Dorier, Jean-Luc (Coordinator):

L'Enseignement de l'Algèbre Linéaire en Question

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This review in English contains a detailed overview in order to make the book – which is written in French – more accessible for English speaking teachers and researchers.

In the preface André Revuz points out the paradox of linear algebra: although linear algebra may look the simplest of all theories in mathematics, the problems encountered in teaching it are out of proportion with its intrinsic difficulties.

Until recently little research was done on the teaching of linear algebra. This book gives an overview of research on the teaching and learning of linear algebra in the first year of university, and this from an international perspective (France, USA, Canada). It is somewhat a counterpart for the many reports about the teaching and learning of calculus that have been published in the beginning of the nineties.

The book is intended to provide teachers of linear algebra with information obtained by analysis of experiments, to discuss difficulties and to present alternatives, but without giving definite answers. It also addresses researchers in didactics of mathematics in general who are interested in experiments and their evaluation at university level.

1. Organisation of the book

The book consists of two parts. The first part (78 pages), written by the coordinator Jean-Luc Dorier, presents an epistemological analysis of the historical genesis of the theory of vector spaces, from the 17th up to the middle of the 20th century. The author shows that the modern theory is the result of a long process of generalisation and unification, reaching into many domains of mathematics. At the end of the process the concept of vector space emerges as an essential tool for linearity. This historical analysis is useful for teachers in order to understand the failure of instruction of linear algebra for a considerable number of students, and to allow for a better epistemological control.

In the second part several authors present results on almost ten years of research about the teaching and learning of linear algebra. The first four chapters (111 pages) give a synthesis of the work done by a French group consisting of Jean-Luc Dorier, Aline Robert, Jacqueline Robinet and Marc Rogalski. The next three chapters (54 pages) present an analysis of problems related to the learning of linear algebra in the USA (Guershon Harel) and Canada (Joel Hillel and Anna Sierpinska). In the last chapter (21 pages) Jean-Luc Dorier presents four more French research papers.

We give a detailed description of the content and add personal comments.

2. Content of part 1

The author describes the evolution of linear algebra from the first theoretical studies on linear equations until the development of the modern theory of vector spaces and its integration in the curricula. He points out and analyses the different stages of unification and generalisation. He also shows the sometimes heavy resistance after each unification, and especially against the axiomatic theory. Some questions on the teaching of linear algebra are added to this historical overview. The historical-epistemological-didactic interactions serve as a common element in the different research projects described in part 2.

Chapter 1 describes the analytic and geometric origins of linear algebra, resulting in a first unification.

The first paragraph shows the development of the first concepts of linear algebra within the framework of linear equations. Although techniques of substitution and elimination have been used since antiquity, it is only in 1750 that a first descriptive and qualitative approach to linear equations is made by Euler, using a notion of "inclusive dependence". In the same year Cramer publishes a text marking the beginning of the theory of determinants. But here emphasis is again on computation and not on a qualitative study of linear equations, for which we have to wait until the middle of the 19th century, when the concept of rank (in terms of determinants) is introduced. In 1875 Frobenius introduces the notion of linear dependence simultaneously for n-tupels and linear equations, a unification which is a step in the direction of the modern notions of vector, duality and rank.

The second paragraph describes the epistemological relationship between geometry and linear algebra, which originates in the first half of the 17th century with the development of analytic geometry by Descartes and Fermat. This "algebraisation" of geometry is a success as it allows to simplify and unify. Nevertheless critique is formulated because the role played by intuition is lost and because computations do not explain geometric results. Leibniz, Möbius and Bellavitis each develop a geometric calculus . The work of Gauss on complex numbers (1831) is accepted as a more intrinsic model for geometric computations in the plane than Cartesian coordinates. After several attempts to generalise the geometric representation of complex numbers to three dimensions, it is finally Hamilton who publishes his theory of quaternions in 1844.

