\sigma_s, i_{s+1}\dots i_q}\, y^{\sigma_{k+1}}_{I_{k+1} i_{k+1}} \cdots y^{\sigma_s}_{I_s i_s}, \quad \mathrm{alt}(i_{k+1}, \dots, i_q).$$

(The summations over multiindices I_p are taken over all independent choices of indices in each multiindex.) The proof of the existence and uniqueness of the decomposition (3) and the relation (5) can be found in [49]. It can be immediately seen from the relation (5) that for $q > n$ every $q-$form is contact. Moreover, in such a case it holds $h\varrho = p_1\varrho = \cdots = p_{q-n-1}\varrho = 0$. Let $q > n$. A form $\varrho \in \Omega^r_q W$ is called *strongly contact* if $p_{q-n}\varrho = 0$. A form $\varrho \in \Omega^r_q W$ is called *decomposable* if $h\varrho$ (or $p_{q-n}\varrho$) is $\pi^{r+1,r}-$projectable for $1 \le q \le n$, (or $q > n$, respectively).

The decomposition of forms (3), and especially contact and strongly contact forms, plays an important role in the theory of variational functionals. We anticipate that all such basic concepts as a lagrangian, the Euler–Lagrange form and the Helmholtz–Sonin form are based on the decomposition (3) combined with the exterior derivative operator. Let us present the local structure of contact forms more precisely (for detailed discussion see e.g. [47], [49]).

Let $W \subset Y$ be an open set and let $\varrho \in \Omega^r_q W$ be a $q-$form. Let (V,ψ) be again a fibered chart on Y for which $V \subset W$, $\psi = (x^i, y^\sigma)$. Then it holds:

(a) For $q = 1$ a form ϱ is contact if and only if it can be expressed in (V,ψ) as

$$\varrho = \Phi^J_\sigma \omega^\sigma_J, \quad 0 \le |J| \le r-1, \tag{6}$$

where $\Phi^J_\sigma \in \Omega^r_0 V$ are some functions.

(b) For $2 \le q \le n$ a form ϱ is contact if and only if it can be expressed in (V,ψ) as

$$\varrho = \omega^\sigma_J \wedge \Psi^J_\sigma + \mathrm{d}\Psi, \quad 0 \le |J| \le r-1, \tag{7}$$

where $\Psi^J_\sigma \in \Omega^r_{q-1}V$ are some $(q-1)-$forms and $\Psi \in \Omega^r_{q-1}V$ is some contact $(q-1)-$form for which $\Psi = \omega^\sigma_I \wedge \chi^I_\sigma$, $|I| = r-1$, $\chi^I_\sigma \in \Omega^r_{q-2}V$.

(c) For $n < q \leq N_r$ a form ϱ is strongly contact if and only if it can be expressed in (V, ψ) as

$$\varrho = \omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_p}_{J_p} \wedge \mathrm{d}\omega^{\sigma_{p+1}}_{I_{p+1}} \wedge \ldots \wedge \mathrm{d}\omega^{\sigma_{p+s}}_{I_{p+s}} \wedge \Phi^{J_1}_{\sigma_1} \cdots {}^{J_p I_{p+1}}_{\sigma_p \sigma_{p+1}} \cdots {}^{I_{p+s}}_{\sigma_{p+s}}, \tag{8}$$

where $\Phi^{J_1}_{\sigma_1} \cdots {}^{J_p I_{p+1}}_{\sigma_p \sigma_{p+1}} \cdots {}^{I_{p+s}}_{\sigma_{p+s}} \in \Omega^r_{q-p-2s}V$ are some forms, $0 \leq |J| \leq r-1$ for $1 \leq l \leq p$, $|I_j| = r-1$ for $p+1 \leq j \leq p+s$, and summation is taken over all such p for which $p+s \geq q-n-1$, $p+2s \leq q$. It is evident that for $q > P_r$, where $P_r = \sum_{k=0}^{r-1} M_k + 2n - 1$, the relation (8) gives the identically zero form. Furthermore, for convenience in most calculations we denote by $\omega_0 = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ the volume element on X and $\omega_i = i_{\frac{\partial}{\partial x^i}} \omega_0 = (-1)^{i-1} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \ldots \wedge \mathrm{d}x^n$.

2.2. **The finite order variational sequence.** In this section we give a relatively complete exposition of the theory of higher order variational sequence including comments concerning the proofs. The main ideas and results are based on the theory of sheaves e.g. in [51].

Let $q \geq 0$ be an integer. Let Ω^r_q be the *direct image* of the sheaf of smooth $q-$forms over J^rY by the jet projection $\pi^{r,0}$. We denote

$$\Omega^r_{q,c} = \ker p_0 = \ker h \quad \text{for} \quad 0 \leq q \leq n,$$
$$\Omega^r_{q,c} = \ker p_{q-n} \quad \text{for} \quad n < q \leq N_r,$$

where p_0 and p_{q-n} are morphisms of sheaves induced by mappings $p_0 : \varrho \to p_0\varrho$ and $p_{q-n} : \varrho \to p_{q-n}\varrho$ for $0 \leq q \leq n$ and $n < q \leq N_r$, respectively. So, for $0 \leq q \leq n$, $\Omega^r_{q,c}$ is the sheaf of contact $q-$forms and for $n < q \leq N_r$ it is the sheaf of strongly contact $q-$forms. (Recall that the functions are considered as $0-$forms and thus $\Omega^r_{0,c} = \{0\}$. Moreover, $\Omega^r_q = \{0\}$ for $q > N_r$.) Let $\mathrm{d}\Omega^r_{q-1,c}$ be the image sheaf of $\Omega^r_{q-1,c}$ by the exterior derivative d. Let $W \subset Y$ be an open set. Then $\Omega^r_q W$ is the Abelian group of $q-$forms on W^r and $\Omega^r_{q,c}W$ is its Abelian subgroup of contact or strongly contact $q-$forms on W^r, for $0 \leq q \leq n$ or $n < q \leq N_r$, respectively. Let us denote

$$\Theta^r_q = \Omega^r_{q,c} + \mathrm{d}\Omega^r_{q-1,c}, \quad \Theta^r_q W = \Omega^r_{q,c}W + \mathrm{d}\Omega^r_{q-1,c}W. \tag{9}$$

Note that $\Theta^r_q W$ is a subgroup of the group $\Omega^r_q W$. Let us consider the well-known de Rham sequence of sheaves

$$\{0\} \to \Omega^r_1 \to \cdots \to \Omega^r_n \to \Omega^r_{n+1} \to \Omega^r_{n+2} \to \cdots \to \Omega^r_{N_r} \to \{0\} \tag{10}$$

in which the arrows (with the exception of the first one) represent the exterior derivative d. The sequence (10) is exact. Furthermore, let us consider the sequence

$$\{0\} \to \Theta^r_1 \to \cdots \to \Theta^r_n \to \Theta^r_{n+1} \to \Theta^r_{n+2} \to \cdots \to \Theta^r_{P_r} \to \{0\} \tag{11}$$

with arrows having the same meaning as in (10). The following lemma ensures that (11) is the exact subsequence of de Rham sequence (10):

Lemma 1. *Let $W \subset Y$ be an open set and let $\varrho \in \Theta^r_q W$ be a form, $1 \leq q \leq N_r$. Then the decomposition $\varrho = \varrho_c + \mathrm{d}\overline{\varrho}_c$, where $\varrho_c \in \Omega^r_{q,c} W$ and $\overline{\varrho}_c \in \Omega^r_{q-1,c} W$, is unique.*

