ENSAIOS MATEMÁTICOS
SOCIEDADE BRASILEIRA DE MATEMÁTICA

The Dynamics of Inner Functions

CLAUS I. DOERING
Universidade Federal do Rio Grande do Sul

RICARDO MAÑÉ
Instituto de Matemática Pura e Aplicada

VOLUME 3 1991

Ensaios Matemáticos

Esta série objetiva publicar textos sobre temas atuais da Matemática, permitindo ao leitor, através de exposições acessíveis, adquirir ampla perspectiva do assunto em questão, inclusive no que se refere a seus aspectos em aberto. Os trabalhos devem ser enviados a um dos editores.

This series is intended as a vehicle for survey papers in all areas of Mathematics. The aim of the series is to offer readers an accessible overview of current topics in Mathematics. The Brazilian Mathematical Society invites authors to submit papers for inclusion in the series by sending their work to one of the editors.

Full instructions on the preparation of copy are available from the editors upon request.

How to order Ensaios Matemáticos

Ensaios Matemáticos are available directly from the Brazilian Mathematical Society by writing to the following address:

Ensaios Matemáticos
Sociedade Brasileira de Matemática
Estrada Dona Castorina, 110
22460 Rio de Janeiro RJ
Brazil

Financed by the Programa de Apoio a Publicações Científicas CNPq/FINEP.
Partial support from Instituto de Matemática Pura e Aplicada.

CONTENTS

Introduction and statement of results

Let D denote the open unit disc $\{z \in \mathbf{C} \mid |z| < 1\}$ in the complex plane, ∂D its boundary, the unit circle, and λ the Lebesgue probability on the Borel σ-algebra of ∂D. A classical result of Fatou states that a bounded holomorphic function $f\colon D \to \mathbf{C}$ possesses a radial limit $f^*(w) = \lim_{r\to 1} f(rw)$ at λ-almost every $w \in \partial D$. An *inner function* is a holomorphic function $f\colon D \hookleftarrow$ such that $f^*(w) \in \partial D$ for a.e. $w \in \partial D$. The map $f^*\colon \partial D \hookleftarrow$ thus induced is called the *boundary map* of f.

Our subject here is the dynamics of inner functions; more specifically, the ergodic theory of the boundary map of inner functions. Actually, the dynamics of the inner function itself is quite simple and fully explained by the following result due to Denjoy [**De**] and Wolff [**W**] (to be proved in Section 2): *Given a holomorphic map $f\colon D \hookleftarrow$ that is not a Möbius transformation, there exists a point $p \in \overline{D}$ (called the Denjoy-Wolff point of f) such that $\lim_{n\to\infty} f^n(z) = p$ uniformly on compact sets.* Obviously this implies that if a non-Möbius inner function $f\colon D \hookleftarrow$ has a fixed point, then it is unique and is the Denjoy-Wolff point of f, whereas if f has no fixed points, its Denjoy-Wolff point belongs to the boundary ∂D.

On the other hand, the boundary map $f^*\colon \partial D \hookleftarrow$ of an inner function $f\colon D \hookleftarrow$ very frequently exhibits interesting non-trivial forms of recurrence and ergodic phenomena. Since such boundary maps can easily be highly discontinuous, their ergodic theory falls, in general, far beyond the range of the theory of smooth maps of the circle. But, due to its extremely special origin, boundary maps of inner functions can be studied through the tools of classical complex analysis and a satisfactory ergodic theory can be obtained of them. Our first aim is to present an exposition of this theory, through proofs often different from those in the original papers, and afterwards to relate it to rational functions. Given a rational map $f: \overline{\mathbf{C}} \hookleftarrow$, where $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere, an important aspect of its dynamics are the fixed components U of the complement of the Julia set, i.e., such that $f(U) = U$. By virtue of the uniformization theorem, $f|U: U \hookleftarrow$ lifts to an inner function whose properties yield relevant information on the ergodic theory (in the harmonic class) of $f|\partial U$.

In this introduction we present and comment on the results whose proof is the objective of the following sections. We begin with a brief review of the fundamental results of the classical theory of inner functions.

The simplest examples of inner functions are the finite Blaschke products

$$f(z) = \alpha \prod_{n=0}^{N} \frac{z - a_n}{1 - \overline{a}_n z} \quad ,$$

with $\alpha \in \partial D$ and $a_n \in D$ for $0 \leq n \leq N$. More generally, if $\{a_n\}$ is a sequence in D such that $\sum_{n \geq 0} (1 - |a_n|) < \infty$, it is easy to see that the (infinite) Blaschke product

$$B(z) = z^m \prod_{a_n \neq 0} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z} \tag{1}$$

converges (m is the number of a_n equal to zero). Moreover, $B: D \hookleftarrow$ is holomorphic, $B \not\equiv 0$ and (not trivially) B is an inner function. (For a proof, see [**Du**; Thm. 2.4] or [**G**; Thm. II.2.2].) Obviously not every inner function is a Blaschke product: simply take a Möbius map T of D onto the half plane $\{z \in \mathbf{C} \mid \operatorname{Re} z < 0\}$ and set $f(z) = \exp T(z)$. Then f is an inner function without zeros, hence not a Blaschke product. This example belongs to the class of holomorphic functions defined by

$$S_\mu(z) = \exp\left(-\int_{\partial D} \frac{w+z}{w-z} d\mu(w)\right), \tag{2}$$

where μ is a finite measure on ∂D. For the example above, take the unit mass μ concentrated at $w = 1$ and $f(z) = S_\mu(z) = \exp\left(\frac{z+1}{z-1}\right)$; observe that μ is singular with respect to λ, that is, there exists a set of full Lebesgue measure and zero μ measure.

Blaschke products and these functions S_μ with singular μ are the two conspicuous examples into which any inner function can be factored: the general analytical expression of inner functions is given by the following characterization theorem ([**Du**; Thm. 2.8], [**G**; Thm. II.5.5]): *If $f: D \hookleftarrow$ is holomorphic, then its sequence of zeros $\{a_n\}$, which may be empty, satisfies $\sum_{n \geq 0}(1 - |a_n|) < \infty$ and f can be written as*

$$f(z) = \alpha B(z) S_\mu(z), \tag{3}$$

where $|\alpha| \leq 1$, $B(z)$ *is the Blaschke product* (1) *and* $S_\mu(z)$ *is given by* (2), *with* μ *a finite measure on* ∂D. *Conversely, given a sequence* $\{a_n\}$ *in* D *satisfying* $\sum_{n\geq 0}(1-|a_n|) < \infty$, $|\alpha| \leq 1$ *and a finite measure* μ *on* ∂D, (3) *defines a holomorphic function* $f: D \hookleftarrow$. *This map* (3) *is an inner function if and only if* $|\alpha| = 1$ *and* μ *is singular with respect to Lebesgue measure.*

But *every inner function is conjugate to a Blaschke product.* More specifically, let $T_a: D \hookleftarrow$ denote the Möbius map $T_a(z) = (z-a)/(1-\bar{a}z)$, with $a \in D$; then, if f is an inner function, the composition $T_a \circ f$ is a Blaschke product for all $a \in D$, except possibly for a set of zero logarithmic capacity (Frostman; [**G**; Thm. II.6.4]). We shall not discuss the important concept of logarithmic capacity here, but we recall that sets of logarithmic capacity zero are extremely thin: they cannot contain connected sets with more than one point and their Hausdorff dimension is zero. Since $B \circ T_a$ is still a Blaschke product whenever B is one, it follows that if f is an inner function, then $T_a \circ f \circ T_a^{-1}$ is a Blaschke product except for a set of zero logarithmic capacity of values of $a \in D$.

The boundary map of an inner function $f: D \hookleftarrow$ can be highly discontinuous. In fact, the following radical dichotomy holds: *Given* $w \in D$, *either* f *has a holomorphic extension to* $D \cup V$ *for some open neighborhood* V *of* w *in* $\mathbf{C}$ *or else* f^* *maps every neighborhood* W *of* w *in* ∂D *onto almost all of* ∂D *(in the sense of Lebesgue measure)* ([**He**]). In the second case, we say that w is a *singular point* of f. Going back to the example $f(z) = \exp((z+1)/(z-1))$, at its unique singular point $w = 1$ we have $f^*(W) = \partial D$ for every neighborhood W of $w = 1$ in ∂D, a fact that can be read off the expression $f^*(e^{i\theta}) = \exp(-i\cot\frac{\theta}{2})$, $0 < \theta < 2\pi$, of the boundary map. In general, the singular points of an inner function given by (3) above, are the points in the closure of the support of the measure μ and the accumulation points of the zeros of f ([**G**; Thms. II.6.1,2]). Therefore it must be regarded as normal that every point of ∂D is a singular point of an inner function. Much more pathological behaviors may arise in special cases; for instance, there exist inner functions f such that f^* maps sets of zero measure onto sets of positive Lebesgue measure ([**He**]). These extremes of pathology can be avoided under supplementary hypotheses. We say that a holomorphic $F: D \to \mathbf{C}$ is a *Nevanlinna function* if

$$\sup_{0 \le r < 1} \int_{\partial D} \log^+ |F(rw)| \, d\lambda(w) < \infty;$$

if F is a Nevanlinna function, the radial limit $F^*(w) = \lim_{r \to 1} F(rw)$ exists for λ-a.e. $w \in \partial D$ ([**Du**; Thm. 2.2]). When $f: D \hookleftarrow$ is an inner function and $f': D \to \mathbf{C}$ is Nevanlinna, f^* maps zero measure sets onto zero measure sets and ∂D can be covered, except for a set of measure zero, by a countable family $A_1, A_2, \ldots$ of disjoint Borel sets such that, for each n, $f^*|A_n$ is injective and maps Borel sets onto Borel sets ([**He**]). Moreover, because f' is Nevanlinna, $(f')^*$ exists and is a sort of derivative of f, in the following very weak sense: There exist Borel sets $S \subseteq \partial D$, with complement S^c in ∂D having arbitrarily small Lebesgue measure, such that

$$\lim_{z \to w} \frac{f^*(z) - f^*(w)}{z - w} = (f')^*(w)$$

for all $w \in S$, if $z \to w$ in the set S.

For the study of the ergodic properties of the boundary map, a key concept is that of harmonic measure. Given $p \in D$, the *harmonic measure* λ_p associated to p is the probability on the Borel σ-algebra of ∂D defined by

$$\lambda_p(A) = \int_A \frac{1 - |p|^2}{|w - p|^2} d\lambda(w).$$

Clearly $\lambda_0 = \lambda$ is the Lebesgue probability. Moreover, if $\psi: \partial D \to \mathbf{R}$ is a continuous function and $\hat{\psi}: D \to \mathbf{R}$ denotes its harmonic extension (i.e., $\hat{\psi}$ is harmonic and for each $w \in \partial D$, $\lim_{z \to w} \hat{\psi}(z) = \psi(w)$), then Poisson's formula implies

$$\int \psi d\lambda_p = \hat{\psi}(p).$$

It is not difficult to prove, using this property, that λ_p is invariant under the boundary map of inner functions having $p \in D$ as fixed point (see Corollary 1.5 of Section 1 below). More than this, λ_p has good ergodic properties, as the next results will show. Before stating them, let us recall some definitions of abstract ergodic theory ([**Ho**]). Let $(X, \mathcal{A}, \mu)$ be a measure space and $T: X \hookleftarrow$ a measurable map, i.e., $A \in \mathcal{A} \Rightarrow T^{-1}(A) \in \mathcal{A}$. We say that T is *ergodic* if $A \in \mathcal{A}$ and $T^{-1}(A) = A$ imply $\mu(A) = 0$ or $\mu(A^c) = 0$, and that T is *exact* if every $A \in \bigcap_{n \geq 0} T^{-n}(\mathcal{A})$ satisfies $\mu(A) = 0$ or $\mu(A^c) = 0$; alternatively, we sometimes say that μ is ergodic (or exact) instead of T. Clearly exactness implies ergodicity, because $A \in \mathcal{A}$ and $T^{-1}(A) = A$ imply $T^{-n}(A) = A$ for all $n \geq 0$ and then $A \in \bigcap_{n \geq 0} T^{-n}(\mathcal{A})$. The map T is said to be *recurrent* if for every $A \in \mathcal{A}$ and μ-a.e. $x \in A$, there exist infinitely many values of $n \geq 0$ such that $T^n(x) \in A$. Finally, if $\mu(T^{-1}(A)) = \mu(A)$

for each $A \in \mathcal{A}$, we say that μ is *invariant* under T or that T preserves μ. When μ is finite, i.e., $\mu(X) < \infty$, T is recurent whenever T preserves μ (Poincaré), but if $\mu(X) = \infty$, not even from T being exact and measure preserving follows that T is recurrent. We shall find examples of this antagonistic coexistence in the boundary behavior of certain inner functions.

The harmonic measure is unique in the following sense.

THEOREM A. *If $f: D \hookleftarrow$ is an inner function, then $f^*: \partial D \hookleftarrow$ has an invariant probability μ on the Borel σ-algebra of ∂D which is absolutely continuous with respect to λ if and only if f has a fixed point $p \in D$; in this case, $\mu = \lambda_p$ and f^* is exact.*

This theorem follows from a stronger result to be stated below (Theorem G).

When the inner function $f: D \hookleftarrow$ has a fixed point $p \in D$ a very complete picture of the ergodic theory of $f^*: \partial D \hookleftarrow$ with respect to λ_p is given by the following results of Craizer.

THEOREM B. *If $f: D \hookleftarrow$ is an inner function with a fixed point $p \in D$, then*

*a) ([**Cr2**]) with respect to λ_p, $f^*: \partial D \hookleftarrow$ is Bernoulli, i.e., its natural extension is isomorphic, in the measure theoretical sense, to a Bernoulli shift;*

*b) ([**Cr1**]) the entropy $h_{\lambda_p}(f^*)$ of λ_p is finite if and only if f' is Nevanlinna, in which case,*

$$h_{\lambda_p}(f^*) = \int_{\partial D} \log |(f')^*| \, d\lambda_p.$$

We shall not give here the proof of these important (and difficult) results. Our presentation will be mostly oriented toward the other alternative, when the Denjoy-Wolff point belongs to ∂D.

Given $p \in \partial D$, we define a measure μ_p on the Borel σ-algebra of ∂D by

$$\mu_p(A) = \int_A \frac{1}{|w-p|^2} d\lambda(w).$$

Observe that μ_p is precisely the image of the Lebesgue measure on $\mathbf{R}$ (multiplied by a scalar) under any Möbius map transforming the upper half plane onto D and mapping ∞ to p. Clearly $\mu_p(\partial D) = \infty$.

The interesting property of this measure is that, if p is the Denjoy-Wolff point of an inner function f, then f^* transforms μ_p via multiplication by a fixed constant depending upon the behavior of f near p.

THEOREM C. *Let $f: D \hookleftarrow$ be an inner function with Denjoy-Wolff point $p \in \partial D$. There exists $0 < c \leq 1$ such that*

a) $\mu_p(f^{*-1}(A)) = c\mu_p(A)$ for every Borel set $A \subseteq \partial D$;

b) $\lim_{n\to\infty} (1 - |f^n(z)|)^{1/n} = c$ for all $z \in D$;

c) $\lim \dfrac{1-|f(z)|}{1-|z|} = c$, when $z \to p$ nontangentially.

For a proof of this theorem, see Section 4.

It is sometimes convenient to consider inner functions of the upper half plane $\mathbf{R}^2_+ = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ instead of inner functions of the disc. An inner function of $\mathbf{R}^2_+$ is a holomorphic function $f: \mathbf{R}^2_+ \hookleftarrow$ such that the boundary map $f^*(x) = \lim_{t\to 0} f(x+ti)$, which is well defined for a.e. $x \in \mathbf{R}$, satisfies $f^*(x) \in \mathbf{R}$ for a.e. $x \in \mathbf{R}$ with respect to Lebesgue measure λ on $\mathbf{R}$. Obviously the class of inner functions of D coincides, up to conjugacy by a conformal representation of D onto $\mathbf{R}^2_+$, with the class of inner functions of $\mathbf{R}^2_+$. Thus, formally, no advantage can be expected from replacing the disc by the upper half plane. However, there are interesting examples of inner functions of the upper half plane that admit easy analytical expressions. For instance, the map $f(z) = z + \tan z$ (whose ergodic properties will presently be discussed) or the map $f(z) = z - z^{-1}$, that goes back to Boole who, in 1857, proved that its restriction to $\mathbf{R}$ preserves the Lebesgue measure ([**Bo**], [**Ad-W**]). The reader may check this by brute force, i.e., verifying that $1 = |f'(x_1)|^{-1} + |f'(x_2)|^{-1}$, where x_1 and x_2 are the two roots of $f(x) = c$, for all $c \in \mathbf{R}$. More subtlety is required to answer the problem, posed by Pólya in 1931, of characterizing all the rational maps f such that $f(\mathbf{R}) \subseteq \mathbf{R}$ and $f|\mathbf{R}$ preserves the Lebesgue measure. The answer (Pólya [**Pa**], Szegö [**Sz**]; see also Section 5) is that f has these properties if and only if f is of the form

$$f(z) = \varepsilon \left(z + \beta - \sum_{i=1}^{n} \frac{\gamma_i}{z - a_i} \right),$$

where $\varepsilon = 1$ or $= -1$, $\beta \in \mathbf{R}$ and $a_i \in \mathbf{R}$, $\gamma_i > 0$ for all $1 \leq i \leq n$. When $\varepsilon = 1$, this is an inner function of the upper half plane. Observe that $f'(x) > 1$ for all $x \in \mathbf{R}$ and, if to study f nearby ∞ we consider $g(z) = -(f(-z^{-1}))^{-1}$, the Taylor series of g at $z = 0$ is

$$g(z) = z + \beta z^2 - \Big(\sum_{i=1}^{n} \gamma_i\Big) z^3 + \dots .$$

Using these properties, the standard theory of rational functions shows that, for $\beta = 0$, $\mathbf{R} \cup \{\infty\}$ is the Julia set of f and $\lim_{n\to\infty} f^n(z) = \infty$ for all $z \notin \mathbf{R}$, whereas for $\beta \neq 0$, the Julia set of f is a Cantor set Λ of $\mathbf{R} \cup \{\infty\}$ (containing the parabolic fixed point ∞) and $\lim_{n\to\infty} f^n(z) = \infty$ for all $z \notin \Lambda$. The ergodic properties of $f|\mathbf{R}$ with

$\beta = 0$ will be discussed below.