The third paragraph discusses the extension theory of Grassmann (1844), which may be considered as a first formal theory of finite dimensional linear algebra, several decades ahead of all other such developments. It is only appreciated however around 1920 by E. Cartan who uses the external product to create an external algebra, the foundation of multilinear algebra. The extension theory is based on geometric intuition, takes a very formalistic position right from the start, but the approach is not axiomatic. It anticipates many results in linear algebra that are only discovered almost a century later and it clarifies the role played by geometry and algebra in the development of linear algebra.

In the fourth paragraph the author describes how a first stage of unification in finite dimension is realised in the second half of the 19th century when the relationship between linear problems and methods in algebra, geometry and physics becomes clear. Successive contributions by Euler, Cauchy and Cayley on matrix representation are discussed. This unification is essentially centered on determinants and geometric analogy and is still far away from a formal approach in terms of algebraic structure.

Chapter 2 analyses the development of a formal axiomatic theory.

The first paragraph describes the early axiomatic approaches in linear algebra. The approaches by Peano (1888), Pincherle (1901), Burali-Forti and Marcolongo (1909) all have their origin in Grassmann's work. They are linked to the geometric space model, except for Pincherle's theory which is more complete and advanced, centered on the study of operators, and includes ideas for generalisation to infinite dimension. Pioneering work is also done by H. Weyl (1918) in higher dimensions. A different approach to an axiomatic theory originates in the extension of fields by Dedekind (1893). His work is one of the starting points of the development of modern algebra in Germany during the first three decades of the 20th century. From the many contributions of modern algebra to linearity only the contribution by Steinitz (1910) is discussed in the book. In the second edition of van der Waerden's survey book on Modern Algebra (1937), based on the courses taught by Noether and Artin, the notion of vector space becomes central, the theory of equations is treated as an application and the importance of determinants is reduced. All these axiomatic approaches can be considered as a way to give a solid foundation to results that have been obtained at the end of the 19th century in the framework of coordinates and to lay a theoretical basis for a vector calculus freed from the model of Cartesian coordinates.

The next paragraph discusses the evolution of linear problems in infinite dimension, marking the beginning of functional analysis. It is remarkable that for a period of thirty years, until 1920, there is no interaction with the axiomatic approaches in finite dimension. Functions are identified with series, allowing for methods and tools based on an analogy with the finite dimensional case, generalising coordinates and determinants. The works by Wronski and Cauchy at the beginning of the 19th century allow to establish a theoretical basis for systems of linear differential equations towards the end of the century. Fundamental results on infinite linear systems are obtained by Fourier (1822), Hill (1877), Poincaré (1886 and 1900) and von Koch (1891), extending the theory of determinants to infinite dimension, with certain restrictions for convergence. Integral equations are also a source of infinite dimensional linear problems. Fredholm (1903) works in analogy with the finite case, but he introduces tools of infinite dimension, using operators to study function equations. Hilbert (1904–1910) continues Fredholm's work, with a diversity of new results and methods, and he justifies the analogy with finite dimension. His main interest is in solving analysis problems, not in trying to formalise his methods as a means of unification, or to give a geometric interpretation. Important progress toward the definition of Hilbert space is made by Riesz and Fischer (1907), with independent proofs of the isomorphism between l^2 and L^2 and generalisations by Riesz (1910) to L^p spaces. The topological nature of the problems allows Riesz to give a geometric interpretation of Euclidean distance on a space with an infinite number of coordinates, drawing a parallel with the duality analytic/synthetic geometry. In 1908 Schmidt publishes most of the now classical results on the geometry of Hilbert spaces. Another implicit approach to the topology of function spaces has its origin in the calculus of variations, with the work of Weierstrass, Volterra, Pincherle, Ascoli, Arzela, Hadamard and also Fréchet. The latter defines different topologies on one space, a decisive step towards the axiomatisation of topological vector spaces. So there is a convergence of ideas from algebra, geometry and analysis. Procedures which are repeated in more and more general settings and the introduction of a geometric language show similarities between the different works and lead to the concept of function space. But the formalisation concerns the topological structure, whereas the algebraic structure remains implicit. It finally results in an axiomatic definition of a complete normed vector space with contributions by Wiener (1920), Hahn (1922) and especially Banach (1920). This work is continued by Fréchet (1925) and elaborated further by Banach (1929). When working on the analogy (established by Schrödinger) of two totally different approaches in quantum mechanics, each resulting in an eigenvalue problem, von Neumann (1927) gives an axiomatic definition of a Hilbert space. The establishment of theories in infinite dimensional finally allows, by analogy, to rework the finite dimension case and give it a more formal basis. At this point Hilbert spaces play an important role. In 1932 von Neumann publishes a general study of Hilbert spaces starting from the axiomatic definition of a finite dimensional hermitian space.