Proof–comments: The proof of lemma 1 is done by the direct coordinate calculations and its idea is as follows: For $1 \leq q \leq n$ it holds $\mathrm{d}\Omega^r_{q-1,c} W \subset \Omega^r_q W$ and thus only the case $n < q \leq N_r$ needs proof. Let $q > n$ and let $\varrho_c \in \Theta^r_q W$. Let $\varrho = 0$, i.e. $\varrho_c = -\mathrm{d}\overline{\varrho}_c$. Then $\mathrm{d}\varrho_c = 0$. Moreover, it holds $p_{q-n}\varrho_c = 0$, $p_{q-n-1}\overline{\varrho}_c = 0$. Using the decomposition (3) for ϱ_c and the chart expression (5), we can calculate the chart expression of $(\pi^{r+2,r+1})^* p_k \,\mathrm{d}\varrho_c$. Then we use two mentioned conditions $p_{q-n}\varrho_c = 0$ and $p_{q-n-1}\overline{\varrho}_c = 0$. By some recursive calculations we show that the conditions $p_k \,\mathrm{d}\varrho_c = 0$ for $q-n+1 \leq k \leq q+1$ imply that all coefficients in the chart expression of ϱ_c vanish, i.e. $\varrho_c = 0$. Thus, $\mathrm{d}\overline{\varrho}_c = 0$. In an completely analogous way we prove that also $\overline{\varrho}_c = 0$.

$\diamondsuit$

Thus, the sequence (11) is the exact subsequence of the de Rham sequence (10). The quotient sequence

$$\{0\} \to \mathbf{R}_Y \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \cdots \to \Omega^r_n/\Theta^r_n \to \Omega^r_{n+1}/\Theta^r_{n+1} \to \Omega^r_{n+2}/\Theta^r_{n+2} \to$$

$$\to \cdots \to \Omega^r_{P_r}/\Theta^r_{P_r} \to \Omega^r_{P_r+1} \to \cdots \to \Omega^r_{N_r} \to \{0\} \tag{12}$$

is called the *$r-th$ order variational sequence* on Y. It is exact too. Elements of Ω^r_q/Θ^r_q are *classes of forms* defined by the following equivalence relation: Forms ϱ, $\eta \in \Omega^r_q W$ are called *equivalent* if $\varrho - \eta \in \Theta^r_q W$. The quotient mappings are defined by the relation

$$E^r_q : \Omega^r_q/\Theta^r_q \ni [\varrho] \longrightarrow E^r_q([\varrho]) = [\mathrm{d}\varrho] \in \Omega^r_{q+1}/\Theta^r_{q+1}, \quad 0 \leq q \leq N_r. \tag{13}$$

In the standard sense, the quotient spaces are determined up to an isomorphism. This enables us to interpret the classes of equivalent forms as elements of different sheaves. This means that we could describe each of the quotient sheaves Ω^r_q/Θ^r_q by means of a certain subsheaf of the sheaf of forms Ω^s_q, generally for $s \geq r$. Within this approach a class of equivalent forms will be represented by an element of Ω^s_q. More precisely: Let us consider the diagram

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \Theta^{r+1}_q & \longrightarrow & \Omega^{r+1}_q & \longrightarrow & \Omega^{r+1}_q/\Theta^{r+1}_q & \longrightarrow & \{0\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{0\} & \longrightarrow & \Theta^r_q & \longrightarrow & \Omega^r_q & \longrightarrow & \Omega^r_q/\Theta^r_q & \longrightarrow & \{0\}
\end{array}$$

in which the first two "uparrows" represent the immersions by pullbacks and the third one defines the quotient mapping

$$Q^{r+1,r}_q : \Omega^r_q/\Theta^r_q \longrightarrow \Omega^{r+1}_q/\Theta^{r+1}_q$$

defined by

$$Q^{r+1,r}_q([\varrho]) = [(\pi^{r+1,r})^* \varrho]. \tag{14}$$

The following lemma ensures the injectivity of mappings $Q_q^{r+1,r}$:

Lemma 2. *Let us consider the diagram*

$$
\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \Theta_q^r & \longrightarrow & \Theta_q^{r+1} & \longrightarrow & \Theta_q^{r+1}/\Theta_q^r & \longrightarrow & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & \Omega_q^r & \longrightarrow & \Omega_q^{r+1} & \longrightarrow & \Omega_q^{r+1}/\Omega_q^r & \longrightarrow & \{0\}
\end{array} \tag{15}
$$

in which the last downarrow denotes the quotient mapping and the remaining ones are inclusions. Then the quotient mapping is injective.

Proof–comments: Let $W \subset Y$ be an open set and let $\varrho \in \Theta_q^{r+1}W$ be a form, $1 \leq q \leq N_r$. Let us suppose that the form ϱ is $\pi^{r+1,r}-$projectable, i.e. there exists a form $\eta \in \Omega_q^r W$, such that $\varrho = (\pi^{r+1,r})^*\eta$. We need to show that $\eta \in \Theta_q^r W$. The proof of this property can be made again by direct coordinate calculations: We express both forms $\varrho_c \in \Omega_{q,c}^{r+1}W$ and $\overline{\varrho}_c \in \Omega_{q-1,c}^{r+1}W$ in the decomposition $\varrho = \varrho_c + \mathrm{d}\overline{\varrho}_c$ in agreement with (5) and we calculate the corresponding chart expression of ϱ. Taking into account the $\pi^{r+1,r}-$projectability of the resulting expression we can conclude, after somewhat tedious calculations, that the forms ϱ_c and $\overline{\varrho}_c$ themselves are $\pi^{r+1,r}-$projectable, i.e. $\varrho_c \in \Omega_{q,c}^r W$, $\overline{\varrho}_c \in \Omega_{q-1,c}^r W$.

This ensures the injectivity of the quotient mapping in the scheme (15). Then the 3×3 lemma ensures the exactness of the sequence $\{0\} \to \Omega_q^r/\Theta_q^r \to \Omega_q^{r+1}/\Theta_q^{r+1} \to \Psi \to \{0\}$, in which $\Psi = (\Omega_q^{r+1}/\Theta_q^{r+1})/(\Omega_q^r/\Theta_q^r)$, as well as the injectivity of $Q_q^{r+1,r}$. We can define the mappings

$$Q_q^{s,r} : \Omega_q^r/\Theta_q^r \ni [\varrho] \to [(\pi^{s,r})^*\varrho] \in \Omega_q^s/\Theta_q^s, \quad s > r$$

in a quite analogous way. These mappings are injective as well.

◊

Now, let us discuss the cohomology of the variational sequence.

Theorem 1. *Each of the sheaves Ω_q^r is fine.*

Proof–comments: It is sufficient to show that Θ_q^r admits a sheaf partition of unity. However, this property is the immediate consequence of lemma 1 (the details of the proof see e.g. in [43] or [50].)

◊

The following theorem describes global properties of the variational sequence. It is the direct consequence of theorem 1.

Theorem 2. *The variational sequence is an acyclic resolution of the constant sheaf $\mathbf{R}_Y$ over $\mathcal{V}$.*

Proof–comments: It has been proved that the variational sequence is exact and thus it is a resolution of the constant sheaf $\mathbf{R}_Y$. On the other hand, by theorem 1, each of the sheaves Θ^r_q is fine and thus soft. The sheaves Ω^r_q are soft too, and thus the same holds for the quotient sheaves Ω^r_q/Θ^r_q. Thus, the resolution is acyclic.