The general formula for inner functions of $\mathbf{R}^2_+$ is the following: *Every holomorphic function* $f\colon \mathbf{R}^2_+ \hookleftarrow$ *can be written as*

$$f(z) = \alpha z + \beta + \int_{\mathbf{R}} \frac{1+zw}{w-z} d\mu(w), \tag{4}$$

where $\alpha \geq 0$, $\beta \in \mathbf{R}$ *and* μ *is a finite measure on the Borel* σ*-algebra of* $\mathbf{R}$. *Moreover,* f *is an inner function if and only if* μ *is singular with respect to the Lebesgue measure* λ *on* $\mathbf{R}$ ([**T**]). Theorem D below is an upper half plane version of Theorem C, with $\alpha = c^{-1}$ (to be proved in Section 5).

THEOREM D. *If the inner function* f *of the upper half plane is given by* (4) *with* $\alpha > 0$, *then*

$$\lambda(f^{*-1}(A)) = \frac{1}{\alpha}\lambda(A)$$

for every Borel set $A \subseteq \mathbf{R}$. *Moreover,* ∞ *is the Denjoy-Wolff point of* f *if and only if* $\alpha \geq 1$.

Let us now address the question of the *recurrence* of the boundary map of an inner function. Recurrence is completely characterized by the following results of Aaronson [**Aa2**] (to be proved in Section 4).

THEOREM E. *If* $f\colon D \hookleftarrow$ *is an inner function with Denjoy-Wolff point* $p \in \partial D$, *then either* f^* *is recurrent or* $\lim_{n\to\infty} f^{*n}(w) = p$ *for a.e.* $w \in \partial D$.

THEOREM F. *If* $f\colon D \hookleftarrow$ *is an inner function with Denjoy-Wolff point* $p \in \partial D$, *then the following conditions are equivalent:*

a) f^* *is recurrent;*

b) $\sum_{n\geq 0}(1-|f^n(z)|)=\infty$ *for all* $z \in D$*;*

c) There exists $z \in D$ *with* $\sum_{n\geq 0}(1-|f^n(z)|)=\infty$.

However, with or without recurrence, if the Denjoy-Wolff point p is in ∂D, it acts as an "attractor in the mean"; more specifically,

$$\lim_{n\to\infty}\frac{1}{n}\#\{0\leq j<n|f^{*j}(w)\in W\}=1 \tag{5}$$

for every neighborhood W of p in ∂D and a.e. $w \in \partial D$. This is proved in Section 2, but also follows from a stronger result (Theorem H) to be stated below.

As an example of application, let us prove that, restricted to $\mathbf{R}$, the inner function $f(z) = z + \tan z$ of the upper half plane preserves the Lebesgue measure and is recurrent. The function f preserves the imaginary axis because

$$f(it)=i\left(t-\frac{e^{-t}-e^{t}}{e^{-t}+e^{t}}\right);$$

if $g(z)=-(f(-z^{-1}))^{-1}$, the Taylor series of $g(it)$, for small t, is

$$g(it)=i(t-t^2+\dots).$$

Hence

$$\lim_{t\to 0}\frac{|g(it)|}{t}=1,$$

thus proving that g preserves μ_0 and then that f preserves λ. Moreover, the above expression for g implies that $g^n(it)\cong\frac{1}{n}$; it follows that

$$\sum_{n\geq 0}|g^n(it)|=\infty$$

and therefore g and also f are recurrent.

Assume now that f is a Pólya-Szegö map with $\varepsilon = 1$ and $\beta = 0$ and let us see that $f|\mathbf{R}$ (which we already know is measure preserving) is recurrent. Since the Taylor series of $g(z) = -(f(-z^{-1}))^{-1}$ at $z = 0$ is

$$g(z) = z - \Big(\sum_{i=1}^{n} \gamma_i\Big) z^3 + \dots,$$

standard methods concerning parabolic points imply that there exists a sector $S = \{z \in \mathbf{C} |\, |z| < \delta,\ |\mathrm{Arg}\,(-iz)| < \varepsilon\}$ such that $g(S) \subseteq S$ and, for all $z \in S$, $g^n(z) \to 0$ with $n \to \infty$ and more: $g^n(z) \cong \frac{1}{\sqrt{n}}$; thus $\sum_{n \geq 0} |g^n(z)| = \infty$.

Now let us consider the question of the *ergodicity* and *exactness* of the boundary map of an inner function. The next result was proved (with a somehow different statement) by Aaronson in [**Aa3**], relying on his previous result [**Aa2**] and on a theorem of Pommerenke [**Pe**]. For its statement, let $d_P(\cdot,\cdot)$ denote the Poincaré metric on D (or on $\mathbf{R}^2_+$).

THEOREM G. *Let $f\colon D \hookleftarrow$ be an inner function. The following conditions are equivalent:*

a) f^* is ergodic;

b) f^* is exact;

c) $\lim_{n\to\infty} d_P(f^n(x), f^n(y)) = 0$ for all $x, y \in D$;

d) There exist $a, b \in D$ such that $f^n(a) \neq f^n(b)$ for $n \geq 0$ and $\lim_{n\to\infty} d_P(f^n(a), f^n(b)) = 0$.

COROLLARY 1. *Recurrence implies ergodicity.*

COROLLARY 2. *If $f: D \hookleftarrow$ is an inner function with Denjoy-Wolff point $p \in \partial D$ and f^* ergodic, then f^* preserves μ_p.*

The proof of Theorem G is in Section 3; the proof of Corollaries 1 and 2 is straightforward (by Theorems F and C, respectively) and can be found in Section 4.

Let us show here how Theorem A follows from Theorems G and E. Suppose that the boundary map $f^*: \partial D \hookleftarrow$ of an inner function $f: D \hookleftarrow$ admits a λ-absolutely continuous and invariant probability μ. If f has a fixed point $p \in D$, then λ_p is f^*-invariant, f^* is recurrent and, by Corollary 1, λ_p is ergodic. Hence $\mu << \lambda_p$ implies $\mu = \lambda_p$. Let us assume that f has no fixed point (and seek for a contradiction). Then the Denjoy-Wolff point p belongs to ∂D and, moreover, μ-invariance implies that μ-a.e. $w \in \partial D$ is recurrent for f^*. Since $\mu << \lambda$, this implies that the set of recurrent points has positive Lebesgue measure. Theorem E now guarantees that f^* is recurrent and then μ_p is f^*-invariant and ergodic by the two corollaries above. Since $\mu << \mu_p$, it follows that $\mu = \mu_p$, contradicting that μ is a probability and showing that f has a fixed point in D.

The main virtue of ergodicity of finite invariant measures, however, thoroughly fails with μ_p, as the following theorem (to be proved in Section 4) shows: there is no equality between time and space averages.

THEOREM H. *Let $f: D \hookleftarrow$ be an inner function with Denjoy-Wolff point $p \in \partial D$ and such that μ_p is f^*-invariant. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^{*j}(w)) = 0$$

for every μ_p-integrable function $\psi: \partial D \to \mathbf{R}$ and a.e. $w \in \partial D$.

From this theorem, property (5) follows as a corollary. In fact, (5) is obvious when f^* is not recurrent, since then $\lim_{n\to\infty} f^{*n}(w) = p$ for a.e. $w \in \partial D$ by Theorem E. If f^* is recurrent, then the two corollaries above imply that μ_p is f^*-invariant; applying Theorem H to the characteristic function ψ of the complement of a neighborhood W of p in ∂D, we get (5).

The weak nature of the ergodicity of f^* when its Denjoy-Wolff point is in ∂D, is further enhanced by an example of Aaronson [**Aa3**] of an inner function $f: D \hookleftarrow$ with Denjoy-Wolff point $p \in \partial D$ and such that *f^* is ergodic but nevertheless $\lim_{n\to\infty} f^{*n}(w) = p$ for a.e. $w \in \partial D$*. Let us roughly describe this example. Let $\{a_n\}$ be an increasing sequence of positive real numbers with $\sum_{n\geq 0} a_n^{-2} < \infty$ and consider the function

$$f(z) = z - \sum_{n\geq 0} \frac{2z}{z^2 - a_n^2} = z + \sum_{n\geq 0} \left(-\frac{1}{z - a_n} - \frac{1}{z + a_n} \right).$$

The series converges uniformly on compact sets of $\mathbf{C} - \{a_n\}$, defines a meromorphic function and obviously maps $\mathbf{R} - \{a_n\}$ onto $\mathbf{R}$ and $\mathbf{R}^2_+$ into $\mathbf{R}^2_+$ (because every map $z \mapsto -(z-\alpha)^{-1}$, $\alpha \in \mathbf{R}$, maps $\mathbf{R}^2_+$ into $\mathbf{R}^2_+$). Then f is an inner function. Moreover, $f'(t) > 1$ for all $t \in \mathbf{R} - \{a_n\}$ and f maps every interval $(-a_{n+1}, -a_n), (-a_0, a_0)$ and (a_n, a_{n+1}) onto $\mathbf{R}$. Hence, by standard techniques, it can be proved that f is *transitive*, i.e., for a residual subset of $\mathbf{R}$, its forward orbit is dense in $\mathbf{R}$. To show that ∞ is the Denjoy-Wolff point of f, consider $g(z) = -(f(-z^{-1}))^{-1}$. Then

$$g(it) = i\frac{t}{1 + \sum_{n\geq 0} \frac{2t^2}{1 + a_n^2 t^2}}$$

shows that g leaves the imaginary axis invariant and $\operatorname{Im} g(it) < t$, implying that $\lim_{n\to\infty} g^n(it) = 0$ and proving that ∞ is the Denjoy-Wolff point of f. To study the ergodic properties of f, we have to specify the sequence $\{a_n\}$. Set $a_n = n^{\frac{1}{2}\gamma}$ with $\gamma > 1$. Then it is not hard to obtain

$$\sum_{n\geq 0} \frac{2t^2}{1 + n^\gamma t^2} \cong t^{2-\frac{2}{\gamma}}$$

for small t; it follows that $\beta = 3 - \frac{2}{\gamma}$ satisfies

$$g(it) \cong i(t - t^\beta).$$

This implies $\lim_{n\to\infty} d_P(g^n(it), g^{n+1}(it)) = 0$ (and then that f is ergodic), since $d_P(g^n(it), g^{n+1}(it))$ is proportional to

$$\frac{|g^{n+1}(it) - g^n(it)|}{|g^n(it)|} \cong \frac{|g^n(it)|^\beta}{|g^n(it)|} = |g^n(it)|^{\beta-1}.$$

But the asymptotic expression for g implies that

$$g^n(it) \cong \left(\frac{1}{n}\right)^{\frac{1}{\beta-1}}$$

and therefore

$$\sum_{n\geq 0} |g^n(it)| \cong \sum_n \left(\frac{1}{n}\right)^{\frac{1}{\beta-1}}$$

converges if $(\beta-1)^{-1} > 1$, i.e., if $\gamma < 2$ and diverges if $\gamma \geq 2$. Hence f is recurrent for $\gamma \geq 2$ but, when $\gamma < 2$, $\lim_{n\to\infty} f^n(w) = \infty$ for a.e. $w \in \mathbf{R}$.

Now we shall present some applications of these results to rational maps of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. First we recall ([T]) the concepts of Dirichlet regularity and harmonic measures. An open connected set $U \subseteq \overline{\mathbf{C}}$ is said to be *regular* (in the sense of Dirichlet) if every continuous function $\varphi\colon \partial U \to \mathbf{R}$ has an harmonic extension $\hat{\varphi}\colon \overline{U} \to \mathbf{R}$, i.e., $\hat{\varphi}$ is continuous, harmonic on U and $\hat{\varphi}|\partial U = \varphi$. If $U \subseteq \overline{\mathbf{C}}$ is an open regular set and $p \in U$, we define the *harmonic measure* λ_p on the Borel σ-algebra of ∂U as the unique one such that

$$\int \varphi d\lambda_p = \hat{\varphi}(p)$$

for every continuous $\varphi\colon \partial U \to \mathbf{R}$ with harmonic extension $\hat{\varphi}$. Harmonic measures may be defined even when U is not regular via more subtle methods but, for our purposes, this definition will suffice. It is easy to see that λ_p is positive on open sets. Moreover, the harmonic measures λ_p and λ_q are equivalent if $p, q \in U$. Even more, there exists $k > 0$ such that

$$k^{-1} \leq \frac{d\lambda_p}{d\lambda_q}(w) \leq k$$

for every $w \in \partial U$. To check this, it is enough to show the existence of $k > 0$ such that

$$k^{-1} \leq \frac{V(p)}{V(q)} \leq k$$

for every strictly positive harmonic function $V\colon U \to \mathbf{R}$. But the existence of such a constant is an immediate corollary of Harnack's inequality. The *harmonic class* of ∂U is the class of measures on the Borel σ-algebra of ∂U which are equivalent to a harmonic measure λ_p for some $p \in U$.

Let $\psi: D \to U$ be a uniformization mapping, i.e., a holomorphic covering map of U; it always exists if U^c contains at least three points, as in our case, since U is regular. The regularity of U also implies that the logarithmic capacity of U^c is positive and then the radial limit $\psi^*(w) = \lim_{r \to 1} \psi(rw)$ exists for a.e. $w \in \partial D$ and, since ψ is a covering map, $\psi^*(w) \in \partial D$ for a.e. $w \in \partial D$ (in fact, whenever $\psi^*(w)$ exists). Moreover ψ^* transforms a harmonic measure λ_p on ∂U in the harmonic measure λ_q on ∂D, where $q \in \psi^{-1}(\{p\})$. More precisely,

$$\lambda_q((\psi^*)^{-1}(A)) = \lambda_p(A) \tag{6}$$

for every Borel set $A \subseteq \partial U$. To see this, take a continuous function $\varphi: \partial U \to \mathbf{R}$ and let $\hat{\varphi}: \overline{U} \to \mathbf{R}$ be its harmonic extension. Then $\lim_{r \to 1} (\hat{\varphi} \circ \psi)(rw) = \varphi(\psi^*(w))$ for a.e. $w \in \partial D$ and, by a theorem of Fatou,

$$\int (\varphi \circ \psi^*) d\lambda_q = (\hat{\varphi} \circ \psi)(q) = \hat{\varphi}(p) = \int \varphi d\lambda_p \ .$$

From this, (6) follows applying standard approximation methods.

Now let us consider a rational map $f: \overline{\mathbf{C}} \hookleftarrow$ and let $J(f)$ be its Julia set. It is easy to see that f maps connected components of $J(f)^c$ onto connected components of $J(f)^c$. A fundamental and deep result of Sullivan [**Su**] states that if U is a connected component of $J(f)^c$, then, for some $N > 0$, the connected component $f^N(U)$ is periodic, that is, for some $m > 0$, $f^m(f^N(U)) = f^N(U)$;

more than this, that there are finitely many periodic connected components of $J(f)^c$. Sullivan also proved that if U is a fixed connected component of $J(f)^c$, i.e., $f(U) = U$, then $f|U$ belongs to one of the following four classes:

a) U is a *Siegel disc*: U is a disc and $f|U\colon U \hookleftarrow$ is conjugate, via a conformal representation, to an irrational rotation of D, i.e., $z \mapsto \alpha z$ with $|\alpha| = 1$ and $\alpha^n \neq 1$ for all n.

b) U is a *Herman ring*: $f|U\colon U \hookleftarrow$ is conjugate, via conformal representation, to an irrational rotation of an annulus $\{z \in \mathbf{C}|\ r < |z| < R\}$.

c) U is the *immediate basin of a sink*: there exists a sink $p \in U$, i.e., $f(p) = p$, $|f'(p)| < 1$ and $\lim_{n\to\infty} f^n(z) = p$ for all $z \in U$.

d) U is a *parabolic basin*: there exists a parabolic fixed point $p \in \partial U$ of f, i.e., $f'(p)$ is a root of unity and $\lim_{n\to\infty} f^n(z) = p$ for all $z \in U$.

Inner functions and harmonic measures have interesting applications to the study of ergodic properties of $f|\partial U$ when U is in the last two classes. In the first two cases, they yield, so far, no relevant contributions.

Suppose first that U is the immediate basin of a sink p. We shall show that *λ_p is f-invariant and exact*. To prove the invariance of λ_p, take any continuous function $\varphi\colon \partial U \to \mathbf{R}$ and let $\hat{\varphi}\colon \overline{U} \to \mathbf{R}$ be its harmonic extension. Recall that every component of $J(f)^c$ is regular; even more, its linear density of logarithmic capacity is positive (Mañé, Rocha [**M-R**]). Then $\hat{\varphi}\circ f$ is the harmonic extension of $\varphi \circ f$ and

$$\int (\varphi \circ f) d\lambda_p = (\hat{\varphi} \circ f)(p) = \hat{\varphi}(f(p)) = \hat{\varphi}(p) = \int \varphi d\lambda_p \ ,$$

proving the f-invariance of λ_p. To prove its exactness, consider a uniformization $\psi: D \to U$ of U and a lifting $\hat{f}: D \hookleftarrow$ of f by ψ: $\psi \circ \hat{f} = f \circ \psi$. Then $\hat{f}$ is an inner function with a fixed point $q \in \psi^{-1}(\{p\})$. By Theorem A, λ_q is an exact probability for $\hat{f}^*: \partial D \hookleftarrow$. But since $\psi^* \circ \hat{f}^* = f \circ \psi^*$, it follows from (6) that λ_p on ∂U is also exact.

Many other interesting properties of λ_p are known; for instance, its relation with the entropy maximizing measure of $f|\partial U$. It is known (Freire, Lopes, Mañé [**F-L-M**], Ljubich [**Lj**], Lopes [**Lo**] and Mañé [**M**]) that the topological entropy of $f|\partial U$ is $\log m$, where m is the degree of $f|U: U \hookleftarrow$ and that there exists a unique invariant probability μ^+ on ∂U such that $h_{\mu^+}(f|\partial U) = \log m$ and even more: μ^+ is exact. It is also known that the invariant probabilities λ_p and μ^+ are singular unless $f^{-1}(\{p\}) \cap U = \{p\}$, in which case they coincide (Mañé, Rocha [**M-R**]). In [**Pi**], Przytycki presents a very detailed analysis of the harmonic class on the boundary of an immediate basin of a sink.

For parabolic basins, the ergodic theory of the harmonic class is quite different. The following result will be proved in Section 6.

THEOREM I. *Let U be a fixed parabolic basin of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$ with parabolic fixed point $p \in \partial U$. Let $\psi: D \to U$ be a uniformization and $\hat{f}: D \hookleftarrow$ be a lifting (i.e., $\psi \circ \hat{f} = f \circ \psi$). Then, for all $z \in D$ and all $\alpha > \frac{1}{2}$, the inequality*

$$|1 - \hat{f}^n(z)| \geq \frac{1}{n^\alpha}$$

holds for sufficiently large n.

This implies that the boundary map $\hat{f}^*$ is recurrent and that $\hat{f}$ has no fixed points (otherwise f would have a fixed point in U).

Using the boundary map $\psi^*: \partial D \to \partial U$ and (6), we easily obtain the following corollary.

COROLLARY. *If U is a fixed parabolic basin of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$, then $f|\partial U: \partial U \hookleftarrow$ is exact and recurrent with respect to the harmonic class of ∂U.*

The following result (to be proved in Section 6) further characterizes the ergodic theory of parabolic basins.

THEOREM J. *Let U be a fixed parabolic basin of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$. If $q \in U$, $\varphi \in L^\infty(\lambda_q)$ is positive and $\not\equiv 0$, then*

$$\lim_{n\to\infty} \frac{\int \Big(\sum_{i=0}^{n-1} (\varphi \circ f^i)\psi\Big) d\lambda_q}{\sum_{i=0}^{n-1} \int (\varphi \circ f^i) d\lambda_q} = \int \psi d\lambda_q$$

for every $\psi \in L^1(\lambda_q)$.

In other words, with $n \to \infty$,

$$\sum_{i=0}^{n-1} \varphi \circ f^i \Big/ \sum_{i=0}^{n-1} \int (\varphi \circ f^i) d\lambda_q \ \to \ 1$$

weakly in the dual of $L^1(\lambda_q)$. To this property, a.e. convergence ***cannot be added*** because of the following result of Aaronson [**Aa1**]: If $T: X \hookleftarrow$ is an ergodic and recurrent measure preserving map of a σ-finite non-atomic infinite measure space $(X, \mathcal{A}, \mu)$, then for every positive $\varphi \in L^1(X, \mathcal{A}, \mu)$ and every sequence $\{a_n\}$ of positive real numbers, one (or both) of the following properties holds for a.e. $x \in X$:

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{j=0}^{n-1} (\varphi \circ T^j)(x) = \infty,$$

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{j=0}^{n-1} (\varphi \circ T^j)(x) = 0.$$

Then, considering the map $f(z) = z - z^{-1}$, for which $U = \mathbf{R}^2_+$ is a parabolic basin, and applying this result to $f|\partial U$, which preserves Lebesgue measure and is ergodic and recurrent, it is clear that a.e. convergence is impossible in Theorem J.

Theorem J has an interesting consequence for Pólya-Szegö maps

$$f(z) = z - \sum_{i=1}^{k} \frac{\gamma_i}{z - a_i} \ ,$$

with $a_i \in \mathbf{R}$ and $\gamma_i > 0$ for all $1 \le i \le k$, namely, that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \lambda(f^{-j}(A) \cap B) = \frac{2}{\pi\sqrt{2\gamma}} \lambda(A)\lambda(B), \tag{7}$$

where $\gamma = \sum_{i=1}^{k} \gamma_i$, for all bounded Borel sets $A, B \subseteq \mathbf{R}$.

To prove this, let ψ_A, ψ_B stand for the characteristic functions of $A, B \subseteq \mathbf{R}$; then $(\psi_A \circ f^j)\psi_B$ is the characteristic function of $f^{-j}(A) \cap B$ and therefore,

$$\lambda(f^{-j}(A) \cap B) = \int (\psi_A \circ f^j)\psi_B \frac{d\lambda}{d\lambda_z} d\lambda_z$$

for $z \in \mathbf{R}^2_+$. Since B is bounded, $\psi_B \frac{d\lambda}{d\lambda_z} \in L^1(\lambda_z)$ and Theorem J implies that

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \lambda(f^{-j}(A) \cap B)}{\sum_{j=0}^{n-1} \int (\psi_A \circ f^j) d\lambda_z} = \int \psi_B \frac{d\lambda}{d\lambda_z} d\lambda_z = \lambda(B). \tag{8}$$

On the other hand, the Taylor development of $g(z) = -(f(-z^{-1}))^{-1}$ at $z = 0$ is

$$g(z) = z - \gamma z^3 + \dots .$$

Therefore every $z \in \mathbf{R}^2_+$ approaches the Denjoy-Wolff point of g (which is 0) satisfying

$$\lim_{n\to\infty} \sqrt{n}\, |g^n(z)| = \frac{1}{\sqrt{2\gamma}},$$

which for f decodes to

$$\lim_{n\to\infty} \frac{f^n(z)}{i\sqrt{n}} = \sqrt{2\gamma}.$$

From this limit we obtain, trivially

$$\lim_{j\to\infty} \frac{\text{Im } f^j(z)}{\sqrt{2\gamma}\sqrt{j}} = 1$$

and also, less obviously (but only using the expression of the derivative $d\lambda_z/d\lambda$; see Section 5 for details),

$$\lambda(A) = \lim_{j\to\infty} \pi \ \text{Im} f^j(z) \ \lambda_{f^j(z)}(A),$$

and therefore,

$$\lambda(A) = \lim_{j\to\infty} \pi\sqrt{2\gamma}\sqrt{j} \ \lambda_{f^j(z)}(A).$$

But then

$$\lambda(A) = \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \lambda_{f^j(z)}(A)}{\frac{1}{\pi\sqrt{2\gamma}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}}} = \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \lambda_{f^j(z)}(A)}{\frac{2}{\pi\sqrt{2\gamma}}\sqrt{n}},$$

which, together with (8), proves (7), since

$$\lambda_{f^j(z)}(A) = \lambda_z(f^{-j}(A)) = \int (\psi_A \circ f^j) d\lambda_z.$$

§1 Nontangential limits and harmonic extensions

In this section we gather some basic results on holomorphic and harmonic functions, defined in the open unit disc D, which will be used in the forthcoming sections and also prove that the harmonic measure associated to the fixed point of an inner function is invariant under the boundary map.

For $w \in \partial D$ and $0 < \alpha < \frac{\pi}{2}$, let

$$S_\alpha(w) = \{z \in D \,|\, |\mathrm{Arg}\left(\tfrac{w-z}{w}\right)| < \alpha\}$$

denote the inner sector with vertex w and inner angle 2α. Given any function $f: D \to X$, with $X = \mathbf{C}$ or $\mathbf{R}$, we say that it has a nontangential limit at $w \in \partial D$ if

$$\lim_{\substack{z \to w \\ z \in S_\alpha(w)}} f(z)$$

exists for every $0 < \alpha < \frac{\pi}{2}$; it is easy to see that the limit is independent of α. Its value is denoted by $f^*(w)$ and called the *nontangential limit* of f at w.

The classical theorem of Fatou states that if $f: D \to \mathbf{C}$ is holomorphic and bounded then the nontangential limit of f exists at a.e. point of ∂D, with respect to the Lebesgue probability λ of the unit circle. (For a proof, see [**T**; Thm. IV.7].) (The same is true if instead of a bounded we have a Nevanlinna function, i.e., one for which

$$\sup_{0\le r<1}\int_{\partial D}\log^+|f(rw)|\,d\lambda(w)<\infty,$$

and it also holds if $\mathbf{C}-f(D)$ has positive logarithmic capacity.) We shall later have use for the following uniqueness theorem, due to F. and M. Riesz ([**T**; Thm. IV.9]).

THEOREM 1.1. *If $f\colon D\to\mathbf{C}$ is a bounded holomorphic function and $f^*(w)=0$ for each w in a set of positive Lebesgue measure, then $f\equiv 0$.*

Let us take a quick look at the behavior of harmonic functions with respect to nontangential limits. Given a continuous function $\psi\colon\partial D\to\mathbf{R}$, its *harmonic extension* is a harmonic function $\hat\psi\colon D\to\mathbf{R}$ such that $\lim_{z\to w}\hat\psi(z)=w$ for all $w\in\partial D$. It is well known that every continuous function in D has one and only one harmonic extension. Moreover, introducing the Poisson kernel $P\colon D\times\partial D\to\mathbf{R}$, defined by

$$P(z,w)=\frac{1-|z|^2}{|w-z|^2}=\operatorname{Re}\left(\frac{w+z}{w-z}\right),$$

the unique harmonic extension of ψ is given by

$$\hat\psi(z)=\int_{\partial D}\psi(w)P(z,w)d\lambda(w)\qquad(1)$$

for each $z\in D$. More generally, we *define* the harmonic extension $\hat\psi\colon D\to\mathbf{R}$ of any $\psi\in L^1(\lambda)$ by (1); one checks easily that $\hat\psi$ is indeed harmonic.

The following results, due to Fatou and Schwarz, respectively, contain the essential properties of harmonic extensions ([**T**; Thms. IV.1,2]).

THEOREM 1.2. *Given* $\psi \in L^1(\lambda)$,

a) for a.e. $w \in \partial D$ *the nontangential limit* $\hat{\psi}^*(w)$ *exists and* $\hat{\psi}^* = \psi$ *in* $L^1(\lambda)$;

b) if ψ *is continuous at some* $w_0 \in \partial D$, *then* $\lim_{z \to w_0} \hat{\psi}(z) = \psi(w_0)$; *if* $J \subseteq \partial D$ *is an open set and* ψ *is continuous on* J, *then* $\lim_{z \to w} \hat{\psi}(z) = \psi(w)$ *uniformly in* $w \in J$.

For bounded functions, we again have a uniqueness result, also due to Fatou ([**T**; Thm. IV.6]).

THEOREM 1.3. *If* $u: D \to \mathbf{R}$ *is a bounded harmonic function, then the nontangential limit* $u^*(w)$ *exists at a.e.* $w \in \partial D$ *and* u *is the harmonic extension of* u^*, *i.e.,* $\widehat{u^*} = u$. *In particular, for* $\psi \in L^\infty(\lambda)$, $\hat{\psi}$ *is the unique bounded harmonic function defined in* D *and such that*

$$\lim_{r \to 1} \hat{\psi}(rw) = \psi(w)$$

for a.e. $w \in \partial D$.

Observe that this theorem is false for unbounded functions: simply take the harmonic function $u(z) = \mathrm{Re}((z+1)/(z-1))$, $z \in D$, for which $\lim_{z \to w} u(z) = 0$ at every $w \neq 1$ in ∂D and therefore $u^* = 0$ in $L^\infty(\lambda)$.

Given $z \in D$, we define the *harmonic measure* λ_z as the unique probability on ∂D satisfying

$$\frac{d\lambda_z}{d\lambda}(w) = P(z, w)$$

for $w \in \partial D$. Thus, for every Borel set $A \subseteq \partial D$,

$$\lambda_z(A) = \int_A P(z, w) d\lambda(w)$$

and, for every $\psi \in L^1(\lambda)$,

$$\int \psi d\lambda_z = \hat{\psi}(z),$$

since $\int \psi d\lambda_z = \int \psi \frac{d\lambda_z}{d\lambda} d\lambda = \int \psi(w) P(z,w) d\lambda(w) = \hat{\psi}(z)$.

Let us now prove that λ_z is invariant under the boundary map of an inner function having z as fixed point. In this and the next sections we shall (drop the asterisk and) simply write $f\colon \partial D \hookleftarrow$ for the boundary map of an inner function $f\colon D \hookleftarrow$, since there is hardly any risk involved. The essential property relating nontangential limits and harmonc extensions is the following commutative property.

THEOREM 1.4. *If $f\colon D \hookleftarrow$ is an inner function and $\psi \in L^1(\lambda)$, then $\psi \circ f \in L^1(\lambda)$ and*

$$\hat{\psi} \circ f = \widehat{\psi \circ f}.$$

COROLLARY 1.5. *If $f\colon D \hookleftarrow$ is an inner function and $z \in D$, then the harmonic measure satisfies:*

a) $\int (\psi \circ f) d\lambda_z = \int \psi d\lambda_{f(z)}$ for every $\psi \in L^1(\lambda)$;

b) $\lambda_z(f^{-1}(A)) = \lambda_{f(z)}(A)$ for every Borel set $A \subseteq \partial D$;

c) λ_z is f-invariant if and only if $f(z) = z$.

The corollary is an immediate consequence of the theorem: $\int (\psi \circ f) d\lambda_z = \widehat{\psi \circ f}(z) = (\hat{\psi} \circ f)(z) = \hat{\psi}(f(z)) = \int \psi d\lambda_{f(z)}$ proves a); if ψ is the characteristic function of $A \subseteq \partial D$, then $\psi \circ f$ is the characteristic function of $f^{-1}(A)$ and a) implies

$$\lambda_z(f^{-1}(A)) = \int (\psi \circ f) d\lambda_z = \int \psi d\lambda_{f(z)} = \lambda_{f(z)}(A),$$

which proves b); finally, c) follows from a) and b).

To prove the theorem we need two preliminary results.

LEMMA 1. *For* $\psi \in L^1(\lambda)$ *and* $z \in D$, $|\hat{\psi}(z)| \leq \frac{1+|z|}{1-|z|} \|\psi\|_1$.

PROOF: Given $\psi \in L^1(\lambda)$ and $z \in D$,

$$|\hat{\psi}(z)| = \left| \int \psi(w) P(z,w) d\lambda(w) \right| \leq \int |\psi(w)| \, P(z,w) d\lambda(w)$$

$$\leq \|\psi\|_1 \max_{w \in \partial D} P(z,w) \leq \frac{1+|z|}{1-|z|} \|\psi\|_1 .$$

□

LEMMA 2. *Suppose* $f: D \hookleftarrow$ *is an inner function and let* $\psi: \partial D \to \mathbf{R}$ *be given.*

a) *If* ψ *is continuous then* $\psi \circ f \in L^\infty(\lambda)$ *and* $\hat{\psi} \circ f = \widehat{\psi \circ f}$.

b) *If* $\psi \in L^1(\lambda)$ *then* $\psi \circ f \in L^1(\lambda)$ *and*

$$\|\psi \circ f\|_1 \leq \frac{1+|f(0)|}{1-|f(0)|} \|\psi\|_1 .$$

PROOF: a) Given an inner function $f: D \hookleftarrow$, we have $\lim_{r \to 1} f(rw) = f(w)$ for a.e. $w \in \partial D$. Given a continuous $\psi: \partial D \to \mathbf{R}$, $\hat{\psi}$ is harmonic and $\lim_{z \to w} \hat{\psi}(z) = \psi(w)$ for each $w \in \partial D$ by Theorem 1.2.b). Therefore

$$\lim_{r \to 1} (\hat{\psi} \circ f)(rw) = \lim_{r \to 1} \hat{\psi}(f(rw)) = \psi(f(w)) = (\psi \circ f)(w)$$

for a.e. $w \in \partial D$. But $\hat{\psi} \circ f$ is harmonic and bounded. Thus the uniqueness property of Theorem 1.3 guarantees that $\hat{\psi} \circ f$ is the harmonic extension $\widehat{\psi \circ f}$ of $\psi \circ f$.

To prove b) it is sufficient to prove the inequality for continuous ψ; the result then follows by density. For continuous ψ,

$$\|\psi \circ f\|_1 = \int |\psi \circ f| \, d\lambda = \int (|\psi| \circ f)(w) P(0,w) d\lambda(w)$$

$$= \widehat{|\psi| \circ f}(0) = \widehat{|\psi|}(f(0)),$$

where the last equality is secured by part a). Applying Lemma 1 we get

$$\|\psi \circ f\|_1 = \widehat{|\psi|}(f(0)) \leq \frac{1 + |f(0)|}{1 - |f(0)|} \|\psi\|_1 .$$

□

Proof of Theorem 1.4: Given $\psi \in L^1(\lambda)$, take a sequence of continuous $\psi_n \colon \partial D \to \mathbf{R}$ converging to ψ in $L^1(\lambda)$. By Lemma 1,

$$\sup_{|z|<r} \left| (\hat{\psi}_n - \hat{\psi})(z) \right| \leq \frac{1 + r}{1 - r} \|\psi_n - \psi\|_1$$

for all $0 < r < 1$. Hence $\hat{\psi}_n \to \hat{\psi}$ uniformly on compact sets and then also $\hat{\psi}_n \circ f \to \hat{\psi} \circ f$ uniformly on compact sets. Now for every $n \geq 0$, Lemma 2.a) yields

$$\hat{\psi}_n \circ f = \widehat{\psi_n \circ f}. \tag{2}$$

Moreover, $(\psi_n - \psi) \circ f \in L^1(\lambda)$ and

$$\|(\psi_n - \psi) \circ f\|_1 \leq \frac{1 + |f(0)|}{1 - |f(0)|} \|\psi_n - \psi\|_1$$

by Lemma 2.b). Therefore, for all $0 < r < 1$, Lemma 1 gives

$$\begin{aligned} \sup_{|z|<r} \left| (\widehat{\psi_n \circ f} - \widehat{\psi \circ f})(z) \right| &\leq \frac{1 + r}{1 - r} \|(\psi_n - \psi) \circ f\|_1 \\ &\leq \frac{1 + r}{1 - r} \, \frac{1 + |f(0)|}{1 - |f(0)|} \|\psi_n - \psi\|_1 . \end{aligned}$$

Hence, $\widehat{\psi_n \circ f} \to \widehat{\psi \circ f}$ uniformly on compact sets. Taking limits in (2) we obtain $\hat{\psi} \circ f = \widehat{\psi \circ f}$. □

§2 The Denjoy-Wolff theorem

The dynamics of a holomorphic function of D into itself is described by the following result due to Denjoy [**De**] and Wolff [**W**].