$\diamondsuit$

Let us use the following shortened notation for the variational sequence (12): $0 \to \mathbf{R}_Y \to \mathcal{V}$. Let $\Gamma(Y, \Omega^r_0)$ be the cochain complex of global sections

$$0 \to \Gamma(Y, \mathbf{R}_Y) \to \Gamma(Y, \Omega^r_0) \to \Gamma(Y, \Omega^r_1) \to \cdots \to \Gamma(Y, \Omega^r_{N_r}) \to 0.$$

Let $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V}))$ be the $q-$th comohology group of this complex. As the immediate consequence of theorem 2 and the abstract de Rham theorem applied to the variational sequence $0 \to \mathbf{R}_Y \to \mathcal{V}$ we can identify the cohomology groups $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V}))$ for every $q \geq 0$ with the corresponding standard cohomology group $H^q(Y, \mathbf{R})$ of the manifold Y, i.e. $H^q(\Gamma(\mathbf{R}_Y, \mathcal{V})) = H^q(Y, \mathbf{R})$. This is an important result for the discussion of global properties of variational functionals.

3. Fundamental concepts of the calculus of variations

This part of the paper is devoted to the presentation of basic concepts of higher order calculus of variations, such as higher order variational functionals, Lepage equivalents of forms (especially of $n-$forms and lagrangians), the Euler–Lagrange mapping and the Helmholtz–Sonin mapping. All considerations are based on the theoretical background presented in [32] and [35], and on the theory of Lepage forms (see e.g. [7], [15], [17], [18], [32], [36], [65], and especially [49] for Lepage equivalents of lagrangians).

3.1. Variational functionals.

In this section we introduce the definition of higher order variational functionals and their variational derivatives.

Let $W \subset Y$ be an open set. Let Ω be a compact $n-$dimensional submanifold of X with boundary, such that $\Omega \subset \pi(W)$ and let $\partial\Omega$ be its boundary. Let $\varrho \in \Omega^r_n W$ be an $n-$form. Then the mapping

$$\Gamma_\Omega(\pi) \ni \gamma \longrightarrow \varrho_\Omega(\gamma) = \int_\Omega J^r\gamma^*\varrho \in \mathbf{R} \tag{16}$$

defines a *variational functional induced by* ϱ. Note that in this definition the variational functional is connected with an arbitrarily chosen $n-$form and thus it is more general than the one obviously defined by a lagrangian $\lambda \in \Omega^r_{n,X} W$. On the other hand, it holds $J^r\gamma^*\varrho = J^{r+1}\gamma^* h\varrho$ and thus the lagrangian $h\varrho \in \Omega^{r+1}_{n,X} W$ defines the same functional as the form ϱ. Hence, the generalized $r-$th order variational functional (16) connected with an arbitrary $n-$form ϱ can be defined by means of the specially chosen lagrangian of the $(r+1)-$st order (polynomial in variables of the highest order, $y^\sigma_{j_1\dots j_{r+1}}$). If, as a special case, the form ϱ itself is an $r-th$ *order* lagrangian $\lambda \in \Omega^r_{n,X} W$, we obtain from (16) the standard definition of the corresponding variational functional: $\lambda_\Omega = \int_\Omega J^r\gamma^*\lambda$.

Let $U \subset X$ be an open set and let $\gamma \in \Gamma_U(\pi)$ be a section. Let Ξ be a $\pi-$projectable vector field on an open set $W \subset Y$ for which $\gamma(U) \subset W$. If α_t is the local one-parameter group of Ξ and α_{0t} is its projection, we define by $\gamma_t = \alpha_t\gamma(\alpha_{0t})^{-1}$ a one-parameter family of sections of the projection π, called the *variation (deformation) of* γ *induced by the vector field* Ξ. Let $\varepsilon > 0$ be such a real number for which $\Omega \subset \operatorname{dom}\gamma_t$ for all $t \in (-\varepsilon, \varepsilon)$. We define the (smooth) mapping

$$(-\varepsilon,\varepsilon) \ni t \longrightarrow \varrho_{\alpha_{0t}(\Omega)}(\alpha_t\gamma\alpha_{0t}^{-1}) = \int_{\alpha_{0t}(\Omega)} \left(J^r(\alpha_t\gamma\alpha_{0t}^{-1})\right)^* \varrho \in \mathbf{R}.$$

Using the transformation integral theorem and the definition of Lie derivative we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\varrho_{\alpha_{0t}(\Omega)}(\alpha_t\gamma\alpha_{0t}^{-1})\right)_{t=0} = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\varrho \Longrightarrow (\partial_{J^r\Xi}\varrho)_\Omega(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\varrho.$$

We call the mapping $\Gamma_\Omega(\pi) \ni \gamma \to (\partial_{J^r\Xi}\varrho)_\Omega(\gamma) \in \mathbf{R}$ the *variational derivative* or *first variation of* ϱ_Ω *by the vector field* Ξ. Note that the direct generalization of this definition is possible for obtaining higher order variational derivatives of the starting variational function (for more details see [49]).

We say that the section γ is the stationary point of the variational function ϱ_Ω if $(\partial_{J^r\Xi}\varrho)(\gamma) = 0$, i.e. $\int_\Omega J^r\gamma^*\partial_{J^r\Xi}\varrho = 0$ for all admissible variations Ξ of γ. Let $\lambda \in \Omega^r_{n,X}W$ be a lagrangian. Stationary points of the variational function λ_Ω are called the *extremals* of the $r-$*th order Lagrange structure* (π, λ). Let $\varrho \in \Omega^r_n W$ be a form. It is evident that the stationary points of the variational function ϱ_Ω are just the extremals of the $(r+1)-$th order Lagrange structure $(\pi, h\varrho)$.

3.2. **Lepage forms and Lepage equivalents.** Let us now briefly introduce the concept of a Lepage form. Let $W \subset Y$ be an open set and let $\varrho \in \Omega^r_n W$. The form ϱ is called the *Lepage* $n-$*form* if the 1-contact component $p_1\mathrm{d}\varrho$ of its exterior derivative is $\pi^{r+1,0}-$horizontal, i.e. $hi_\Xi\mathrm{d}\varrho = 0$ for every $\pi^{r,0}-$vertical vector field Ξ on W^r. The following theorem describes the local structure of Lepage $n-$forms:

Theorem 3. *Let $W \subset Y$ be an open set and let $\varrho \in \Omega^r_n W$ be an $n-$form. Then ϱ is the Lepage $n-$form if and only if for every fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$ on Y for which $V \subset W$, it has the following chart expression*

$$(\pi^{r+1,r})^*\varrho = \Theta_P + \mathrm{d}\chi + \mu, \tag{17}$$

where $\chi \in \Omega^{r+1}_{n-1,c}V$ is a contact $(n-1)$-form, $\mu \in \Omega^{r+1}_{n,c}V$ is a form with the degree of contactness at least 2, and Θ_P is expressed as

$$\Theta_P = f_0\omega_0 + \sum_{k=0}^{r}\left(\sum_{l=0}^{r-k}(-1)^l \mathrm{d}_{s_1}\dots\mathrm{d}_{s_l}\frac{\partial f_0}{\partial y^\sigma_{j_1\dots j_k s_1\dots s_l i}}\right)\omega^\sigma_{j_1\dots j_k}\wedge\omega_i, \tag{18}$$

where $f_0 \in \Omega^{r+1}_0 V$ is a function.

Proof–comments: Theorem 3 can be proved by tedious calculations in three steps (see [49]):

Step 1: Every Lepage $n-$form $\varrho \in \Omega_n^r W$ has the chart expression

$$(\pi^{r+1,r})^* \varrho = f_0 \omega_0 + \sum_{k=0}^{r} f_\sigma^{i,j_1 \ldots j_k} \omega^\sigma_{j_1 \ldots j_k} \wedge \omega_i + \eta,$$

where $\eta \in \Omega_{n,c}^{r+1} V$ has the degree of contactness at least 2 and functions f_0, $f_\sigma^{i,j_1 \ldots j_k} \in \Omega_0^{r+1} V$ are connected by the relations

$$\frac{\partial f_0}{\partial y^\sigma_{j_1 \ldots j_k}} - \mathrm{d}_i f_\sigma^{i,j_1 \ldots j_k} - f_\sigma^{j_k, j_1 \ldots j_{k-1}} = 0, \quad \mathrm{sym}\,(j_1, \ldots, j_k), \quad 1 \leq k \leq r, \tag{19}$$

$$\frac{\partial f_0}{\partial y^\sigma_{j_1 \ldots j_{r+1}}} - f_\sigma^{j_{r+1}, j_1 \ldots j_r} = 0, \quad \mathrm{sym}\,(j_1, \ldots, j_{r+1}).$$

Step 2: The system of equations (19) is solved by means of the decomposition of functions

$$f_\sigma^{i,j_1 \ldots j_k} = F_\sigma^{i,j_1 \ldots j_k} + G_\sigma^{i,j_1 \ldots j_k}$$

into their symmetric and complementary parts, $F_\sigma^{i,j_1 \ldots j_k}$ and $G_\sigma^{i,j_1 \ldots j_k}$, respectively. Functions $F_\sigma^{i,j_1 \ldots j_k}$ symmetrized over $(j_1, \ldots, j_k, i)$ are finally expressed by means of f_0. This enables us to express the form $(\pi^{r+1,r})^* \varrho$ as the sum $\Theta_P + \nu + \mu$ where Θ_P has exactly the form (18), $\mu \in \Omega_{n,c}^{r+1} V$ is the form of the degree of contactness at least 2 and the contact form $\nu \in \Omega_{n,c}^{r+1} V$ is expressed by means of functions $G_\sigma^{i,j_1 \ldots j_k}$.

Step 3: There exists a contact $(n-1)-$form χ for which $p_1 \mathrm{d}\chi = \nu$. This can be proved by the direct solution of this equation supposing the form χ to have the chart expression

$$\chi = \frac{1}{2} \sum_{k=0}^{r} H_\sigma^{i_1 i_2, j_1 \ldots j_k} \, \omega^\sigma_{j_1 \ldots j_k} \wedge \omega_{i_1 i_2}$$

with unknown coefficients $H_\sigma^{i_1 i_2, j_1 \ldots j_k}$, $0 \leq k \leq r$, where $\omega_{ij} = i_{\frac{\partial}{\partial x^i}} \omega_j$.

$\diamondsuit$

The form Θ_P is called the *principal component* of the Lepage form ϱ with respect to the considered fibered chart (V, ψ). Note that Θ_P is not in general coordinate invariant.

Let $\varrho \in \Omega_n^r W$ be an $n-$form. A Lepage $n-$form $\Theta_\varrho \in \Omega_{n,Y}^s W$, $s \geq r$ in general, is called the *Lepage equivalent of* ϱ, if it obeys the condition $h\Theta_\varrho = h\varrho$, up to a possible projection. Note that if ϱ is a lagrangian we obtain the standard concept of Lepage equaivalent of lagrangian (see e.g. [49]). Let $\lambda \in \Omega_{n,X}^r W$ be a lagrangian for which $\lambda = \mathcal{L}\omega_0$ in a fibered chart (V, ψ) such that $V \subset W$. Then, as the immediate consequence of the relation (18), a Lepage form $\Theta \in \Omega^s_{n,Y} V$ is its Lepage equivalent if and only if its principal component is of the form

$$\Theta_P = \mathcal{L}\omega_0 + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-k-1} (-1)^l \mathrm{d}_{j_1} \dots \mathrm{d}_{j_l} \frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 \dots i_k j_1 \dots j_l i}} \right) \omega^\sigma_{i_1 \dots i_k} \wedge \omega_i .$$

This assertion can be reformulated for an arbitrary form $\varrho \in \Omega^r_n W$ taking its horizontal component as the corresponding lagrangian. It is evident that for every form $\varrho \in \Omega^r_n W$ there exists its Lepage equivalent, $\Theta_\varrho = \Theta_{h\varrho}$. It is not unique, in general. (Note that the principal component $(\Theta_\varrho)_P$ itself gives a Lepage equivalent of the form ϱ which is in general defined only locally, because of the non-invariance of the splitting (17) with respect to various fibered charts.)

The corresponding reformulation of the well-known first variational formula, in its integral or infinitesimal version, reads:

$$\int_\Omega J^{r+1}\gamma^* \partial_{J^{r+1}\Xi} h\varrho = \int_\Omega J^s\gamma^* i_{j^s\Xi}\, \mathrm{d}\Theta_\varrho + \int_{\partial\Omega} J^s\gamma^* i_{J^s\Xi} \Theta_\varrho, \quad \text{or}$$

$$(\pi^{s+1,r+1})^* \partial_{J^{r+1}\Xi} h\varrho = h i_{J^s\Xi}\, \mathrm{d}\Theta_\varrho + h\, \mathrm{d} i_{j^s\Xi} \Theta_\varrho$$

for every $\pi-$projectable vector field Ξ on W. (In a special case in which ϱ is a lagrangian this gives the standard first variational formula.)

It is well-known that the concept of Lepage equivalents of lagrangians is closely related to equations of motion of variational physical systems. Moreover, we shall see that the concept of Lepage forms in somewhat generalized sense plays an important role in the problem of representation of variational sequence by forms. So, let us now present some examples of Lepage equivalents of lagrangians.

Example 1. In mechanics, every lagrangian $\lambda \in \Omega^r_{1,X} W$ has unique Lepage equivalent. In a fibered chart (V, ψ), $V \subset W$, a lagragian is expressed as $\lambda = \mathcal{L}\, \mathrm{d}t$. Then its Lepage equivalent is an element of $\Omega^{2r-1}_{n,Y} W$ and has the form

$$\Theta_\lambda = \mathcal{L}\, \mathrm{d}t + \sum_{k=0}^{r} \left(\sum_{l=0}^{r-k-1} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l} \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{(k+l+1)}} \right) \right) \omega^\sigma_{(k)} .$$

For $r = 1$ we obtain the well-known *Poincaré–Cartan form.*

In the field theory the situation is not so simple, because of the fact that every lagrangian has a family of Lepage equivalents which are not necessarily globally defined. Nevertheless one can construct some special types of Lepage equivalents:

Example 2. Let $\lambda \in \Omega^1_{n,X} W$. The family of corresponding Lepage equivalents contains the uniquely defined one, such that its degree of contactness is at most 1. It is given by the chart expression (see also [49]).

$$\Theta_\lambda = \mathcal{L}\omega_0 + \frac{\partial \mathcal{L}}{\partial y^\sigma_i} \omega^\sigma \wedge \omega_i$$

and it is called the *Poincaré–Cartan equivalent of* λ.