THEOREM 2.1. *Let $f\colon D \hookleftarrow$ be a holomorphic function that is not a Möbius transformation. Then there exists a point $p \in \overline{D}$ such that $\lim_{n\to\infty} f^n(z) = p$ for all $z \in D$, uniformly on compact sets. Moreover, if $p \in D$ then $|f'(p)| < 1$.*

The point $p \in \overline{D}$ given by the theorem is obviously unique and we shall call it the *Denjoy-Wolff point* of f. Thus, given any holomorphic function $f\colon D \hookleftarrow$, the sequence of its iterates $f^n\colon D \hookleftarrow$ converges, uniformly on compact sets, to the constant p; the only exception (besides the identity) is presented by Möbius transformations with elliptic fixed point in D, since clearly the other two possibilities of Möbius transformations $T\colon D \hookleftarrow$ (parabolic or hyperbolic) also possess a unique Denjoy-Wolff point $p \in \partial D$.

In the proof of Theorem 2.1, as well as in subsequent sections, we shall use the conformally invariant *Poincaré metric* d_P on D; $d_P(x, y)$ is defined as the infimum of the hyperbolic lenghts of the arcs in D joining x to y ([**G**; pg. 5]) and we have

$$\lim_{n\to\infty} d_P(x_n, y_n) = 0 \Leftrightarrow \lim_{n\to\infty} \frac{|x_n - y_n|}{|1 - x_n \overline{y}_n|} = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{|x_n - y_n|}{1 - |x_n|^2} = 0,$$

for all sequences of points $x_n, y_n \in D$.

PROOF: Let $f: D \hookleftarrow$ be a holomorphic function and let us assume that f is not a Möbius transformation.

Suppose first that there exist $x \in D$ and $n_1 < n_2 < \dots$ such that $f^{n_j}(x)$ converges with $j \to \infty$ to a point $p \in D$. By virtue of Pick's invariant formulation of Schwarz's lemma,

$$d_P(f(x), f(y)) < d_P(x, y)$$

for all $x \neq y$ in D. Therefore, for all $n \geq 0$,

$$\begin{aligned} d_P(f^{n+2}(x), f^{n+1}(x)) &= d_P(f(f^{n+1}(x)), f(f^n(x))) \\ &\leq d_P(f^{n+1}(x), f^n(x)) \end{aligned}$$

and it follows that the sequence $d_P(f^{n+1}(x), f^n(x))$ converges with $n \to \infty$. Then

$$\begin{aligned} \lim_{n\to\infty} d_P(f^{n+1}(x), f^n(x)) &= \lim_{n\to\infty} d_P(f^{n_j+1}(x), f^{n_j}(x)) \\ &= d_P(f(p), p) \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} d_P(f^{n+1}(x), f^n(x)) &= \lim_{n\to\infty} d_P(f^{n_j+2}(x), f^{n_j+1}(x)) \\ &= d_P(f^2(p), f(p)) \end{aligned}$$

imply $d_P(f^2(p), f(p)) = d_P(f(p), p)$. This gives $f(p) = p$, because otherwise

$$d_P(f^2(p), f(p)) = d_P(f(f(p)), f(p)) < d_P(f(p), p).$$

To prove that $\lim_{n\to\infty} f^n(z) = p$ uniformly on compact subsets of D, we will assume that $p = 0$. (If not, we conjugate f with an appropriate

Möbius transformation.) Then $f(0) = 0$ and Schwarz's lemma gives $|f'(0)| < 1$ and $|f(z)| < |z|$ for all $0 < |z| < 1$, since f is not Möbius. As in the proof of Schwarz's lemma, consider $F: D \to \mathbf{C}$ defined by $F(z) = \frac{1}{z}f(z)$ for $z \neq 0$ and $F(0) = f'(0)$. Then $\lim_{|z|\to 1} |F(z)| \leq 1$ and, since F cannot be constant because f is not a Möbius map, the maximum principle gives $|F(z)| < 1$ for $z \in D$. Therefore

$$c_R = \max_{|z|<R} |F(z)| < 1$$

and $|f(z)| \leq c_R\,|z| \leq c_R R < R$ for all $|z| < R < 1$; then we also have

$$\left|f^2(z)\right| = |f((f(z))| \leq c_R\,|f(z)| \leq c_R^2\,|z| < R$$

when $|z| < R$. We conclude that

$$|f^n(z)| \leq c_R^n\,|z|$$

for all $n \geq 0$ and $|z| < R$, proving that $\lim_{n\to\infty} f^n(z) = p$ uniformly on compact subsets of D.

Now suppose that for all $z \in D$, the sequence $\{f^n(z)\}$ has no accumulation points in D. Then either there exists a point $p \in \partial D$ such that $\lim_{n\to\infty} f^n(z) = p$ for all $z \in D$ or else there exist $x, y \in D$, $p_1 \neq p_2$ in ∂D and sequences $n_1 < n_2 < \ldots$ and $m_1 < m_2 < \ldots$ such that $\lim_{j\to\infty} f^{n_j}(x) = p_1$ and $\lim_{j\to\infty} f^{m_j}(y) = p_2$. Let us derive a contradiction from this second possibility.

We start with the following: since for all $z \in D$ the sequence $\{f^n(z)\}$ accumulates nowhere in D, we have $\lim_{n\to\infty} |f^n(z)| = 1$. We claim that this convergence is uniform on compact subsets of D. If it is not, we can choose a sequence $\{z_j\}$ in D, bounded away from

∂D, and integers $k_1 < k_2 < \dots$ such that $|f^{k_j}(z_j)| \leq c < 1$. Since the sequence of iterates of f is uniformly bounded, we may assume that the sequence $\{f^{k_j}\}$ converges uniformly on compact sets to a holomorphic function $F: D \to \mathbf{C}$. Without loss of generality, we may also assume that z_j converges with $j \to \infty$ to some point $z \in D$. Then

$$\lim_{j\to\infty} f^{k_j}(z) = F(z) = \lim_{j\to\infty} f^{k_j}(z_j) \in D$$

shows that the sequence $f^n(z)$ has accumulation points in D, contrary to our hypothesis.

Assume now that $x, y \in D$, $p_1 \neq p_2$ in ∂D and $n_1 < n_2 < \dots$ and $m_1 < m_2 < \dots$ satisfy $\lim_{j\to\infty} f^{n_j}(x) = p_1$ and $\lim_{j\to\infty} f^{m_j}(y) = p_2$. Take $0 < r < 1$ sufficiently large to guarantee that x, y belong to

$$B = \{z \in D|\, |z| < r\}$$

and $f(B) \cap B \neq \phi$; it follows that $f^{n+1}(B) \cap f^n(B) \neq \phi$ for each $n \geq 0$. Then, for every $N \geq 0$, $\bigcup_{n\geq N} f^n(B)$ is an open connected set that accumulates on p_1 and p_2. Moreover, given any $\varepsilon > 0$, the annulus $A_\varepsilon = \{z \in \mathbf{C}|\, 1 - \varepsilon < |z| < 1\}$ contains $\bigcup_{n\geq N} f^n(B)$ for sufficiently large N, by the uniform convergence of $\lim_{n\to\infty} |f^n(z)| = 1$, $z \in B$. Hence, given any $q_1, q_2 \in \partial D$ which separate p_1 and p_2, $\bigcup_{n\geq N} f^n(B)$ intersects the interval $S_1 = \{tq_1 | 1 - \varepsilon < t < 1\}$ or the interval $S_2 = \{tq_2 | 1 - \varepsilon < t < 1\}$, for sufficiently large N. Since this holds for all $\varepsilon > 0$, it follows that $\bigcup_{n\geq N} f^n(B)$ accumulates radially on q_1 or q_2. But q_1 and q_2 are arbitrary points separating

p_1 and p_2 and $\bigcup_{n\geq N} f^n(B)$ is connected. Thus there exists an interval $J = [p_1, p_2]$ in ∂D all of whose points are radially accumulated by $\bigcup_{n\geq N} f^n(B)$. Given any $w \in J$ such that the nontangential limit $f(w)$ exists, we choose points $z_j \in B$ and integers $k_1 < k_2 < \dots$ such that $f^{k_j}(z_j)$ converges radially to w with $j \to \infty$. Without loss of generality we may assume that z_j converges to $\tilde{z} \in D$ with $j \to \infty$ and that f^{k_j} converges uniformly on compact sets to a holomorphic function $F: D \to \mathbf{C}$. Since $\lim_{n\to\infty} |f^n(z)| = 1$ for all $z \in D$, we obtain $F(D) \subseteq \partial D$ and then F must be constant: $F(z) = w$ for all $z \in D$. It follows that

$$
\begin{aligned}
f(w) &= \lim_{j\to\infty} f(f^{k_j}(z_j)) = \lim_{j\to\infty} f^{k_j+1}(z_j) \\
&= \lim_{j\to\infty} f^{k_j}(f(z_j)) = F(f(\tilde{z})) = w,
\end{aligned}
$$

proving that the boundary map of f satisfies $f(w) = w$ for a.e. $w \in J$. But J has positive Lebesgue measure; by Theorem 1.1, necessarily $f(z) = z$ for all $z \in D$. Since we precluded this possibility, we arrived at the contradiction we stroved for. Thus there exists $p \in \partial D$ such that $\lim_{n\to\infty} f^n(z) = p$ for all $z \in D$; since f^n is uniformly bounded, the convergence is uniform on compact subsets of D. □

COROLLARY 2.2. *Let $f: D \hookleftarrow$ be an inner function and let $p \in \overline{D}$ be its Denjoy-Wolff point.*

a) If $p \in D$, then the boundary map of f preserves the harmonic measure λ_p.

b) If $p \in \partial D$, then for every neighborhood W of p in ∂D,

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \leq j < n \mid f^j(w) \in W\} = 1$$

for a.e. $w \in \partial D$.

PROOF: a) is part c) of Corollary 1.5. To prove b), let ψ be the characteristic function of W; since $f^j(w) \in W$ if and only if $(\psi \circ f^j)(w) = 1$, we have

$$\#\{0 \leq j < n | f^j(w) \in W\} = \sum_{j=0}^{n-1} (\psi \circ f^j)(w)$$

for a.e. $w \in \partial D$. Denoting $z_n = f^n(0)$, Corollary 1.5 implies

$$\int (\psi \circ f^j) d\lambda = \int \psi d\lambda_{z_j} = \hat{\psi}(z_j).$$

But $z_n \to p$ and therefore $\hat{\psi}(z_n) \to \psi(p) = 1$, with $n \to \infty$, by Theorem 1.2.b). Dominated convergence thence implies

$$\int \Big(\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ f^j) \Big) d\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int (\psi \circ f^j) d\lambda$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\psi}(z_j) = \lim_{n\to\infty} \hat{\psi}(z_n) = \psi(p) = 1,$$

proving that $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ f^j)(w) = 1$ for a.e. $w \in \partial D$. □

§3 Ergodicity of inner functions

Recall that a measurable map $T\colon X \hookleftarrow$ of a measure space $(X, \mathcal{A}, \mu)$ is *ergodic* if $\mu(A) = 0$ or $\mu(X - A) = 0$ for every $A \in \mathcal{A}$ such that $T^{-1}(A) = A$ and is *exact* if the same occurs for $A \in \bigcap_{n\geq 0} T^{-n}(\mathcal{A})$. Exact maps are ergodic because $T^{-1}(A) = A$ implies $T^{-1}(A) = A$ for all $n \geq 0$ and then $A \in \bigcap_{n\geq 0} T^{-n}(\mathcal{A})$. If μ is a probability and T preserves μ then T is μ-recurrent and exactness also implies mixing.

In this section we give necessary and sufficient conditions for the ergodicity of the boundary map of an inner function and prove that, for such maps, ergodicity and exactness are the same. These results are due to Aaronson [**Aa3**] and Pommerenke [**Pe**]. Observe that ergodicity does not imply that a.e. boundary point is recurrent; this is true if the Denjoy-Wolff point belongs to D, since then the harmonic measure associated to it is invariant by Corollary 2.2 above, hence recurrent and, being equivalent to Lebesgue measure, a.e. boundary point is recurrent. Necessary and sufficient conditions for recurrence will be given in the next section.

THEOREM 3.1. *The following conditions are equivalent for inner functions $f\colon D \hookleftarrow$.*

a) The boundary map of f is exact.

b) The boundary map of f is ergodic.

c) There exist $a, b \in D$ *such that* $f^n(a) \neq f^n(b)$ *for* $n \geq 0$ *and* $\lim_{n\to\infty} d_P(f^n(a), f^n(b)) = 0$.

d) $\lim_{n\to\infty} d_P(f^n(x), f^n(y)) = 0$ *for all* $x, y \in D$.

COROLLARY 3.2. *The boundary map is exact whenever the inner function has a fixed point in* D.

The corollary is obvious: recall that in the presence of a fixed point $p \in D$, p is the Denjoy-Wolff point and $\lim_{n\to\infty} |f^n(z) - f^n(p)| = 0$, hence $\lim_{n\to\infty} d_P(f^n(z), f^n(p)) = 0$ for each $z \in D$, implying condition d).

PROOF: c) $\Rightarrow$ d) We shall use the following simple fact: if a sequence of holomorphic functions $g_n D \to \mathbf{C}$ converges uniformly on compact sets to $g: D \to \mathbf{C}$ and satisfies $g_{n+1}^{-1}(\{0\}) \supseteq g_n^{-1}(\{0\})$ for all $n \geq 0$, then either $g \equiv 0$ or $g^{-1}(\{0\}) = \bigcup_{n\geq 0} g_n^{-1}(\{0\})$.

To prove c) $\Rightarrow$ d) we may, and will, assume that $a = 0$. For each $n \geq 0$ choose a Möbius map $T_n: D \hookleftarrow$ satisfying $T_n f^n(0) = 0$ and select $n_0 < n_1 < \dots$ in such a way that the sequence $g_k = T_{n_k} \circ f^{n_k}: D \hookleftarrow$ converges uniformly on compact sets to a map $g: D \hookleftarrow$. Then

$$g_k^{-1}(\{0\}) = f^{-n_k}(T_{n_k}^{-1}(\{0\})) = f^{-n_k}(\{f^{n_k}(0)\})$$

and it is easy to check that $g_{k+1}^{-1}(\{0\}) \supseteq g_k^{-1}(\{0\})$ for $k \geq 0$. Now the hypothesis $f^n(0) \neq f^n(b)$ guarantees $g_k(b) \neq 0$ for each k, therefore $b \notin \bigcup_{k\geq 0} g_k^{-1}(\{0\})$. On the other hand, $g(b) = \lim_{k\to\infty} g_k(b) = 0$, since $d_P(g_k(b), 0) = d_P(T_{n_k} f^{n_k}(b), T_{n_k} f^{n_k}(0)) = d_P(f^{n_k}(b), f^{n_k}(0))$ converges to 0 with $k \to \infty$ by the hypothesis

$\lim_{n\to\infty} d_P(f^n(0), f^n(b)) = 0$. Thus the above stated fact implies $g \equiv 0$ and we obtain, for $z \in D$:

$$\begin{aligned} 0 = d_P(g(z), 0) &= \lim_{k\to\infty} d_P(g_k(z), 0) \\ &= \lim_{k\to\infty} d_P(T_{n_k} f^{n_k}(z), T_{n_k} f^{n_k}(0)) = \lim_{k\to\infty} d_P(f^{n_k}(z), f^{n_k}(0)). \end{aligned}$$

Since the sequence $d_P(f^n(z), f^n(0))$ is decreasing, its limit exists and it follows that $\lim_{n\to\infty} d_P(f^n(z), f^n(0)) = 0$ for all $z \in D$, which clearly implies condition d).

d) $\Rightarrow$ a) Let $\mathcal{A}$ denote the Borel σ-algebra of ∂D and suppose $A \in \bigcap_{n\geq 0} f^{-n}(\mathcal{A})$; then we can write $A = f^{-n}(A_n)$ with $A_n \in \mathcal{A}$, $n \geq 0$. To prove that $\lambda(A) = 0$ or $\lambda(A) = 1$, we consider the characteristic functions ψ and ψ_n of A and A_n respectively. Since $f^{-n}(A_n) = A$, we have $\psi_n \circ f^n = \psi$ and therefore $\hat{\psi} \circ f^n = \widehat{\psi_n \circ f^n} = \hat{\psi}$ by the commutative property of Theorem 1.4. Let $x, y \in D$ be given; for every $n \geq 0$,

$$\hat{\psi}(x) - \hat{\psi}(y) = \hat{\psi}_n(f^n(x)) - \hat{\psi}_n(f^n(y)). \tag{1}$$

For each $n \geq 0$ we now choose a Möbius map $T_n : D \hookleftarrow$ with $T_n(0) = f^n(x)$; since $\widehat{\psi_n \circ T_n} = \hat{\psi}_n \circ T_n$, it follows that

$$\begin{aligned} \hat{\psi}_n(f^n(x)) - \hat{\psi}_n(f^n(y)) &= \widehat{\psi_n \circ T_n}(0) - \widehat{\psi_n \circ T_n}(T_n^{-1} f^n(y)) \\ &= \int \psi_n(T_n w)[1 - P(T_n^{-1} f^n(y), w)] d\lambda(w). \end{aligned}$$

But $\lim_{n\to\infty} T_n^{-1} f^n(y) = 0$, since

$$d_P(T_n^{-1} f^n(y), 0) = d_P(f^n(y), T_n(0)) = d_P(f^n(y), f^n(x))$$

converges to 0 with $n \to \infty$ by hypothesis. Therefore

$$\lim_{n\to\infty} [\sup_{|w|=1} \left|1 - P(T_n^{-1} f^n(y), w)\right|] = 0,$$

which implies

$$\lim_{n\to\infty} [\hat{\psi}(f^n(x)) - \hat{\psi}(f^n(y))] = 0$$

and we obtain $\hat{\psi}(x) = \hat{\psi}(y)$ from (1). This proves that $\hat{\psi}$ is constant, hence $\psi = \hat{\psi}^*$ is constant a.e. by Theorem 1.2.a). Since ψ is the characteristic function of A, this means that $\lambda(A) = 0$ or $\lambda(A) = 1$.

a) $\Rightarrow$ b) Immediate, as above.

b) $\Rightarrow$ c) We shall prove that f is not ergodic when condition c) does not hold. Assume, therefore, that $\lim_{n\to\infty} d_P(f^n(x), f^n(y)) > 0$ for all $x, y \in D$ satisfying $f^n(x) \neq f^n(y)$ for each $n \geq 0$. Clearly we may choose $z \in D$ with $f^{n+1}(z) \neq f^n(z)$, $n \geq 0$; to simplify, we take $z = 0$. For each $n \geq 0$ we choose a Möbius map $T_n: D \hookleftarrow$ such that

$$T_n f^n(0) = 0 \quad \text{and} \quad T_n f^{n+1}(0) > 0.$$

Then

$$\begin{aligned} d_P(T_n T_{n+1}^{-1}(0), 0) &= d_P(T_n T_{n+1}^{-1}(0), T_n f^n(0)) \\ &= d_P(T_{n+1}^{-1}(0), f^n(0)) = d_P(f^{n+1}(0), f^n(0)) \end{aligned}$$

implies that the moduli $\left|T_n T_{n+1}^{-1}(0)\right|$ are decreasing and, since

$$T_n T_{n+1}^{-1}(0) = T_n f^{n+1}(0) > 0,$$

the sequence $\{T_n T_n^{-1}(0)\}$ converges to some $c \geq 0$ with $n \to \infty$. But our hypothesis gives

$$d_P(c, 0) = \lim_{n\to\infty} d_P(T_n T_{n+1}^{-1}(0), 0) = \lim_{n\to\infty} d_P(f^{n+1}(0), f^n(0)) > 0$$

and we have $c > 0$. These facts guarantee that the sequence $T_n \circ T_{n+1}^{-1}: D \hookleftarrow$ of Möbius maps converges to a Möbius map $T: D \hookleftarrow$; notice that $T(0) = c > 0$.