Example 3. Some other important type of Lepage equaivalent of first order lagrangians is so called *fundamental Lepage equivalent* discovered by Krupka [36], [42] and Betounes [7]. It has the chart expression

$$\Theta_\lambda = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \left(\frac{\partial^k \mathcal{L}}{\partial y^{\sigma_1}_{j_1} \dots \partial y^{\sigma_k}_{j_k}} \right) \epsilon_{j_1 \dots j_k i_{k+1} \dots i_n} \, \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge \mathrm{d}x^{i_{k+1}} \wedge \dots \wedge \mathrm{d}x^{i_n}.$$

Note that this Lepage equivalent is defined on $J^r Y$, i.e. it is of the same order as the lagrangian.

Example 4. The family of Lepage equivalents of every second order lagrangian contains an invariant Lepage equivalent given by:

$$\Theta_\lambda = \mathcal{L}\omega_0 + \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_i} - \mathrm{d}_j \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{ji}} \right) \right) \omega^\sigma \wedge \omega_i + \frac{\partial \mathcal{L}}{\partial y^\sigma_{ji}} \omega^\sigma_j \wedge \omega_i$$

(see [32]). As an example let us show the second order *Hilbert–Einstein lagrangian* depending on second order derivatives of the metric tensor, which has been studied in details by Krupková [61] and Novotný [64] (see also [56]):

$$\lambda = R\sqrt{|\det g_{ij}|}\omega_0,$$

where (g_{ij}) is the metric tensor and R is the scalar curvature

$$R = g^{ik} g^{jp} R_{ijkp}, \quad 0 \le i, j, k, p \le 3,$$

$$R_{ijkp} = \frac{1}{2}(g_{ip,jk} + g_{jk,ip} - g_{ik,jp} - g_{jp,ik}) + g_{sq}(\Gamma^s_{jk}\Gamma^q_{ip} - \Gamma^s_{jp}\Gamma^q_{ik}),$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{is}(g_{sj,k} + g_{sk,j} - g_{jk,s}).$$

Moreover, this lagrangian is affine in second order variables $(g_{ij,kl})$ and it is of the special type $\lambda = \left(L_0(x^i, y^\sigma) + G^{jk}_\nu(x^i, y^\sigma)\, y^\nu_{jk} \right) \omega_0$ (see [61]). There exists the global first order Lepage equivalent of λ (see [61], [64]):

$$\Theta_\lambda = \sqrt{|\det g_{ij}|} g^{ip} \left(\Gamma^j_{ip}\Gamma^k_{jk} - \Gamma^j_{ik}\Gamma^k_{jp} \right) \omega_0 + \left(g^{jp}g^{iq} - g^{pq}g^{ij} \right) \left(\mathrm{d}g_{pq,j} + \Gamma^k_{pq}\, \mathrm{d}g_{jk} \right) \wedge \omega_i.$$

By some calculations we can make sure that the coefficients of the chart expression of $p_1 \mathrm{d}\Theta_\lambda$ in the fibered chart (V, ψ) are exactly the left-hand sides of the vacuum Einstein equations.

The concept of a Lepage form was extended to the case of $(n+1)-$forms by Krupková in [57], [61] for mechanics (n=1) and recently also for the field theory ($n > 1$, see [62]):

Let $E \in \Omega^r_{n+1,Y} W$ be a form. In a fibered chart (V, ψ) on Y, such that $V \subset W$, it has the chart expression

$$E = E_\sigma \omega^\sigma \wedge \omega_0, \quad E_\sigma \in \Omega^r_0 V.$$

A *closed* form $\alpha \in \Omega^{r-1}_{n+1}W$ is called the *Lepage* $(n+1)$*−form* if it can be decomposed as $(\pi^{r,r-1})^*\alpha = E + F$, where $E \in \Omega^r_{n+1,Y}W$ and $F \in \Omega^r_{n+1,c}W$ is a strongly contact form. For every $\pi^{r,0}$-horizontal $(n+1)-$form E there exists the class of $(n+1)-$forms $[\alpha]$ for which $p_1\alpha = E$. It is well-known that a form $E \in \Omega^r_{n+1,Y}W$ expresses the equations of motion $E_\sigma = 0$ of a physical system. The concept of Lepage $(n+1)$-forms given by Krupková enables us to answer the question whether a physical system given by its equations of motion is variational, i.e. whether it moves along extremals of a lagrangian: It can be proved (see [61] and [62]) that the class $[\alpha]$ corresponding to a given $\pi^{r,0}$-horizontal $(n+1)-$form E contains a Lepage representative if and only if E is variational. Such representative is then unique and $\pi^{r,r-1}-$projectable. In this generalized appropach, the variational form $E \in \Omega^r_{n+1,Y}W$ which represents the variational equations of motion is directly related to the Lepage $(n+1)-$form (instead of a lagrangian). The advantage of this approach lies in the fact that various equivalent lagrangians give the same system of equations for extremals of the corresponding Lagrange structure and, as we shall see in the section 3.3, the same Euler–Lagrange form.

3.3. **Euler–Lagrange and Helmholtz–Sonin form.** In this section we extend the definition of the well-known Euler–Lagrange mapping of calculus of variations which assigns to every lagrangian λ its Euler–Lagrange form E_λ.

By direct calculation we can prove the following theorem which is closely related to the concept of Euler–Lagrange mapping:

Theorem 4. *Let $\varrho \in \Omega^r_{n,Y}$ be a Lepage $n-$form. Then there exists the unique decomposition of its exterior derivative $(\pi^{r+1,r})^*\mathrm{d}\varrho = E + F$, where $E = p_1\mathrm{d}\varrho$ is the 1-contact $\pi^{r+1,0}-$horizontal $(n+1)-$form which depends on $h\varrho$ only, and $F$is a form such that its degree of contactness is at least 2. Moreover, it holds*

$$E = p_1\mathrm{d}\varrho = E_\sigma\,\omega^\sigma \wedge \omega_0 = \left(\sum_{k=0}^{r}(-1)^k \mathrm{d}_{j_1}\dots\mathrm{d}_{j_k}\frac{\partial f_0}{\partial y^\sigma_{j_1\dots j_k}}\right)\omega^\sigma \wedge \omega_0. \tag{20}$$

The form E is called the *Euler–Lagrange form* of ϱ.

Following the standard first variation procedure we can assign to every lagrangian $\lambda \in \Omega^r_{n,X}W$ its Euler–Lagrange form E_λ given by the relation (20) applied to $\varrho = \Theta_\lambda$. This form is defined on $J^{2r}Y$, in general, i.e. $E_\lambda \in \Omega^{2r}_{n+1,Y}W$. (Recall that the Euler–Lagrange form $E_\lambda = p_1\mathrm{d}\Theta_\lambda$ of the lagrangian λ is unique and it is independent of the concrete choice of the Lepage equivalent Θ_λ.) This correspondence defines the *Euler–Lagrange mapping* in the standard way:

$$\Omega^r_{n,X}W \ni \lambda \longrightarrow E_\lambda \in \Omega^{2r}_{n+1,Y}W.$$

The importance of Euler–Lagrange mapping is evident from the following theorem the proof of which is based on the first variational formula and on the fact that $p_1\,\mathrm{d}\Theta_\lambda = E_\lambda$.

Theorem 5. *Let $W \subset Y$ be an open set. Let $\lambda \in \Omega^r_{n,X} W$ be a lagrangian and let E_λ be its Euler–Lagrange form. Let $\gamma \in \Gamma_U(\pi)$ be a section of π, $\Omega \subset U$ a compact $n-$dimensional submanifold of X with boundary $\partial\Omega$. Denote as $\Theta_\lambda \in \Omega^s_{n,Y} W$ a Lepage equivalent of λ. Then the following four conditions are equivalent:*

(a) γ is an extremal of (π, λ) on Ω.

(b) For every π-vertical vector field Ξ defined on a neighborhood of $\gamma(U)$, such that $\operatorname{supp}(\Xi \circ \gamma) \subset U$, *it holds $J^s\gamma^* i_{J^s\Xi}\, \mathrm{d}\varrho = 0$.*

(c) For any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$ on Y, such that $V \subset W$, γ satisfies the system of differential equations (equations of motion) $E_\sigma(\lambda) \circ J^{2r}\gamma = 0$, $1 \le \sigma \le m$.

(d) The Euler–Lagrange form E_λ vanishes along $J^{2r}\gamma$, i.e. $E_\lambda \circ J^{2r}\gamma = 0$.

Following the generalized concept of a Lepage equivalent presented in Section 3.2, we can also extend the concept of the Euler–Lagrange form and the Euler–Lagrange mapping.

It is evident that the structure of the "classical" Euler–Lagrange mapping has the key importance for the variationally trivial problem, because its kernel gives all trivial lagrangians. On the other hand, knowing the structure of its image, we can characterize all variational $\pi^{r,0}-$horizontal $(n+1)-$forms by the well known Helmholtz–Sonin expressions (see e.g. [43]): Let $W \subset Y$ be an open set. Let a form $E \in \Omega^r_{n+1,Y} W$ be, in a fibered chart (V, ψ) on Y, such that $V \subset W$, expressed as $E = E_\sigma\, \omega^\sigma \wedge \omega_0$. The functions

$$\mathcal{H}^{j_1 \ldots j_k}_{\sigma\nu} = \frac{1}{2} \left(\frac{\partial E_\nu}{\partial y^\sigma_{j_1 \ldots j_k}} - (-1)^k \frac{\partial E_\sigma}{\partial y^\nu_{j_1 \ldots j_k}} - \sum_{l=k+1}^{r} (-1)^l \binom{l}{k} \mathrm{d}_{j_{k+1}} \ldots \mathrm{d}_{j_l} \frac{\partial E_\sigma}{\partial y^\nu_{j_1 \ldots j_l}} \right)$$

are called its *Helmholtz–Sonin expressions*. Recall that a form $E = E_\sigma\, \omega^\sigma \wedge \omega_0$ is called variational if it is the Euler–Lagrange form of a lagrangian λ, i.e. $E = E_\lambda$. A given $\pi^{r,0}-$horizontal $(n+1)-$form E is variational if and only if the corresponding Helmholtz–Sonin expression vanish identically.

For the purposes of the variational sequence we define the following $(n+2)-$form associated with a given form $E \in \Omega^r_{n+1,Y} W$:

$$H_E = \mathcal{H}^{j_1 \ldots j_k}_{\sigma\nu}\, \omega^\sigma_{j_1 \ldots j_k} \wedge \omega^\nu \wedge \omega_0, \quad 0 \le k \le 2r. \tag{21}$$

Note that this definition is only local for present. Its coordinate invariance will be mentioned later, in section 4.1. The arising mapping

$$\mathcal{H} : \Omega^r_{n+1,Y} W \ni E \longrightarrow \mathcal{H}(E) = H_E \in \Omega^{2r}_{n+2} W$$

is called ("classical") *Helmholtz–Sonin mapping*.

4. Variational sequence and calculus of variations

In this part of the paper we shall demonstrate the possibility of an effective interpretation of fundamental concepts of the calculus of variations, such as Euler–Lagrange mapping and Helmholtz–Sonin mapping, as well as their relations, by means of the variational sequence. We also introduce the concept of the representation of the variational sequence by forms and we construct such appropriate representation which leads to the generalized concept of lagrangian and its Lepage equivalents, Euler–Lagrange form and Helmholtz–Sonin form.

4.1. **Representation of the variational sequence by forms.** In this section we use the injectivity of mappings $Q^{s,r}_q$ to discuss the problem of the representation of the variational sequence (12) by the appropriately chosen (exact) sequence of mappings of spaces of forms. This problem is completely solved for the first order mechanics in [50]. Its solution for the part of the variational sequence closely related to the standard calculus of variations in higher order mechanics is presented in [30], [63] and [72]. For the field theory see e.g. [24] (first order field theory) and [43] or [31] (higher order field theory).

Any mapping

$$\Phi^{s,r}_q : \Omega^r_q W/\Theta^r_q W \ni [\varrho] \longrightarrow \Phi^{s,r}_q([\varrho]) = \varrho_0 \in \Omega^s_q W$$

with $\varrho_0 \in [(\pi^{s,r})^*\varrho]$ is called *representation of* $\Omega^r_q W/\Theta^r_q W$. Because of the injectivity of mappings $Q^{s,r}_q$ (see definition (14) and lemma 2) the representation mappings $\Phi^{s,r}_q$ are injective too. Thus, we can define the *representation of the variational sequence by forms* as the lower row of the following scheme:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Omega^r_q/\Theta^r_q & \longrightarrow & \Omega^r_{q+1}/\Theta^r_{q+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Omega^s_q & \longrightarrow & \Omega^s_{q+1} & \longrightarrow & \cdots
\end{array}$$

in which the upper row is the variational sequence, the "downarrows" represent the mappings $\Phi^{s,r}_q$ and mappings of the lower row are defined by

$$E^{s,r}_q : \Omega^s_q \longrightarrow \Omega^s_{q+1}, \quad E^{s,r}_q = \Phi^{s,r}_{q+1} \circ E^r_q \circ (\Phi^{s,r}_q)^{-1}, \quad E^{s,r}_0 = \Phi^{s,r}_1 \circ E^r_0. \tag{22}$$

In the following considerations we shall show that there exists such a representation of the variational sequence by forms for $q = n$, $n+1$, $n+2$ for which $E^{s,r}_n$ is the Euler–Lagrange mapping and $E^{s,r}_{n+1}$ is the Helmholtz–Sonin mapping (see section 3.3).

Lemma 3. *Let $W \subset Y$ be an open set, and let $q \geq 1$ be an integer. Let (V, ψ) be a fibered chart on Y for which $V \subset W$.*

(a) *Let $1 \leq q \leq n$ and let $\varrho \in \Omega^r_q W$ be a form. Then the mapping*

$$\Phi^{s,r}_q : \Omega^r_q V/\Theta^r_q V \ni \varrho \longrightarrow \Phi^{s,r}_q([\varrho]) = (\pi^{s,r})^* h\varrho \in \Omega^s_q V, \quad s \geq r+1 \tag{23}$$

is the representation of $\Omega^r_q V/\Theta^r_q V$.