Next we prove that the sequence $g_n = T_n \circ f^n$ converges uniformly on compact sets. We have $g_n(0) = 0$ and, for every $z \in D$,

$$(2) \qquad |g_n(z)| \le |g_{n-1}(z)|$$

because

$$\begin{aligned} d_P(g_n(z), 0) &= d_P(T_n f^n(z), T_n f^n(0)) \\ &= d_P(f^n(z), f^n(0)) \le d_P(f^{n-1}(z), f^{n-1}(0)) \\ &= d_P(T_{n-1} f^{n-1}(z), T_{n-1} f^{n-1}(0)) = d_P(g_{n-1}(0), 0). \end{aligned}$$

Since the sequence $\{g_n\}$ is uniformly bounded, to prove its convergence it is sufficient to show that any two convergent subsequences $g_{n_\ell} \to g^{(1)}$ and $g_{m_k} \to g^{(2)}$ have the same limit, i.e., $g^{(1)} = g^{(2)}$. But (2) implies that the sequence $\{|g_n|\}$ is convergent; hence $\lim_{\ell \to \infty} |g_{n_\ell}| = \left|g^{(1)}\right|$ and $\lim_{k \to \infty} |g_{m_k}| = \left|g^{(2)}\right|$ coincide with $\lim_{n \to \infty} |g_n|$. Then $\left|g^{(1)}\right| = \left|g^{(2)}\right|$. This implies that $g^{(1)} = \alpha g^{(2)}$ with $|\alpha| = 1$; if we show that $\alpha = 1$, we are done. Above we proved $0 < c = \lim_{n \to +\infty} T_n f^{n+1}(0)$, therefore

$$0 < c = \lim_{n \to +\infty} T_n f^n(f(0)) = g^{(1)}(f(0)) = g^{(2)}(f(0))$$

implies $\alpha = 1$ and completes the proof of the convergence of $\{g_n\}$. We define $g = \lim_{n \to \infty} g_n$ and obtain

$$g \circ f = T \circ g,$$

because

$$g(f(z)) = \lim_{n\to\infty} T_n f^n(f(z)) = \lim_{n\to\infty} T_n T_{n+1}^{-1} T_{n+1} f^{n+1}(z)$$
$$= Tg(z).$$

Finally we claim that

$$T = \lim_{n\to\infty} T_n \circ f \circ T_n^{-1}$$

uniformly on compact sets. Since the sequence $T_n \circ f \circ T_n^{-1}$ is uniformly bounded, it is sufficient to show that $T = \lim_{k\to\infty} T_{n_k} \circ f \circ T_{n_k}^{-1}$ for every convergent subsequence $T_{n_k} \circ f \circ T_{n_k}^{-1}$. Suppose $S = \lim_{k\to\infty} T_{n_k} \circ f \circ T_{n_k}^{-1}$ for some choice $n_1 < n_2 < \dots$. Since $f^{n_k}(0) = T_{n_k}^{-1}(0)$ and $g \circ f = T \circ g$, we get

$$S(0) = \lim_{k\to\infty} T_{n_k} f(T_{n_k}^{-1}(0)) = \lim_{k\to\infty} T_{n_k} f^{n_k}(f(0))$$
$$= g(f(0)) = Tg(0) = T(0)$$

and

$$S(T(0)) = \lim_{k\to\infty} T_{n_k} \circ f \circ T_{n_k}^{-1}(T(0))$$
$$= \lim_{k\to\infty} T_{n_k} \circ f \circ T_{n_k}^{-1} \circ T_{n_k} \circ T_{n_k+1}^{-1}(0)$$
$$= \lim_{k\to\infty} T_{n_k} \circ f \circ T_{n_k+1}^{-1}(0) = \lim_{k\to\infty} T_{n_k} \circ f \circ f^{n_k+1}(0)$$
$$= \lim_{k\to\infty} T_{n_k} \circ T_{n_k+1}^{-1} \circ T_{n_k+1} \circ f^{n_k+2}(0)$$
$$= T(\lim_{k\to\infty} T_{n_k+1} \circ f^{n_k+2}(0))$$
$$= T(\lim_{k\to\infty} T_{n_k+1} \circ T_{n_k+2}^{-1}(0)) = T^2(0).$$

Hence $T^{-1} \circ S\colon D \hookleftarrow$ has two fixed points, 0 and $T(0) = c > 0$; Schwarz's lemma now implies $T = S$ and this concludes the proof of the claim. In particular, T has no fixed points, because otherwise $T_n \circ f \circ T_n^{-1}$ would have a fixed point (for large n) and then its image under T_n^{-1} would be a fixed point of f. But this is impossible, because if f has a fixed point then $f^n(0)$ convergs to it, by the Denjoy-Wolff theorem, and therefore $\lim_{n\to\infty} d_P(f^{n+1}(0), f^n(0) = 0$, contradicting our hypothesis.

Thus we have constructed a holomorphic function $g: D \hookleftarrow$ and a Möbius map T of D onto D without fixed points satisfying $g \circ f = T \circ g$. Let us see, to complete the proof of the theorem, that these properties imply that f is not ergodic. Since it has no fixed points, the Möbius map $T: D \hookleftarrow$ is either parabolic (one fixed point in ∂D) or hyperbolic (two fixed points in ∂D) and in both cases it is not hard to find a Borel set $A \subseteq \partial D$ such that $0 < \lambda(A) < 1$ and $T^{-1}(A) = A$. Then the characteristic function ψ of A satisfies $\psi \circ T = \psi$ and, writing $h = \hat{\psi} \circ g$, we get

$$h \circ f = (\hat{\psi} \circ g) \circ f = \hat{\psi} \circ T \circ g = \widehat{\psi \circ T} \circ g = \hat{\psi} \circ g = h$$

by the commutative property of Theorem 1.4. Since h is bounded, Theorem 1.3 gives $h = \widehat{h^*}$ and the same commutative property implies $\widehat{h^* \circ f} = \widehat{h^*} \circ f = h \circ f = h$; therefore, taking nontangential limits, we obtain

$$h^* \circ f = h^*.$$

Thus h^* is f-invariant; if we prove that it is not constant a.e., the non-ergodicity of f follows. Now T has no fixed points; since $g \circ f = T \circ g$, g is not constant and $g(D)$ is a non-empty open set. But $0 < \lambda(A) < 1$, therefore ψ is not constant a.e., which implies that $\hat{\psi}$ and therefore $\hat{\psi}|g(D)$ are not constant. Hence $h = \hat{\psi} \circ g$ is not constant; being bounded, it follows that h^* is not constant. □

§4 Recurrence of inner functions

A measurable map $T: X \hookleftarrow$ of a measure space $(X, \mathcal{A}, \mu)$ is *recurrent* if the set

$$\tilde{A} = \bigcap_{N \geq 0} \bigcup_{n \geq N} T^{-n}(A)$$

of all points $x \in X$ such that $T^n(x) \in A$ for infinitely many values of $n \geq 0$ satisfies $\mu(A - \tilde{A}) = 0$ for every $A \in \mathcal{A}$. Equivalentely, T is recurrent if $\mu(S - T^{-1}(S)) = 0$ for every $S \in \mathcal{A}$ such that $T^{-1}(S) \subseteq S$. To see that recurrence implies this property it suffices to observe that $\tilde{S} = \bigcap_{n \geq 0} T^{-n}(S)$ and therefore $\mu(S - T^{-1}(S)) \leq \mu(S - \tilde{S})$ whenever $T^{-1}(S) \subseteq S$; the converse is obtained following the usual proof of Poincaré's recurrence theorem for measure preserving maps of probability spaces.

Here we shall consider the question of the recurrence of the boundary map of an inner function $f: D \hookleftarrow$. We will show that the series

$$\sum_{n \geq 0} (1 - |f^n(z)|)$$

either converges for all $z \in D$ or else diverges for all $z \in D$. In the first case, the Denjoy-Wolff point p of f necessarily belongs to the boundary (obviously, because $\lim_{n \to \infty} |f^n(z) - p| = 0$ and then $p \in D$ would imply $\inf_{n \geq 0} (1 - |f^n(z)|) > 0$) and

$$\lim_{n \to \infty} f^n(w) = p$$

for a.e. $w \in \partial D$. In the second case, we shall prove that the boundary map is recurrent and exact. Moreover, whenever the Denjoy-Wolff point p of f belongs to ∂D, then the infinite measure μ_p, given by its derivative

$$\frac{d\mu_p}{d\lambda}(w) = \frac{1}{|w-p|^2}, \quad w \in \partial D,$$

with respect to the Lebesgue probability λ on ∂D, is f-invariant and, for every μ_p-integrable function $\psi : \partial D \to \mathbf{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(w)) = 0$$

for a.e. $w \in \partial D$.

THEOREM 4.1. *If $f: D \hookleftarrow$ is an inner function and there exists a point $z \in D$ such that*

$$\sum_{n\geq 0} (1 - |f^n(z)|) < \infty,$$

then the Denjoy-Wolff point p of f belongs to ∂D and $\lim_{n\to\infty} f^n(w) = p$ for a.e. $w \in \partial D$.

PROOF: Assume $z \in D$ is such that $\sum_{n\geq 0}(1-|f^n(z)|) < \infty$; obviously this implies that the Denjoy-Wolff point p of f belongs to ∂D. Given a Borel set $S \subseteq \partial D$, let

$$\hat{S} = \bigcup_{N\geq 0} \bigcap_{n\geq N} f^{-n}(S)$$

denote the set of all points $w \in \partial D$ for which there exists an $N \geq 0$ such that $f^n(w) \in S$ for all $n \geq N$. It is enough to prove that

$\lambda(\hat{S}) = 1$ for every open neighborhood S of p, because taking a decreasing sequence of open neighborhoods S_m of p such that $\bigcap_{m\geq 0} S_m = \{p\}$, we then obtain

$$\lambda(\bigcap_{m\geq 0} \hat{S}_m) = 1;$$

all that remains to do is to observe that $w \in \bigcap_{m\geq 0} \hat{S}_m$ implies that for each neighborhood S_m of p there exists $N_m \geq 0$ such that $f^n(w) \in S_m$ for all $n \geq N_m$, which means that $\lim_{n\to\infty} f^n(w) = p$. Since $\lambda(\bigcap_{m\geq 0} \hat{S}_m) = 1$, this will prove the theorem.

To estimate $\lambda(\hat{S})$, we recall that the harmonic measure λ_z is equivalent to the Lebesgue probability λ and consider the characteristic function ψ of S^c. Then $\psi \circ f^n$ is the characteristic function of $f^{-n}(S^c)$ and Corollary 1.5 gives

$$\lambda_z(f^{-n}(S^c)) = \int (\psi \circ f^n) d\lambda_z = \int \psi d\lambda_{f^n(z)} = \hat{\psi}(f^n(z)).$$

Since

$$\hat{S}^c = \bigcap_{N\geq 0} \bigcup_{n\geq N} f^{-n}(S^c),$$

we obtain

$$\lambda_z(\hat{S}^c) \leq \inf_{N\geq 0} \sum_{n\geq N} \hat{\psi}(f^n(z)). \tag{1}$$

We also have

$$\begin{aligned} \frac{\hat{\psi}(f^n(z))}{1-|f^n(z)|} &= \frac{1}{1-|f^n(z)|} \int_{S^c} P(f^n(z), w) d\lambda(w) \\ &= (1+|f^n(z)|) \int_{S^c} \frac{1}{|w - f^n(z)|^2} d\lambda(w). \end{aligned}$$

Let us now assume that S is an open neighborhood of p. Then $\lim_{n\to\infty} f^n(z) = p \notin \overline{S^c}$ implies

$$\lim_{n\to\infty} \frac{\hat{\psi}(f^n(z))}{1-|f^n(z)|} = 2\int_{S^c} \frac{1}{|w-p|^2} d\lambda(w).$$

Hence there exists $C > 0$ such that $\hat{\psi}(f^n(z)) \leq C(1-|f^n(z)|)$ for large $n \geq 0$ and therefore

$$\sum_{n\geq N} \hat{\psi}(f^n(z)) \leq C \sum_{n\geq N} (1-|f^n(z)|)$$

for large $N \geq 0$. But our hypothesis is $\sum_{n\geq 0}(1-|f^n(z)|) < \infty$, from which it follows that $\inf_{N\geq 0} \sum_{n\geq N} \hat{\psi}(f^n(z)) = 0$. By (1) we get $\lambda_z(\hat{S}^c) = 0$, implying $\lambda(\hat{S}^c) = 0$ and proving $\lambda(\hat{S}) = 1$. □

THEOREM 4.2. *If $f\colon D \hookleftarrow$ is an inner function and there exists a point $z \in D$ such that*

$$\sum_{n\geq 0}(1-|f^n(z)|) = \infty,$$

then the boundary map of f is recurrent.

PROOF: Assume $z \in D$ is such that $\sum_{n\geq 0}(1-|f^n(z)|) = \infty$. We shall prove that $\lambda(S - f^{-1}(S)) = 0$ for every Borel set $S \subseteq \partial D$ such that $f^{-1}(S) \subseteq S$; by our introductory remark, this ensures recurrence. Given a Borel set $S \subseteq \partial D$ such that $f^{-1}(S) \subseteq S$, assume $A \subseteq S - f^{-1}(S)$ is a Borel subset and ψ its characteristic function; then, as in the above proof,

$$\lambda_z(f^{-n}(A)) = \hat{\psi}(f^n(z))$$

and

$$\frac{\hat{\psi}(f^n(z))}{1-|f^n(z)|} = (1+|f^n(z)|)\int_A \frac{1}{|w-f^n(z)|^2}d\lambda(w).$$

Let $p = \lim_{n\to\infty} f^n(z)$ be the Denjoy-Wolff point of f; again $p \notin \overline{A}$ implies

$$\lim_{n\to\infty}\frac{\hat{\psi}(f^n(z))}{1-|f^n(z)|} = (1+|p|)\int_A \frac{1}{|w-p|^2}d\lambda(w). \tag{2}$$

Now $A \subseteq S - f^{-1}(S)$ and $f^{-1}(S) \subseteq S$; therefore

$$f^{-j}(A)\cap f^{-i}(A) = \phi$$

for all $0 \le j < i$ and we have

$$1 \ge \lambda_z(\bigcup_{n\ge 0} f^{-n}(A)) = \sum_{n\ge 0}\lambda_z(f^{-n}(A)) = \sum_{n\ge 0}\hat{\psi}(f^n(z)). \tag{3}$$

Suppose that $\lambda(A) > 0$. Then from (2) it follows that

$$\lim_{n\to\infty}\frac{\hat{\psi}(f^n(z))}{1-|f^n(z)|} > 0,$$

which guarantees the existence of $C > 0$ and $N \ge 0$ such that $\hat{\psi}(f^n(z)) \ge C(1-|f^n(z)|)$ for all $n \ge N$. But then, from (3) and our hypothesis, we get

$$1 \ge \sum_{n\ge N}\hat{\psi}(f^n(z)) \ge C\sum_{n\ge N}(1-|f^n(z)|) = \infty.$$

This contradiction proves that $\lambda(A) = 0$ for every Borel set $A \subseteq S - f^{-1}(S)$ satisfying $p \notin \overline{A}$. Since $S - f^{-1}(S)$ (as, in fact, every subset of ∂D) is a countable union of such sets A, plus (maybe) the point p itself, it follows that $\lambda(S - f^{-1}(S)) = 0$, thus proving the theorem. □

COROLLARY 4.3. *If the boundary map of an inner function is recurrent then it is exact.*

PROOF: Let $f\colon D \hookleftarrow$ be an inner function with recurrent boundary map and let us prove it is exact; by virtue of Theorem 3.1, it suffices to prove ergodicity. Suppose $A \subseteq \partial D$ is a Borel set such that $f^{-1}(A) = A$ and let us consider its characteristic function ψ. Then $\psi \circ f = \psi$ and the commutative property gives $\hat{\psi} \circ f = \widehat{\psi \circ f} = \hat{\psi}$. Therefore, if $g\colon D \to \mathbf{C}$ is any holomorphic function with $\operatorname{Re} g = \hat{\psi}$, then $\operatorname{Re}(g \circ f) = (\operatorname{Re} g) \circ f = \hat{\psi} \circ f = \hat{\psi} = \operatorname{Re} g$ and we may choose $b \in \mathbf{R}$ such that $g \circ f = g + ib$. Let us first assume that $b \neq 0$. Defining $G\colon D \to \mathbf{C}$ by $G = \exp(-2\pi \left|b\right|^{-1} g)$, we obtain

$$|G(z)| = \exp\left(-\frac{2\pi}{|b|} \operatorname{Re} g(z)\right) = \exp\left(-\frac{2\pi}{|b|}\hat{\psi}(z)\right) \leq 1$$

and

$$\begin{aligned}(G \circ f)(z) &= \exp\left(-\frac{2\pi}{|b|} g(f(z))\right) = \exp\left(-\frac{2\pi}{|b|} g(z) \pm 2\pi i\right) \\ &= \exp\left(-\frac{2\pi}{|b|} g(z)\right) = G(z),\end{aligned}$$

that is, G is bounded and f-invariant. It follows that $G \circ f^n = G$ for all $n \geq 0$, which means that $G - G(a)$ is zero at every $f^n(a)$, for all $n \geq 0$ and every $a \in D$. Given $a \in D$, Blaschke's theorem ([**Du**; Thm. 2.3]) ensures that either $G - G(a) \equiv 0$ or else its sequence of zeros $a_0, a_1, \ldots$ satisfies $\sum_{n\geq 0}(1 - |a_n|) < \infty$. If the latter were the case, we would obtain $\sum_{n\geq 0}(1 - |f^n(z)|) < \infty$ and then, by Theorem 4.1, the Denjoy-Wolff point p of f would belong to ∂D and $\lim_{n\to\infty} f^n(w) = p$ for a.e. $w \in \partial D$. Since our hypothesis is the recurrence of f, this cannot happen and we are left with $G - G(a) \equiv 0$. But this implies that G, and then g and $\operatorname{Re} g = \hat{\psi}$, are all constant. Hence also $\psi = \hat{\psi}^*$ is constant, proving that $\lambda(A) = 0$ or $\lambda(A) = 1$. The other case, $b = 0$, is proved applying the same argument to $G = \exp(-g)$, which again is bounded and f-invariant.

□

Recall that, given $p \in \partial D$, we let μ_p denote the measure on ∂D whose derivative with respect to the Lebesgue probability λ is

$$\frac{d\mu_p}{d\lambda}(\omega) = \frac{1}{|\omega - p|^2}, \quad \omega \in \partial D.$$

Clearly $\mu_p(\partial D) = +\infty$.

THEOREM 4.4. *Let $f\colon D \hookleftarrow$ be an inner function with Denjoy-Wolff point $p \in \partial D$. There exists $0 < c \leq 1$ such that*

a) *$\mu_p(f^{-1}(A)) = c\mu_p(A)$ for every Borel set $A \subseteq \partial D$;*

b) *$\lim_{n\to\infty}(1 - |f^n(z)|)^{1/n} = c$, for all $z \in D$;*

c) *$(1 - |f(z)|)/(1 - |z|)$ converges to c when z converges non-tangentially to p and $f(z)$ converges to p.*

COROLLARY 4.5. *μ_p is f-invariant if f is ergodic.*

The corollary is obvious: by Theorem 3.1, ergodicity implies $\lim_{n\to\infty} d_P(f^{n+1}(z), f^n(z)) = 0$ for all $z \in D$, hence

$$\lim_{n\to\infty} \frac{|f^{n+1}(z) - f^n(z)|}{1 - |f^n(z)|^2} = 0$$

and then

$$c = \lim_{n\to\infty} \frac{1 - |f^{n+1}(z)|}{1 - |f^n(z)|} = 1.$$

PROOF: Let $f: D \hookleftarrow$ be an inner function with Denjoy-Wolff point $p \in \partial D$. We define

$$\delta = \liminf_{\substack{z \to p \\ f(z) \to p}} \frac{1 - |f(z)|}{1 - |z|}.$$

Observe that $0 \leq \delta \leq 1$, because for $z_n = f^n(0)$ we have $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} f^n(0) = p$, implying

$$\delta \leq \liminf_{n\to\infty} \frac{1 - |f(z_n)|}{1 - |z_n|} = \liminf_{n\to\infty} \frac{1 - |z_{n+1}|}{1 - |z_n|} \leq 1.$$

We claim that

$$\mu_p(f^{-1}(A)) \leq \delta \mu_p(A) \tag{4}$$

for every Borel set $A \subseteq \partial D$ with $p \notin \overline{A}$. To prove this, let us take any sequence $\{z_n\}$ in D with both z_n and $f(z_n)$ converging to p. Then $\lim_{n\to\infty} |f(z_n)| = \lim_{n\to\infty} |z_n| = |p| = 1$ and we have

$$\liminf_{n\to\infty} \frac{1 - |f(z_n)|^2}{1 - |z_n|^2} = \liminf_{n\to\infty} \frac{1 - |f(z_n)|}{1 - |z_n|} \leq \delta.$$

Given any Borel set $A \subseteq \partial D$ we also have

$$\begin{aligned} &\int_{f^{-1}(A)} \frac{1}{|w - z_n|^2} d\lambda(w) \\ &= \frac{1 - |f(z_n)|^2}{1 - |z_n|^2} \int_A \frac{1}{|w - f(z_n)|^2} d\lambda(w) \ , \end{aligned} \tag{5}$$

because

$$\begin{aligned} (1 - |z_n|^2) \int_{f^{-1}(A)} \frac{1}{|w - z_n|^2} d\lambda(w) &= \int_{f^{-1}(A)} P(z_n, w) d\lambda(w) \\ &= \lambda_{z_n}(f^{-1}(A)) = \lambda_{f(z_n)}(A) \\ = \int_A P(f(z_n), w) d\lambda w = (1 - |f(z_n)|^2) &\int_A \frac{1}{|w - f(z_n)|^2} d\lambda(w), \end{aligned}$$

where we used Corollary 1.5 to ensure the middle step. Finally, if $p \notin \overline{A}$, $\lim_{n\to\infty} f(z_n) = p$ gives

$$(6) \qquad \lim_{n\to\infty} \int_A \frac{1}{|w - f(z_n)|^2} d\lambda(w) = \int_A \frac{1}{|w-p|^2} d\lambda(w) = \mu_p(A).$$

Since

$$\begin{aligned} \mu_p(f^{-1}(A)) &= \int_{f^{-1}(A)} \frac{1}{|w-p|^2} d\lambda(w) \\ &= \int_{f^{-1}(A)} \lim_{n\to\infty} \frac{1}{|w-z_n|^2} d\lambda(w) \ , \end{aligned}$$

Fatou's lemma thence gives

$$\begin{aligned} \mu_p(f^{-1}(A)) &\le \liminf_{n\to\infty} \int_{f^{-1}(A)} \frac{1}{|w-z_n|^2} d\lambda(w) \\ &= \liminf_{n\to\infty} \frac{1-|f(z_n)|^2}{1-|z_n|^2} \int_A \frac{1}{|w-f(z_n)|^2} d\lambda(w) \\ &\le \delta \lim_{n\to\infty} \int_A \frac{1}{|w-f(z_n)|^2} d\lambda(w) = \delta \mu_p(A), \end{aligned}$$

proving claim (4).

Now assume that $\{z_n\}$ is a sequence in D which converges nontangentially to p and is such that $f(z_n)$ converges to p and

$$\lim_{n\to\infty} \frac{1-|f(z_n)|}{1-|z_n|} = c$$

exists; we claim that

$$(7) \qquad \mu_p(f^{-1}(A)) = c\mu_p(A)$$

for every Borel set $A \subseteq \partial D$ such that $\mu_p(A) < \infty$. This obviously implies that c is independent of the sequence $\{z_n\}$, thus proving items a) and c) of the theorem. To prove (7), let $A \subseteq \partial D$ be a Borel set such that $p \notin \overline{A}$; by claim (4),

$$\mu_p(f^{-1}(A)) = \int_{f^{-1}(A)} \frac{1}{|w-p|^2} d\lambda(w) < \infty.$$

Since z_n converges nontangentially to p, there exists $K > 0$ such that

$$\frac{1}{|w-z_n|^2} \leq K \frac{1}{|w-p|^2}$$

for all $w \in \partial D$ and sufficiently large n; applying dominated convergence, we get

$$\mu_p(f^{-1}(A)) = \lim_{n\to\infty} \int_{f^{-1}(A)} \frac{1}{|w-z_n|^2} d\lambda(w).$$

Hence, using (5) and (6),

$$\begin{aligned}\mu_p(f^{-1}(A)) &= \lim_{n\to\infty} \frac{1-|f(z_n)|^2}{1-|z_n|^2} \int_A \frac{1}{|w-f(z_n)|^2} d\lambda(w) \\ &= c\mu_p(A).\end{aligned}$$

The general case of claim (7) follows easily, because if $A \subseteq \partial D$ is a Borel set with $\mu_p(A) < \infty$, then we can write it as $A = \bigcup_{n\geq 0} A_n$ with $A_0 \subseteq A_1 \subseteq \ldots$ and each A_n bounded away from p. Thus

$$\begin{aligned}\mu_p(f^{-1}(A)) = \mu_p\Big(\bigcup_{n\geq 0} f^{-1}(A_n)\Big) &= \lim_{n\to\infty} \mu_p(f^{-1}(A_n)) \\ &= c \lim_{n\to\infty} \mu_p(A_n) = c\mu_p(A).\end{aligned}$$

In particular, $\mu_p(f^{-1}(A)) < \infty$; iterating, we obtain

$$\mu_p(f^{-n}(A)) = c^n \mu_p(A) \tag{8}$$

for all $n \geq 0$ and every $A \subseteq \partial D$ with $\mu_p(A) < +\infty$.

To prove item b), we take $z \in D$ and set

$$\alpha^+ = \limsup_{n\to\infty}(1 - |f^n(z)|)^{1/n},$$

$$\alpha^- = \liminf_{n\to\infty}(1 - |f^n(z)|)^{1/n}.$$

Let $A \subseteq \partial D$ a Borel set such that $p \notin \overline{A}$ and $\mu_p(A) > 0$. For some constant $K = K(z) > 0$ we have, for all $n \geq 0$,

$$\begin{aligned} K\mu_p(f^{-n}(A)) &\geq \lambda_z(f^{-n}(A)) = \lambda_{f^n(z)}(A) \\ &= (1 - |f^n(z)|^2)\int_A \frac{1}{|w - f^n(z)|^2}\,d\lambda(w). \end{aligned}$$

But $\limsup_{n\to\infty}(1 - |f^n(z)|^2)^{1/n} \geq \alpha^+$; therefore, using (6) and (8) we obtain

$$c = \limsup_{n\to\infty}(\mu_p(f^{-n}(A)))^{1/n} \geq \alpha^+.$$

Now we take $\varepsilon > 0$ and choose $m_1 < m_2 < \ldots$ such that

$$1 - \left|f^{m_n+1}(z)\right| \leq (\alpha^- + \varepsilon)(1 - |f^{m_n}(z)|).$$

Proceeding as in the proof of claim (4) with the sequence $z_n = f^n(z)$ we get

$$\begin{aligned} c\mu_p(A) = \mu_p(f^{-1}(A)) &\leq \left(\liminf_{n\to\infty}\frac{1 - |f(z_n)|^2}{1 - |z_n|^2}\right)\mu_p(A) \\ &\leq (\alpha^- + \varepsilon)\mu_p(A), \end{aligned}$$

proving $c \leq \alpha^- + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha^- \leq \alpha^+ \leq c \leq \alpha^-,$$

completing the proof of item b).

To close the proof of the theorem, we observe that $0 < c$; otherwise we would have $\mu_p(f^{-1}(A)) = 0$ for some Borel set $A \subseteq \partial D$ with $p \notin \overline{A}$ and $\mu_p(A) > 0$. Then

$$0 = \mu_p(f^{-1}(A)) \geq \lambda_z(f^{-1}(A)) = \lambda_{f(z)}(A)$$

for $z \in D$, implying $\lambda(A) = 0$ and then $\mu_p(A) = 0$. □

THEOREM 4.6. *Let $f: D \hookleftarrow$ be an inner function with Denjoy-Wolff point $p \in \partial D$ and such that μ_p is f-invariant. For every $\psi \in L^1(\mu_p)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(w)) = 0$$

for a.e. $w \in \partial D$.

PROOF: Lef $f: D \hookleftarrow$ be an inner function with Denjoy-Wolff point $p \in \partial D$ and μ_p invariant. It is enough to prove the theorem for positive $\psi \in L^1(\mu_p)$. If $\psi \geq 0$ is zero in a neighborhood of p then $\lim_{n\to\infty} \hat{\psi}(f^n(0)) = 0$ and, using Fatou's lemma, we get

$$0 \leq \int \Big(\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j\Big) d\lambda \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int (\psi \circ f^j) d\lambda$$

$$= \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\psi}(f^j(0)) = 0,$$

hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(w)) = 0 \tag{9}$$

for a.e. $w \in \partial D$. Given any positive $\psi \in L^1(\mu_p)$, we choose a neighborhood W of p in ∂D and write $\psi = \psi_0 + \psi_1$, with $\psi_0(w) = 0$ for $w \in W$ and $\psi_1(w) = 0$ for $w \in W^c$. Since μ_p is f-invariant,

$$\int (\psi_1 \circ f^n) d\mu_p = \int \psi_1 d\mu_p;$$

therefore, Fatou's lemma and (9) imply

$$\int \Big(\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j \Big) d\mu_p = \int \Big(\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_1 \circ f^j \Big) d\mu_p$$

$$\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int (\psi_1 \circ f^j) d\mu_p = \int \psi_1 d\mu_p = \int_W \psi d\mu_p.$$

Since ψ is μ_p-integrable, the last expression can be made arbitrarily small; thus

$$\int \Big(\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j \Big) d\mu_p = 0,$$

proving the theorem. □

§5 Inner functions of the upper half plane

In this section we consider inner functions of the open upper half plane $\mathbf{R}^2_+ = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ and characterize the Pólya-Szegö maps.

All the results of the previous sections translate into results for inner functions of $\mathbf{R}^2_+$ via the Möbius map $S\colon D \to \mathbf{R}^2_+$ given by $S(z) = i(1+z)/(1-z)$; an additional feature of the boundary $\mathbf{R} \cup \{\infty\}$ of $\mathbf{R}^2_+$, however, is the distinguished point at infinity which, for inner functions of $\mathbf{R}^2_+$, is far more conspicuous than its counterpart $w = 1$ is for inner functions of the unit disc. Thus, if the inner function $f\colon \mathbf{R}^2_+ \hookleftarrow$ has no fixed point, we may, naturally, choose ∞ as the Denjoy-Wolff point of f (conjugating by a Möbius map if necessary). In fact, the canonical representation ([**T**]) of inner functions $f\colon \mathbf{R}^2_+ \hookleftarrow$,

$$(1) \qquad f(z) = \alpha z + \beta + \int_{\mathbf{R}} \frac{1+zw}{w-z} \, d\mu(w),$$

with $\alpha \geq 0$, $\beta \in \mathbf{R}$ and μ a singular finite measure on the Borel σ-algebra of $\mathbf{R}$, already enhances this choice for the Denjoy-Wolff point of f: α is characterized by

$$\lim_{t \to \infty} \frac{f(it)}{it} = \alpha$$

and we shall see that ∞ is the Denjoy-Wolff point of f if and only if $\alpha \geq 1$. Before we prove this, let us comment on the upper half plane versions of the two measures which played a major role in the unit disc scenario. The harmonic measure λ_z, for $z = a + ib \in \mathbf{R}^2_+$, $a, b \in \mathbf{R}$, now has, with respect to Lebesgue measure λ on $\mathbf{R}$, derivative

$$\frac{d\lambda_z}{d\lambda}(x) = \frac{b}{\pi}\frac{1}{(x-a)^2 + b^2}$$

for $x \in \mathbf{R}$ ([**G**; pg. 12]). Denoting, as before, the boundary map of an inner function $f\colon \mathbf{R}^2_+ \hookleftarrow$ simply by $f\colon \mathbf{R} \hookleftarrow$ again, Corollary 1.5 gives

$$\lambda_z(f^{-1}(A)) = \lambda_{f(z)}(A)$$

for every Borel set $A \subseteq \mathbf{R}$; if $z \in \mathbf{R}^2_+$ is a fixed point of f then λ_z is f-invariant. Now if $f\colon \mathbf{R}^2_+ \hookleftarrow$ has no fixed point then, associated to its Denjoy-Wolff point $p \in \mathbf{R} \cup \{\infty\}$, we have an infinite measure μ_p which satisfies

$$\mu_p(f^{-1}(A)) = c\mu_p(A)$$

for some $0 < c \leq 1$ and every Borel set $A \subseteq \mathbf{R}$ (Theorem 4.4). If $p = \infty$, we (normalize and) have for μ_p the Lebesgue measure on $\mathbf{R}$: $\lambda = \mu_\infty$. If $z \to \infty$ nontangentially, we have the following (compare with the relation (2) of Section 4):

LEMMA 1. *Let $\{z_n\}$ be any sequence in $\mathbf{R}^2_+$ such that $\operatorname{Im} z_n \to \infty$ and $\frac{|\operatorname{Re}\, z_n|}{\operatorname{Im}\, z_n} \to 0$ with $n \to \infty$. Then*

$$\lambda(A) = \lim_{n\to\infty} \pi \operatorname{Im} z_n \ \lambda_{z_n}(A)$$

for every Borel set $A \subseteq \mathbf{R}$.