(b) *Let $q = n+1$ and let $\varrho \in \Omega^r_{n+1} W$ be a form for which $p_1\varrho$ is in the fibered chart (V,ψ) expressed by the relation*

$$p_1\varrho = P^J_\sigma \, \omega^\sigma_J \wedge \omega_0,$$

in which coefficients $P^J_\sigma \in \Omega^{r+1}_0 V$, $0 \le |J| \le r$, are given by the chart expression of ϱ following eqs. (4-5). Then the mapping

$$\Phi^{s,r}_{n+1} : \Omega^r_{n+1}V/\Theta^r_{n+1}V \ni \varrho \longrightarrow \Phi^{s,r}_{n+1}([\varrho]) = \varrho_0 \in \Omega^s_{n+1}V, \quad s \ge 2r+1$$

assigning to the class $[\varrho]$ the form

$$\varrho_0 = (\pi^{s,2r+1})^* \left(\sum_{l=0}^{r} (-1)^l \mathrm{d}_{j_1} \dots \mathrm{d}_{j_l} P^{j_1 \dots j_l}_\sigma \right) \omega^\sigma \wedge \omega_0 \tag{24}$$

is the representation of $\Omega^r_{n+1}V/\Theta^r_{n+1}V$.

(c) *Let $q = n+2$ and let $\varrho \in \Omega^r_{n+2} W$ be a form for which $p_2\varrho$ is in the fibered chart (V,ψ) expressed by the relation*

$$p_2\varrho = P^{JK}_{\sigma\nu} \, \omega^\sigma_J \wedge \omega^\nu_K \wedge \omega_0,$$

in which coefficients $P^{JK}_{\sigma\nu} \in \Omega^{r+1}_0 V$, $0 \le |J| \le r$, can be obtained from the chart expression (4-5) of the form $(\pi^{r+1,r})^*\varrho$. Then the mapping

$$\Phi^{s,r}_{n+2} : \Omega^r_{n+2}V/\Theta^r_{n+2}V \ni \varrho \longrightarrow \Phi^{s,r}_{n+2}([\varrho]) = \varrho_0 \in \Omega^s_{n+2}V, \quad s \ge 2r+1$$

assigning to the class $[\varrho]$ the form

$$\varrho_0 = \tag{25}$$

$$= (\pi^{s,2r+1})^* \sum_{j=0}^{2r} \left[\sum_{p=0}^{j} \sum_{l=j-p}^{r} (-1)^l \binom{l}{j-p} \mathrm{d}_{i_{j+1}} \dots \mathrm{d}_{i_{p+l}} \, P^{i_1 \dots i_p, i_{p+1} \dots i_{p+l}}_{\sigma\nu} \right] \omega^\sigma_{i_1 \dots i_j} \wedge \omega^\nu \wedge \omega_0,$$

$\mathrm{sym}(i_1, \dots, i_j)$, *$s \ge 2r+1$, is the representation of $\Omega^r_{n+2}V/\Theta^r_{n+2}V$.*

Proof–comments: The equivalence $\Phi^{s,r}_q([\varrho]) = 0 \Leftrightarrow \varrho \in \Theta^r_q V$ is to be proved in cases (a-c). The proof of the part (a) is trivial, because of the fact that $\Theta^r_q \subset \Omega^r_q$. The proofs of parts (b) and (c) are based on tedious coordinate calculations and we present here their idea only. (For more detailed discussion see [31], [43]).

(b) Let $q = n+1$. Let (V,ψ) be a fibered chart on Y and let $\varrho \in \Theta^r_{n+1}V$. Then ϱ is uniquely decomposed as $\varrho = \varrho_c + \mathrm{d}\overline{\varrho}_c$, where $\varrho_c \in \Omega^r_{n+1,c}V$ and $\overline{\varrho}_c \in \mathrm{d}\Omega^r_{n,c}V$ (see lemma 1). Then we have $\Phi^{s,r}_{n+1}([\varrho_c]) = 0$. Now the equation $\Phi^{s,r}_{n+1}([\mathrm{d}\overline{\varrho}_c]) = 0$ needs proof. Taking into account the local structure of contact forms given by the equation (7) we can obtain by exterior derivative of the decomposition $\varrho = \varrho_c + \mathrm{d}\overline{\varrho}_c$

$$\mathrm{d}\overline{\varrho}_c = \mathrm{d}(\omega^\sigma_J \wedge \Psi^J_\sigma) \Rightarrow p_1 \mathrm{d}\overline{\varrho}_c = -\omega^\sigma_{Ji} \wedge \mathrm{d}x^i \wedge h\Psi^J_\sigma - \omega^\sigma_J \wedge h\mathrm{d}\Psi^J_\sigma.$$

Using the chart expressions of Ψ^J_σ in the form

$$(\pi^{r+1,r})^*\Psi^J_\sigma = \sum_{l=0}^{n-1} (B^J_\sigma)^{J_1}_{\sigma_1} \cdots {}^{J_l}_{\sigma_l, i_{l+1} \dots i_{n-1}} \, \omega^{\sigma_1}_{J_1} \wedge \dots \wedge \omega^{\sigma_l}_{J_l} \wedge \mathrm{d}x^{i_{l+1}} \wedge \dots \mathrm{d}x^{i_{n-1}},$$

we obtain the coefficients P^J_σ in the chart expression of $p_1 \mathrm{d}\overline{\varrho}_c = P^J_\sigma \omega^\sigma_J \wedge \omega_0$. Putting them into (25) we obtain, after some technical steps, the equation $\Phi^{s,r}_{n+1}([\mathrm{d}\overline{\varrho}_c]) = 0$.

Conversely, let $\Phi^{s,r}_{n+1}([\varrho]) = 0$. Using lemma 2 we obtain after some coordinate calculations the expected result $\varrho \in \Theta^r_{n+1}V$.

(c) For $q = n+2$ the proof is based again on coordinate calculations and it is quite analogous with (b).

$\diamondsuit$

The expressions (23) and (24) for representatives of classes of $n-$forms and $(n+1)-$forms can be found in [43]. In the same paper, the special case of the expression (25) was obtained, representing only the classes of $(n+2)-$forms expressed as exterior derivatives of $\pi^{r,0}-$horizontal $(n+1)-$forms. The local expression (25) for representatives of general classes of $(n+2)-$forms was presented recently (see [31]). As we shall see from the following theorem, all these expressions fulfil the transformation rules between various fibered charts and thus they are the chart expressions of forms representing classes of $q-$forms for $q = n$, $n+1$, $n+2$.

Theorem 6. *Let (V, ψ) be a fibered chart on Y. Let $1 \le q \le n+2$ and $\varrho \in \Omega^r_q Y$ be a form. Then the class $[\varrho]$ is represented by eqs. (23), (24) and (25) globally, for $1 \le q \le n$, $q = n+1$ and $q = n+2$, respectively.*

Proof–comments: The horizontalization mapping h is coordinate invariant and thus only the cases $q = n+1$, $n+2$ need proof. For the first order field theory the detailed proof can be found in [25] and [26], for the higher order field theory it was given recently in [31]. The idea of the proof is based on some integration procedure considering the coordinate invariance of functions

$$\eta_\Omega = \int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* h i_{J^r\xi} \varrho \quad \text{and} \tag{26}$$

$$\eta_\Omega = \int_\Omega J^s\gamma^* \circ (\pi^{s,r+1})^* h i_{J^r\xi} i_{J^r\zeta} \chi \tag{27}$$

for $\varrho \in \Omega^r_{n+1}W$ and $\chi \in \Omega^r_{n+2}W$, respectively. In these two relations, Ω is a compact piece of manifold X, ξ and ζ are $\pi-$vertical vector fields such that $\operatorname{supp}\xi \subset \pi^{-1}(\Omega)$. For details of the proof we refer the reader to the paper [31] which has been submitted to these Proceedings as well.

$\diamondsuit$

Corollary 1. *$W \subset Y$ be an open set. Let $(\Phi^{2r+1,r}_q)$ for $1 \le q \le n+2$ be the representation of spaces $\Omega^r_q W / \Theta^r_q W$ which is locally given by relations (23-25) following lemma 3.*

(a) Then the mapping

$$E^{2r+1,r}_n : \Omega^{2r+1}_n \longrightarrow \Omega^{2r+1}_{n+1}, \quad E^{2r+1,r}_n = \Phi^{2r+1,r}_{n+1} \circ E^r_n \circ (\Phi^{2r+1,r}_n)^{-1}$$

is the (extended) Euler–Lagrange mapping.

(b) Let $\Omega^{2r+1}_{n+1,\mathrm{dyn}}$ be the set of representatives of classes of $\pi^{r,0}$–horizontal $(n+1)$–forms defined on J^rY. Then mapping

$$E^{2r+1,r}_{n+1} : \Omega^{2r+1}_{n} \longrightarrow \Omega^{2r+1}_{n+1}, \quad E^{2r+1,r}_{n} = \Phi^{2r+1,r}_{n+2} \circ E^r_{n+1} \circ (\Phi^{2r+1,r}_{n+1})^{-1}$$

restricted to $\Omega^{2r+1}_{n+1,\mathrm{dyn}}$ is the Helmholtz–Sonin mapping.

Proof–comments: (a) Let $[\varrho] \in \Omega^r_n W/\Theta^r_n W$ be a class generated by the form $\varrho \in \Omega^r_n W$. Then $\Phi^{2r+1,r}_n([\varrho]) = (\pi^{2r+1,r+1})^* h\varrho$ is the corresponding lagrangian, $h\varrho = \mathcal{L}\omega_0$. On the other hand, we have

$$p_1\,\mathrm{d}(\pi^{r+1,r})^*\varrho = p_1\,\mathrm{d}h\varrho + p_1\,\mathrm{d}p_1\varrho = p_1\,\mathrm{d}(\mathcal{L}\omega_0) + p_1\,\mathrm{d}(B^{J,i}_\sigma \omega^\sigma_J \wedge \omega_i),$$

$$J = (j_1 \dots j_k), \quad 0 \le k \le r,$$

where coefficients $\mathcal{L}$, $B^{J,i}_\sigma \in \Omega^{r+1}_0 V$ can be determined from the chart expression of the form ϱ, given by (4) and (5). For coefficients $B^{J,i}_\sigma$ we obtain

$$p_1\,\mathrm{d}(\pi^{r+1,r})^*\varrho = \frac{\partial \mathcal{L}}{\partial y^\sigma_J}\omega^\sigma_J \wedge \omega_0 - \mathrm{d}_\mathrm{i} B^{J,i}_\sigma \omega^\sigma_J \wedge \omega_0 - B^{J,i}_\sigma \omega_{Ji} \wedge \omega_0, \quad \text{and thus}$$

$$P^{j_1\dots j_k}_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1\dots j_k}} - \mathrm{d}_i B^{j_1\dots j_k,i}_\sigma - B^{j_1\dots j_{k-1},j_k}_\sigma, \quad 1 \le k \le r, \quad \mathrm{sym}\,(j_1,\dots,j_k),$$

$$P^{j_1\dots j_{r+1}}_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1\dots j_{r+1}}} - B^{j_1\dots j_r,j_{r+1}}_\sigma, \quad \mathrm{sym}\,(j_1,\dots,j_{r+1}), \quad P_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma} - \mathrm{d}_i B^{,i}_\sigma.$$

The relation (24) gives

$$\Phi^{2r+1,r}_{n+1}([\mathrm{d}\varrho]) = \sum_{k=0}^{r+1} (-1)^k \mathrm{d}_{j_1}\dots \mathrm{d}_{j_k}\left(\frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1\dots j_k}}\right)\omega^\sigma \wedge \omega_0\,. \tag{28}$$

We can see that this is exactly the relation (20) (see section 3.3) for the Euler–Lagrange form associated with the form ϱ (or, equivalently, the Euler–Lagrange form of the lagrangian $h\varrho$). Thus we have

$$E^{2r+1,r}_n \circ \Phi^{2r+1,r}_{n+1}([\varrho]) = E_{h\varrho}. \tag{29}$$

(b) Let $\mathcal{E} \in \Omega^r_{n+1,Y} W$ be a form given in the fibered chart (V,ψ), $V \subset W$, by the expression

$$\mathcal{E} = \varepsilon_\sigma\,\omega^\sigma \wedge \omega_0, \quad \varepsilon_\sigma \in \Omega^r_0 V.$$

Then

$$\varrho = \mathrm{d}\mathcal{E} = \sum_{0 \le |J| \le r} \frac{\partial \varepsilon_\nu}{\partial y^\sigma_J}\,\omega^\sigma_J \wedge \omega^\nu \wedge \omega_0.$$

On the other hand, in general, we have

$$p_2\varrho = P^{JK}_{\sigma\nu}\,\omega^\sigma_J \wedge \omega^\nu_K \wedge \omega_0, \qquad P^{JK}_{\sigma\nu} + P^{KJ}_{\nu\sigma} = 0.$$

Thus,

$$P^{0J}_{\sigma\nu} = -P^{J0}_{\nu\sigma} = -\frac{1}{2}\frac{\partial\varepsilon_\sigma}{\partial y^\nu_J}, \qquad J = (j_1 \dots j_k), \;\; 1 \le k \le r \;,$$
$$P^{00}_{\sigma\nu} = -P^{00}_{\nu\sigma} = \left(\frac{\partial\varepsilon_\nu}{\partial y^\sigma}\right)_{\mathrm{alt}(\sigma\nu)},$$

other coefficients $P^{JK}_{\sigma\nu}$ being zero. Using the relation (25) we obtain, as the representative of the class $[E]$, exactly the local expression (21) of the Helmholtz–Sonin form of E.

♢

As the immediate consequence of the just presented considerations we can formulate the following corollary:

Corollary 2. *Let $W \subset Y$ be an open set and let (V, ψ) be a fibered chart on Y form which $V \subset W$. Then the mapping*

$$\Psi^{2r+1,r}_n : \Omega^r_n V/\Theta^r_n V \ni [\varrho] \longrightarrow \Psi^{2r+1,r}_n([\varrho]) = \Theta_\varrho \in \Omega^{2r+1}_n V,$$

assigning to the class of $n-$forms generated by a form ϱ its Lepage equivalent Θ_ϱ is a representation of spaces $\Omega^r_n V/\Theta^r_n V$. Moreover, it holds

$$\Phi^{2r+1,r}_{n+1}([\mathrm{d}\varrho]) = p_1\, \mathrm{d}\Psi^{2r+1,r}_n([\varrho]), \quad \text{i.e.} \quad E_{h\varrho} = p_1\, \mathrm{d}\Theta_\varrho. \tag{30}$$

This discloses the close relation of the variational sequence to one of the basic concepts of calculus of variations, the Euler–Lagrange mapping. Considering this relation described by corollaries 1 and 2 we can generalize the concept of the Euler–Lagrange and Helmholtz–Sonin mappings in the following way: We call the arrows E^r_n and E^r_{n+1} in the variational sequence (12) the *generalized Euler–Lagrange mapping* and *generalized Helmholtz–Sonin mapping*, respectively.

Because of the close relation of mappings $E^{2r+1,r}_n$ and $E^{2r+1,r}_{n+1}$ to physical theories we use for the corresponding representation of the variational sequence the name *physical representation.*

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