PROOF: Write $z_n = a_n + ib_n$, $b_n > 0$, $a_n \in \mathbf{R}$. Then $|\frac{x-a_n}{b_n}| \leq \frac{|x|}{b_n} + \frac{|a_n|}{b_n} \to 0$ with $n \to \infty$ for each $x \in \mathbf{R}$. Given a Borel set

$A \subseteq \mathbf{R}$ we have

$$\lambda_{z_n}(A) = \int_A d\lambda_{z_n} = \frac{1}{\pi} \int_A \frac{b_n}{(x - a_n)^2 + b_n^2} d\lambda(x)$$

and therefore

$$\pi b_n \lambda_{z_n}(A) = \int_A \frac{1}{(\frac{x - a_n}{b_n})^2 + 1} d\lambda(x).$$

Dominated convergence then gives

$$\lim_{n \to \infty} \pi b_n \lambda_{z_n}(A) = \int_A d\lambda(x) = \lambda(A).$$

□

LEMMA 2. *If* $f : \mathbf{R}^2_+ \hookleftarrow$ *is an inner function given by* (1), *then*

$$\alpha = \lim_{n \to \infty} \frac{f(it)}{it} \quad .$$

PROOF: For each $t > 0$ we have

$$\begin{aligned} f(it) &= \alpha it + \beta + \int_{\mathbf{R}} \frac{1 + iwt}{w - it} d\mu(w) \\ &= \alpha it + \beta + (1 - t^2) \int_{\mathbf{R}} \frac{w}{w^2 + t^2} d\mu(w) + it \int_{\mathbf{R}} \frac{1 + w^2}{w^2 + t^2} d\mu(w). \end{aligned}$$

For $t \geq 1$ we certainly have $(1 + w^2)/(w^2 + t^2) \leq 1$ for all $w \in \mathbf{R}$, and therefore dominated convergence gives

$$\lim_{t \to \infty} \int_{\mathbf{R}} \frac{1 + w^2}{w^2 + t^2} d\mu(w) = 0.$$

From the above expression for $f(it)$, we can thus read off $\lim_{t \to \infty} \frac{f(it)}{it} = \alpha$, as soon as we prove the following

CLAIM: There exists $C > 0$ such that for all $t > 0$

$$\int_{\mathbf{R}} \frac{|w|}{w^2+t^2} d\mu(w) \leq \frac{C}{t^2} \ .$$

Observe that

$$\begin{aligned}\int_{\mathbf{R}} \frac{|w|}{w^2+t^2} d\mu(w) &= \int_{-1}^{+1} \frac{|w|}{w^2+t^2} d\mu(w) + 2\int_1^{+\infty} \frac{w}{w^2+t^2} d\mu(w) \\ &\leq \frac{1}{t^2}\mu([-1,1]) + 2\int_1^{+\infty} \frac{w}{w^2+t^2} d\mu(w) \ .\end{aligned}$$

The last term we integrate by parts, obtaining

$$\begin{aligned}\int_1^{+\infty} \frac{w}{w^2+t^2} d\mu(w) &= \frac{F(1)}{1+t^2} - \int_1^{+\infty} F(w)\frac{t^2-w^2}{(w^2+t^2)^2} dw \\ &\leq \frac{F(1)}{1+t^2} + F(1)\int_1^{+\infty} \frac{w^2-t^2}{(w^2+t^2)^2} dw \\ &\leq \frac{F(1)}{1+t^2} + \frac{F(1)}{t}\int_{1/t}^{+\infty} \frac{s^2-1}{(s^2+1)^2} ds = \frac{2F(1)}{1+t^2},\end{aligned}$$

where we wrote $F(w) = \mu([w, +\infty))$. □

THEOREM 5.1. *If the inner function $f\colon \mathbf{R}^2_+ \hookleftarrow$ is given by (1) and $\alpha > 0$, then for every Borel set $A \subseteq \mathbf{R}$,*

$$\lambda(f^{-1}(A)) = \frac{1}{\alpha}\lambda(A).$$

Moreover, $\alpha \geq 1$ if and only if ∞ is the Denjoy-Wolff point of f.

PROOF: Let $f\colon \mathbf{R}^2_+ \hookleftarrow$ be given by (1) with $\alpha > 0$. Then Lemma 2 gives $\lim_{b\to\infty} \frac{f(ib)}{ib} = \alpha > 0$, implying $\frac{\operatorname{Im} f(ib)}{b} \to \alpha$, $\operatorname{Im} f(ib) \to \infty$

and $\dfrac{|\mathrm{Re}\, f(ib)|}{\mathrm{Im}\ f(ib)} \to 0$ with $b \to \infty$, $b > 0$. Let $A \subseteq \mathbf{R}$ be a Borel set. Since $\lambda_{f(ib)}(A) = \lambda_{ib}(f^{-1}(A))$ for $b > 0$, Lemma 1 implies

$$\begin{aligned}\lambda(A) &= \lim_{b\to\infty} \pi \ \mathrm{Im}\ f(ib)\ \lambda_{f(ib)}(A) \\ &= \lim_{b\to\infty} \frac{\mathrm{Im}\ f(ib)}{b} \pi b \lambda_{ib}(f^{-1}(A)) = \alpha\lambda(f^{-1}(A)),\end{aligned}$$

proving the first part of the theorem.

Now assume that $\alpha \geq 1$ and let us prove that ∞ is the Denjoy-Wolff point of f. Suppose that it isn't. Let $p \neq \infty$ be the Denjoy-Wolff point of f and let us write ν for either the harmonic measure λ_p, if $p \in \mathbf{R}^2_+$, or the infinite measure μ_p, if $p \in \mathbf{R}$. From the results of the previous sections, we can choose a constant $0 < c \leq 1$ such that

$$\nu(f^{-1}(A)) = c\nu(A)$$

for every Borel set $A \subseteq \mathbf{R}$. Hence

$$\begin{aligned}c\int_A \frac{d\nu}{d\lambda} d\lambda = c\nu(A) = \nu(f^{-1}(A)) &= \int_{f^{-1}(A)} \frac{d\nu}{d\lambda} d\lambda \\ &= \frac{1}{\alpha}\int_A \left(\frac{d\nu}{d\lambda}\circ f\right) d\lambda\end{aligned}$$

for every Borel set $A \subseteq \mathbf{R}$, where in the last equality we used the first part of the theorem, since $\alpha > 0$. Then

$$\frac{d\nu}{d\lambda}\circ f = \alpha c \frac{d\nu}{d\lambda} \qquad \lambda\text{-a.e.}\ . \tag{2}$$

If $\alpha c < 1$, (2) implies

$$\lim_{n\to\infty} \frac{d\nu}{d\lambda}(f^n(x)) = 0,$$

and then

$$\lim_{n\to\infty} f^n(x) = \infty$$

for a.e. $x \in \mathbf{R}$, which means that ∞ is the Denjoy-Wolff point of f. If $\alpha > 1$ we have

$$\lambda(f^{-n}(A)) = \frac{1}{\alpha^n}\lambda(A)$$

for every Borel set $A \subseteq \mathbf{R}$ with $\lambda(A) < \infty$. This implies that for a.e. $x \in A$ there exists $N > 0$ such that $f^n(x) \notin A$ if $n \geq N$ and then, since this holds for every $A = [-a, a]$, that $\lim_{n\to\infty} f^n(x) = \infty$ for a.e. $x \in \mathbf{R}$; again ∞ is the Denjoy-Wolff point of f. Thus our hypotheses $\alpha \geq 1$ and ∞ is not the Denjoy-Wolff point of f imply $\alpha c \geq 1$ and $\alpha \leq 1$. It follows that $c = 1 = \alpha$ and then (2) implies that f =identity, which is the contradiction we wanted.

Conversely, suppose that ∞ is the Denjoy-Wolff point of f and let us prove that $\alpha \geq 1$. But ∞ being the Denjoy-Wolff point of f implies that $\lambda = \mu_\infty$ and there exists $0 < c \leq 1$ such that $\lambda(f^{-1}(A)) = c\lambda(A)$ for every Borel set $A \subseteq \mathbf{R}$. Suppose we prove that $\alpha > 0$; then, by the first part of the theorem, we will get

$$\lambda(A) = \alpha\lambda(f^{-1}(A)) = \alpha c\lambda(A)$$

for every Borel set $A \subseteq \mathbf{R}$, thence $\alpha c = 1$ and $\alpha = \frac{1}{c} \geq 1$. Therefore all we need to prove is that $\alpha > 0$. Suppose that $\alpha = 0$. Let $\{b_n\}$ be any sequence with $b_n > 0$ and $\lim_{n\to\infty} b_n = \infty$. Then

$$f(ib_n) = \beta + (1 - b_n^2)\int_{\mathbf{R}} \frac{w}{w^2 + b_n^2}d\mu(w) + ib_n\int_{\mathbf{R}} \frac{1 + w^2}{w^2 + b_n^2}d\mu(w),$$

where

$$\sup_{n\geq 0}\left|(1 - b_n^2)\int_{\mathbf{R}} \frac{w}{w^2 + b_n^2}d\mu(w)\right| < \infty$$

by the claim in the proof of Lemma 2. Let us assume first that

$$\lim_{n\to\infty} b_n \int_{\mathbf{R}} \frac{1+w^2}{w^2+b_n^2} d\mu(w) = \infty.$$

Then $\operatorname{Re} f(ib_n)$ is bounded and $\operatorname{Im} f(ib_n) \to \infty$ with $n \to \infty$; since also $\dfrac{\operatorname{Im} f(ib_n)}{b_n} \to \alpha = 0$ with $n \to \infty$ (by Lemma 2), Lemma 1 gives

$$\begin{aligned}\lambda(A) &= \lim_{n\to\infty} \pi \operatorname{Im} f(ib_n)\ \lambda_{f(ib_n)}(A)\\ &= \lim_{n\to\infty} \frac{\operatorname{Im} f(ib_n)}{b_n} \pi b_n \lambda_{ib_n}(f^{-1}(A))\\ &= \lambda(f^{-1}(A)) \lim_{n\to\infty} \frac{\operatorname{Im} f(ib_n)}{b_n} = 0,\end{aligned}$$

for every Borel set $A \subseteq \mathbf{R}$, clearly an impossibility. Therefore we are left with

$$\sup_{n\geq 0} b_n \int_{\mathbf{R}} \frac{1+w^2}{w^2+b_n^2} d\mu(w) < \infty.$$

But then $\{f(ib_n)\}$ is bounded and we may suppose that it converges to some $z \in \mathbf{C}$ with $\operatorname{Im} z \geq 0$. Assume that $\operatorname{Im} z > 0$; since ∞ is the Denjoy-Wolff point of f, this implies, by Lemma 1,

$$\begin{aligned}c\lambda(A) = \lambda(f^{-1}(A)) &= \lim_{n\to\infty} \pi b_n \lambda_{ib_n}(f^{-1}(A))\\ &= \lim_{n\to\infty} \pi b_n \lambda_{f(ib_n)}(A) = \lambda_z(A) \lim_{n\to\infty} \pi b_n = \infty\end{aligned}$$

for every Borel set $A \subseteq \mathbf{R}$, again an impossibility. Finally assume that $\operatorname{Im} z = 0$. Then, as above,

$$\lambda(f^{-1}(A)) = \lim_{n\to\infty} \pi b_n \lambda_{f(ib_n)}(A)$$

for every Borel set $A \subseteq \mathbf{R}$, but since now

$$\lim_{n\to\infty} \frac{\text{Im } f(ib_n)}{b_n} = 0 = \lim_{n\to\infty} \text{Re } f(ib_n),$$

we get

$$\lim_{n\to\infty} \pi \text{ Im } f(ib_n) \ \lambda_{f(ib_n)}(A) = \mu_z(A)$$

whenever $z \notin \overline{A}$ and A is bounded. Thus for such Borel sets with $0 < \mu_z(A) < \infty$, we obtain

$$c\lambda(A) = \lambda(f^{-1}(A)) = \lim_{n\to\infty} \frac{b_n}{\text{Im } f(ib_n)} \pi \text{ Im } f(ib_n) \ \lambda_{f(ib_n)}(A) = \infty,$$

which is again impossible. We have run out of possibilities, thence $\alpha > 0$. □

Let us now turn to the problem, posed by Pólya (and answered by Pólya [**Pa**] and Szegö [**Sz**]), of characterizing the rational functions of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ which preserve the real line and its Lebesgue measure.

THEOREM 5.2. *Let $f: \overline{\mathbf{C}} \hookleftarrow$ be a rational map. Then $f(\mathbf{R}) \subseteq \mathbf{R}$ and $f|\mathbf{R}$ preserves the Lebesgue measure if and only if there exist $\varepsilon = \pm 1$, $\beta \in \mathbf{R}$, $n \geq 0$, $a_i \in \mathbf{R}$ and $\gamma_i > 0$ for $1 \leq i \leq n$ such that, for all $z \in \mathbf{C}$,*

$$f(z) = \varepsilon \left(z + \beta - \sum_{i=1}^{n} \frac{\gamma_i}{z - a_i} \right). \tag{3}$$

PROOF: If $f: \overline{\mathbf{C}} \hookleftarrow$ is given by (3) then clearly $f(\mathbf{R}) \subseteq \mathbf{R}$ and $\varepsilon f(z)$, $z \in \mathbf{R}^2_+$, defines an inner function of the upper half plane, with $\alpha = 1$ in (1). Then Theorem 5.1 above implies that λ is εf-invariant, hence also f-invariant.

Let now $f: \overline{\mathbf{C}} \hookleftarrow$ be a rational map such that $f(\mathbf{R}) \subseteq \mathbf{R}$ and $\lambda(f^{-1}(A)) = \lambda(A)$ for every Borel set $A \subseteq \mathbf{R}$. Since f preserves the Lebesgue measure, we have

$$\sum_{f(x)=c} \frac{1}{|f'(x)|} = 1$$

for all $c \in \mathbf{R}$. Hence $|f'(x)| > 1$ for all x. In particular $\lim_{x \to +\infty} f(x) = +\infty$ or $-\infty$. Multiplying f by -1 if necessary, we can assume $\lim_{x \to +\infty} f(x) = +\infty$. This means that for some $a > 0$ and $m \in \mathbf{N}$,

$$f(x) = ax^m + \frac{P(x)}{Q(x)} \ ,$$

where P, Q are polynomials with no common root (in $\mathbf{C}$) and degree $P \leq$ degree Q. Let us prove that $m = 1$ and $a = 1$. Since $\lim_{x \to +\infty} f(x) = +\infty$ we have $m \geq 1$ and $a \geq 0$. Then, for $c > 0$ large, the real roots of $f(x) = c$ can be written as $\varphi_1(c) > \varphi_2(c) > \cdots > \varphi_k(c)$, k independent of c and $\varphi_i(c)$ being a C^∞ function of c. Clearly, when $c \to +\infty$, $\lim_{c \to +\infty} \varphi_1(c) = +\infty$, and $\lim_{c \to +\infty} \varphi_i(c)$ is, for $1 < i < k$, a real root of $Q(x)$. The same property holds for $i = k$ when m is odd and $\lim_{c \to +\infty} \varphi_k(c) = -\infty$ if m is even. Then

$$(4) \qquad 1 = |f'(\varphi_1(c))|^{-1} + |f'(\varphi_k(c))|^{-1} + \sum_{1<i<k} |f'(\varphi_i(c))|^{-1} .$$

When $c \to +\infty$, clearly

$$\lim_{c \to +\infty} |f'(\varphi_i(c))|^{-1} = 0$$

and also for $i = k$ if m is odd. Moreover,

$$\lim_{c \to +\infty} \frac{|f'(\varphi_1(c))|}{ma\,|\varphi_1(c)|^{m-1}} = 1$$

and

$$\lim_{c \to +\infty} \frac{|f'(\varphi_k(c))|}{ma\,|\varphi_k(c)|^{m-1}} = 1$$

when m is even. Hence, if $m > 1$,

$$\lim_{c \to +\infty} |f'(\varphi_1(c))|^{-1} = \lim_{c \to +\infty} |f'(\varphi_k(c))|^{-1} = 0,$$

contradicting (4). Then $m = 1$ and

$$\lim_{c \to +\infty} |f'(\varphi_1(c))|^{-1} = \frac{1}{a} \ ;$$

taking the limit of (4) when $c \to +\infty$, we obtain $a = 1$. From the fact that $a = 1$ and $m = 1$, it follows that there exists a connected component V of $f^{-1}(\mathbf{R}^2_+)$ such that V contains a set of the form $\{z \in \mathbf{C} \mid |z| > \rho, \operatorname{Re} z > 0\}$. We claim that $V = \mathbf{R}^2_+$. Take any bounded interval $J_0 \subset \mathbf{R}$ and set $J_1 = f^{-1}(J_0) \cap \mathbf{R}$. Let ψ_{J_i} be the characteristic function of J_i, $i = 0, 1$. Let $\hat{\psi}_{J_i} \colon \mathbf{R}^2_+ \to \mathbf{R}$ be the harmonic extension of ψ_{J_i}. Then

$$\lambda(J_1) = \lim_{t \to +\infty} \frac{1}{t} \hat{\psi}_{J_1}(ti).$$

Now consider the harmonic function $\hat{\psi}_{J_0} \circ f \colon V \to \mathbf{R}$. Observe that $V \cap \mathbf{R} = \mathbf{R} \cap \partial V$ and $(\hat{\psi}_{J_0} \circ f)|(\mathbf{R} \cap \partial V) = \psi_{J_1}$. Moreover, if $V \neq \mathbf{R}^2_+$, ∂V contains arcs contained in $\mathbf{R}^2_+$, and these arcs contain arcs that are connected components of $f^{-1}(J_0)$. Hence $\hat{\psi}_{J_0} \circ f$ is equal to one on these arcs. On the other hand $0 < \hat{\psi}_{J_1}(z) < 1$ for all $z \in \mathbf{R}^2_+$, in particular on those arcs in ∂V where $\hat{\psi}_{J_0} \circ f$ is equal to one. Therefore $\hat{\psi}_{J_1}|\partial V \leq (\hat{\psi}_{J_0} \circ f)|\partial V$ and the inequality is strict on those arcs. Hence $\hat{\psi}_{J_1}|V < \hat{\psi}_{J_0} \circ f$ and even more:

$$\lim_{t\to+\infty} \frac{1}{t}\hat{\psi}_{J_1}(it) < \lim_{t\to+\infty} \frac{1}{t}(\hat{\psi}_{J_0} \circ f)(it)\cdot$$

$$= \lim_{t\to+\infty} \frac{|f(it)|}{t} \cdot \frac{1}{|f(it)|}(\hat{\psi}_{J_0} \circ f)(it).$$

But, since $m = 1$ and $a = 1$, $\lim_{t\to\infty} f(it)/t = i$. Then

$$\lim_{t\to+\infty} \frac{1}{|f(it)|}(\hat{\psi}_{J_0} \circ f)(it) = \lambda(J_0)$$

and

$$\lambda(J_1) < \lim_{t\to+\infty} \frac{|f(it)|}{t} \cdot \frac{1}{|f(it)|}(\hat{\psi}_{J_0} \circ f)(it) = \lambda(J_0).$$

This contradicts the measure preserving property of $f|\mathbf{R}$ and proves the claim. But $V = \mathbf{R}^2_+$ obviously implies that f is an inner function. Hence all its poles are real and simple, and, recalling that $m = 1$ and $a = 1$, we can write

$$f(z) = z + \beta + \sum_{i=1}^{n} \frac{\lambda_i}{z - a_i}$$

with $\beta \in \mathbf{R}$ and $a_i \in \mathbf{R}$, $1 \leq i \leq n$. Moreover $\lambda_i < 0$ for all i, because if some λ_i doesn't satisfy this condition, f maps the half disk $\{z \in \mathbf{C}|\, |z - a_i| < 1, \operatorname{Re} z > 0\}$ onto a set intersecting the *lower* half plane. □

§6 Inner functions and parabolic basins

Let $f: \overline{\mathbf{C}} \hookleftarrow$ be a rational map of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and let $U \subseteq \overline{\mathbf{C}}$ be a fixed connected component of $\overline{\mathbf{C}} - J(f)$, where $J(f)$ denotes the Julia set of f. In this section we suppose that U is a parabolic basin. (See [**Su**] for the classification of the components of $\overline{\mathbf{C}} - J(f)$.) Let $p \in \partial U$ be the fixed parabolic point of f, that is, $f'(p)$ is a root of unity and $\lim_{n \to \infty} f^n(z) = p$ for all $z \in U$. If we choose a uniformization $\psi: D \to U$ and lift $f: U \hookleftarrow$ to D, we obtain an inner function $\hat{f}: D \hookleftarrow$ of the unit disc such that $\psi \circ \hat{f} = f \circ \psi$, with Denjoy-Wolff point $\hat{p} \in \partial D$. Moreover, since the boundary map $\psi^*: \partial D \to \partial U$ of ψ satisfies

$$\lambda_{\hat{z}}((\psi^*)^{-1}(A)) = \lambda_z(A)$$

for each Borel set $A \subseteq \partial U$ and every $z \in U$, $\psi(\hat{z}) = z$, it is clear that the ergodic properties in the harmonic class of $f|\partial U$ are given by the ergodic properties of the boundary map of $\hat{f}$. Thus exactness and recurrence of $f|\partial U$ with respect to the harmonic class of ∂U are consequences of

$$\sum_{n \geq 0} (1 - |\hat{f}^n(z)|) = \infty$$

being valid for some $z \in D$, by Theorem 4.2 and Corollary 4.3. More than this is granted by the following two theorems.

THEOREM 6.1. *Let U be a fixed parabolic basin of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$ and let $\hat{f}: D \hookleftarrow$ be a lifting of $f: U \hookleftarrow$ via a uniformization $\psi: D \to U : \psi \circ \hat{f} = f \circ \psi$. Then, for every $z \in D$ and all $\alpha > \frac{1}{2}$, the inequality*

$$1 - |\hat{f}^n(z)| \geq \frac{1}{n^\alpha}$$

holds for all sufficiently large n.

PROOF: We shall only prove that there exists a point $z_0 \in D$ such that, for all $\alpha > \frac{1}{2}$,

$$1 - |\hat{f}^n(z_0)| \geq \frac{1}{n^\alpha} \tag{1}$$

holds for n sufficiently large. From this, the same relation holds for all $z \in D$ because (1) implies the recurrence of $\hat{f}^*$ (by Theorem 4.2) and then its ergodicity (by Corollary 4.3); now Theorem 3.1 guarantees that $\lim_{n\to\infty} d_P(\hat{f}^n(z), \hat{f}^n(z_0)) = 0$ and therefore

$$\lim_{n\to\infty} \frac{|\hat{f}^n(z) - \hat{f}^n(z_0)|}{1 - |\hat{f}^n(z_0)|} = 0$$

for all $z \in D$. Hence, given $z \in D$, we have that

$$\begin{aligned} 1 - |\hat{f}^n(z)| &\geq 1 - |\hat{f}^n(z_0)| - |\hat{f}^n(z) - \hat{f}^n(z_0)| \\ &\geq (1 - \varepsilon)(1 - |\hat{f}^n(z_0)|) \end{aligned}$$

holds for all $\varepsilon > 0$ and sufficiently large n. This implies that z also satisfies (1), because (1) holds for all $\alpha > \frac{1}{2}$.

To prove (1), let p be the parabolic fixed point of f; then $p \in \partial U \subseteq \overline{\mathbf{C}}$, $f'(p)$ is a root of unity and $\lim_{n\to\infty} f^n(z) = p$ for all

$z \in U$. Without loss of generality we can assume $p = 0$, $f'(p) = 1$ and that the Taylor series of f at $z = 0$ is

$$f(z) = z - z^k + \sum_{n>k} a_n z^n \tag{2}$$

for some $k \geq 2$. The basic theory of parabolic fixed points (see, for instance, Camacho [**Ca**]) implies that there exists an even $0 \leq m < 2(k-1)$ with the following property: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that U contains the sector

$$S = \left\{ z \in \mathbf{C} \middle| |z| < \delta \ , \ \left| \mathrm{Arg}(z) - \frac{\pi m}{k-1} \right| < \frac{\pi(1-\varepsilon)}{k-1} \right\}$$

and, moreover, $f^n(S) \subseteq S$ for all large n, say $n \geq N$ and $\lim_{n\to\infty} f^n(z) = 0$ and $\lim_{n\to\infty} \left| \mathrm{Arg}(f^n(z)) - \frac{\pi m}{k-1} \right| = 0$ for all $z \in S$. Without loss of generality, we may assume that $m = 0$. From (2) it easily follows that for each $z \in S$ there exists $C > 0$ such that

$$|f^n(z)| \geq C \left(\frac{1}{n} \right)^{\frac{1}{k-1}} \tag{3}$$

for all n. Let $\psi: D \to U$ be the uniformization of U; write $\hat{S}$ for the connected component of $\psi^{-1}(S)$ such that $\hat{f}^n(\hat{S}) \subseteq \hat{S}$ for $n \geq N$. Set $\delta = \frac{1-\varepsilon}{k-1}$ and choose a branch $g: S \to \mathbf{C}$ of $z^{1/2\delta}$ which leaves the positive half line $\{t \in \mathbf{R} \mid t \geq 0\}$ invariant. Then $u(z) = \mathrm{Re}\, g(z)$ is a harmonic function on S that vanishes on the sides of S. Consider the harmonic function $u \circ \psi: \hat{S} \to \mathbf{R}$ and observe that there exists $K > 0$ such that

$$-K \log |z| \geq (u \circ \psi)(z)$$

for all $z \in \hat{S}$. In fact, to check this inequality, it sufficies to verify it at the boundary of $\hat{S}$. It holds in the part of $\partial \hat{S}$ mapped by ψ onto the sides of the sector S, since there $u \circ \psi$ vanishes. Taking K large enough we can make it true also on the part of $\partial \hat{S}$ that ψ maps onto $\left\{z \in \mathbf{C} \middle| |z| = \delta \ , \ |\text{Arg}(z)| < \frac{\pi(1-\varepsilon)}{k-1}\right\}$, since $u \circ \psi$ is bounded there. Now, for $z_0 \in \hat{S}$ we have $\hat{f}^n(z_0) \in \hat{S}$ for $n \geq N$ and

$$\begin{aligned} -K \log |\hat{f}^n(z_0)| &\geq (u \circ \psi)(\hat{f}^n(z_0)) = u(f^n(\psi(z_0))) \\ &= \text{Re}\, g(f^n(\psi(z_0))). \end{aligned}$$

Since $\lim_{n\to\infty} |\text{Arg}(f^n(\psi(z_0)))| = 0$, it follows that also

$$\lim_{n\to\infty} |\text{Arg}(g(f^n(\psi(z_0))))| = 0$$

and therefore

$$\begin{aligned} \text{Re}\, g(f^n(\psi(z_0))) &\geq \frac{1}{2} |g(f^n(\psi(z_0)))| \\ &= \frac{1}{2} |f^n(\psi(z_0))|^{\frac{1}{2\delta}} \end{aligned}$$

for large n. Together with (3) we obtain, for large n,

$$\begin{aligned} -K \log \left|\hat{f}^n(z_0)\right| &\geq \frac{1}{2} C^{\frac{1}{2\delta}} \left(\frac{1}{n}\right)^{\frac{1}{2\delta(k-1)}} \\ &= \frac{1}{2} C^{\frac{1}{2\delta}} \left(\frac{1}{n}\right)^{\frac{1}{2\delta(1-\varepsilon)}}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves (1). □

THEOREM 6.2. *Let U be a fixed parabolic basin of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$. Given $x \in U$ and a positive $\varphi \in L^\infty(\lambda_x)$, $\varphi \not\equiv 0$,*

$$\lim_{n\to\infty} \frac{\int \left(\sum_{i=0}^{n-1} (\varphi \circ f^i)\psi\right) d\lambda_x}{\sum_{i=0}^{n-1} \int (\varphi \circ f^i) d\lambda_x} = \int \psi d\lambda_x$$

holds for every $\psi \in L^1(\lambda_x)$.

PROOF: Since $\varphi > 0$, clearly, for each $\psi \in L^1(\lambda_x)$,

$$\left| \frac{\int \left(\sum_{i=0}^{n-1} (\varphi \circ f^i) \psi \right) d\lambda_x}{\sum_{i=0}^{n-1} (\varphi \circ f^i) d\lambda_x} \right| \leq \|\psi\|_1 .$$

Hence it suffices to prove the theorem for a set of functions $\psi \in L^1(\lambda_x)$ that span a dense subspace of $L^1(\lambda_x)$. Let us see that the Radon-Nikodym derivatives $d\lambda_y/d\lambda_x$, $y \in U$, span such a dense subspace of $L^1(\lambda_x)$. To check its density, it is enough, by the Hahn-Banach theorem, to take any $\eta \in L^\infty(\lambda_x)$ such that

$$\int \eta d\lambda_y = \int \eta \left(\frac{d\lambda_y}{d\lambda_x} \right) d\lambda_x = 0$$

for every $y \in U$ and show that $\eta = 0$ a.e. . Let us first assume $U = D$. Then the harmonic extension $\hat{\eta}: D \to \mathbf{R}$ of η satisfies

$$\hat{\eta}(y) = \int \eta d\lambda_y = 0$$

for every $y \in D$, hence $\eta(w) = \lim_{r \to 1} \hat{\eta}(rw) = 0$ for a.e. $w \in \partial D$, by Theorem 1.3. For the general U the same follows using the uniformization theorem and this case $U = D$. Thus we proved the density in $L^1(\lambda_x)$ of the space spanned by $d\lambda_y/d\lambda_x$, with $y \in U$. To complete the proof of the theorem we have to check it for $\psi = d\lambda_y/d\lambda_x$. For these we have

$$\begin{aligned}\sum_{i=0}^{n-1} \int (\varphi \circ f^i) \frac{d\lambda_y}{d\lambda_x} d\lambda_x &= \sum_{i=0}^{n-1} \int (\varphi \circ f^i) d\lambda_y \\ &= \sum_{i=0}^{n-1} \int \varphi d\lambda_{f^i(y)}\end{aligned}$$

for each $y \in U$. Instead of a general $\varphi \in L^\infty(\lambda_x)$, let us first suppose that φ is continuous and let $\hat{\varphi}: U \to \mathbf{R}$ be its harmonic extension. Then

$$\frac{\displaystyle\int\Big(\sum_{i=0}^{n-1}(\varphi\circ f^i)\frac{d\lambda_y}{d\lambda_x}\Big)d\lambda_x}{\displaystyle\sum_{i=0}^{n-1}\int(\varphi\circ f^i)d\lambda_x} = \frac{\displaystyle\sum_{i=0}^{n-1}\hat{\varphi}(f^i(y))}{\displaystyle\sum_{i=0}^{n-1}\hat{\varphi}(f^i(x))}. \tag{4}$$

We want to prove that this quotient converges to $\int \frac{d\lambda_y}{d\lambda_x} d\lambda_x = \int d\lambda_x = 1$ with $n \to \infty$. But the Poincaré distance $d_P(f^i(x), f^i(y))$ converges to 0 with $n \to \infty$, since the boundary map of $f: U \hookleftarrow$ is ergodic (Theorem 3.1). Hence, by Harnack's inequality,

$$\lim_{n\to\infty}\frac{\hat{\varphi}(f^i(y))}{\hat{\varphi}(f^i(x))} = 1. \tag{5}$$

We claim that

$$\sum_{i\geq 0}\hat{\varphi}(f^i(x)) = \infty. \tag{6}$$

This follows by lifting the problem to the disc D, via a uniformization. In fact, let us denote by $\hat{\varphi}_1$ the lifting of $\hat{\varphi}$ and by $\hat{f}$ the lifting of f. Now $\hat{\varphi}_1$ is harmonic and positive, therefore for some $C > 0$ we get $\hat{\varphi}_1(z) > C(1-|z|)$ for all $z \in D$. Since U is a fixed parabolic basin of f, Theorem 6.1 above gives $1 - |\hat{f}^i(z)| > i^{-\alpha}$ for all $\alpha > \frac{1}{2}$, $z \in D$ and large i, and therefore

$$\sum_{i\geq 0}\hat{\varphi}_1(\hat{f}^i(z)) = \infty$$

for all $z \in D$. This proves (6); since (6) together with (5) implies that (4) converges to 1 with $n \to \infty$, we proved the desired property for continuous φ. For the general case of $\varphi \in L^\infty(\lambda_x)$, we observe that the continuity of φ was used only to grant the existence of a harmonic function $\hat{\varphi}: U \to \mathbf{R}$ which satisfies

$$\hat{\varphi}(p) = \int \varphi d\lambda_p \tag{7}$$

for every $p \in U$. Such a function, that extends the concept of harmonic extension, is easily obtained for every positive (hence for all) $\varphi \in L^1(\lambda_p)$, $p \in U$, by taking a sequence of continuous functions $\varphi_n: \partial U \to \mathbf{R}$ which are a Cauchy sequence in $L^1(\lambda_{p_0})$, for some $p_0 \in U$, and $\varphi_n \to \varphi$ a.e. (in the harmonic class). Then

$$\begin{aligned} |\hat{\varphi}_n(p_0) - \hat{\varphi}_m(p_0)| &= \left| \int \varphi_n d\lambda_{p_0} - \int \varphi_m d\lambda_{p_0} \right| \\ &\leq \int |\varphi_n - \varphi_m| \, d\lambda_{p_0}. \end{aligned}$$

Hence $\{\hat{\varphi}_n(p_0)\}$ is a convergent sequence. By Harnack's inequality, $\{\hat{\varphi}_n\}$ converges uniformly on compact sets to a harmonic function $\hat{\varphi}: U \to \mathbf{R}$ and, for all $p \in U$, we get

$$\hat{\varphi}(p) = \lim_{n \to \infty} \hat{\varphi}_n(p) = \lim_{n \to \infty} \int \varphi_n d\lambda_p = \int \varphi d\lambda_p,$$

thus providing the general $\varphi \in L^\infty(\lambda_p)$ with a harmonic function $\hat{\varphi}: U \to \mathbf{R}$ satisfying (7). $\square$

REFERENCES

[Aa1] J. Aaronson, On the ergodic theory of non-integrable functions and infinite measure spaces. Israel Jour. of Math., 27 (1977), 163–173.

[Aa2] J. Aaronson, Ergodic theory for inner functions of the upper half plane. Ann. Inst. Henri Poincaré, Section B, vol. XIV (1978), 233–253.

[Aa3] J. Aaronson, A remark on the exactness of inner functions. J. London Math. Soc., 23 (1981), 469–474.

[Ad-W] R. Adler, B. Weiss, The ergodic, infinite measure preserving transformation of Boole. Israel Jour. of Math., 16 (1973), 263–278.

[B] G. Boole, On the comparison of transcendents with certain applications to the theory of definite integrals. Philos. Trans. Roy. Soc. London, 147 (1857), 745–803.

[Ca] C. Camacho, On the local structure of conformal mappings and holomorphic vector fields in $\mathbf{C}^2$. Asterisque, 59–60 (1978), 83–94.

[Cr1] M. Craizer, The entropy of inner functions. Preprint, IMPA (1989).

[Cr2] M. Craizer, Inner functions are Bernoulli. Preprint, PUC (1990).

[De] A. Denjoy, Sur l'itération des fonctions analytiques. C.R. Acad. Sci. Paris, 182 (1926), 255–257.

[Du] P. Duren, Theory of H^p spaces. Academic Press, New York, 1970.

[F-L-M] A. Freire, A. Lopes, R. Mañé, An invariant measure for rational maps. Bol. Soc. Bras. Mat., 14 (1983), 45–62.

[G] J. Garnett, Bounded analytical functions. Academic Press, New York, 1981.

[He] M. Heins, On the finite angular derivatives of an analytic function mapping the disk into itself. J. London Math. Soc., 15 (1977), 239–254.

[Ho] E. Hopf, Ergodentheorie. Springer Verlag, Berlin, 1937.

[M] R. Mañé, On the uniqueness of the maximizing measure for rational maps. Bol. Soc. Bras. Mat., 14 (1983), 27–43.

[M-R] R. Mañé, L.F. da Rocha, Julia sets are uniformly perfect. Preprint, IMPA (1990).

[Lj] M.J. Ljubich, Entropy properties of rational endomorphisms of the Riemann sphere. Ergod. Th. & Dynam. Sys., 3 (1983), 351–385.

[Lo] A. Lopes, Equilibrium measures for rational maps. Ergod. Th. & Dynam. Sys., 6 (1986), 393–399.

[Pa] G. Pólya, Problem 107. Jber. Deutsch. Math. Verein. 40, 2 Abt. (1931), p. 81.

[Pe] Ch. Pommerenke, On the iteration of analytic functions in a half plane, I. J. London Math. Soc., 19 (1979), 439–447.

[Pi] F. Przytycki, Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. Invent. Math., 80 (1985), 161–179.

[Su] D. Sullivan, Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains. Ann. Math., 122 (1985), 401–418.

[Sz] G. Szegö, Problem. Jber. Deutsch. Math. Verein. 43, 2 Abt. (1934), 17–20.

[T] M. Tsuji, Potential theory in modern function theory. Maruzen, Tokyo, 1959.

[W] J. Wolff, Sur l'itération des fonctions holomorphes dans une region. C.R. Acad. Sci. Paris, 182 (1926), 42–43; 200-201.