The third paragraph describes the elaboration of modern theory of vector spaces since 1930 and its introduction in French education.

At the beginning of the thirties the axiomatic theory of vector spaces is presented in the books by Banach (1932), van der Waerden (1930–31) and von Neumann (1932). A decisive step to unify the viewpoints of algebraists and analysts is made by Gelfand (1941) when he introduces Banach algebras. American mathematicians, attracted by the modern theory, take over from Germany which is preparing for war. Birkhoff and Mac Lane (1941) publish their Survey of Modern Algebra, intended for university teaching and containing an axiomatic approach of vector spaces and linear transformations. Halmos (1942) publishes his book on finite dimensional vector spaces, showing links between Hilbert spaces, matrix theory and ideas in analysis, and insisting on the geometric aspect of function spaces. In France young mathematicians, impressed by the modern theory published by van der Waerden, react against the conservatism of the older generation. Under impulse of H. Cartan and A. Weil the Bourbaki group is formed in 1934, and innovations in teaching take place at certain universities in the province. Bourbaki's book on linear and multilinear algebra (1947) is not much appreciated abroad because its central notion is the more general concept of module. Important but less radical contributions in France are the course taught by Julia at the Sorbonne (1935) and the book published by Lichnerowicz (1947). So in the period 1930–1950 the axiomatic notion of vector space becomes accepted as a tool.

The appointment of Choquet in Paris (1954) results in a quick modernisation of mathematics teaching in all French universities. Algebraic structures, linear algebra included, are taught from the first year of university. In 1969 the axiomatic theory of finite dimensional vector spaces becomes part of the French secondary school mathematics curriculum and absorbs most of the geometry teaching. Since the beginnings of the eighties however modern algebra has been removed gradually from the curriculum and vector geometry has been reduced. The author considers this to be an explanation for the difficulties encountered in the assimilation of linear algebra in the first year at university. I cannot completely agree with this: Belgian students have the same types of problems and are making the same mistakes as those illustrated in part 2 of the book, while vector space theory is still taught in secondary school.

3. Content of part 2

In the first four chapters of part 2 the authors Jean–Luc Dorier, Aline Robert, Jacqueline Robinet and Marc Rogalski analyse the instruction of a first course in linear algebra in French universities (first year, ages 18–20).

Chapter 1, written by all four authors, is of diagnostic nature. It analyses difficulties and mistakes. It tries to explain how the unifying and generalising nature of linear algebra – as described in the historical analysis – is a source of difficulties as well for teaching as for learning the subject. The main obstacle seems to be the formalism in the theory of vector spaces and the fact that many students do not master any degree of formalism themselves. The authors describe successive studies carried out between 1987 and 1995.

They start with a detailed analysis of a test taken from 379 students in their second year, aimed at revealing knowledge and conceptions after one year of instruction and after passing the first year. The results on the test are poor, with an average of less than 40% correct answers on test questions that are very basic. Although most of us have had similar experiences, looking at such results is always shocking and leads to reflection on one's own teaching. The answers reveal that for most students linear algebra is only a catalogue of very abstract notions, they are submerged under an avalanche of new symbols, new definitions and new theorems.

The second study is based on the analysis of a pre-test and classwork. The level of the tasks in linear algebra over a year of instruction is described, as well as the difficulties and the individual evolution of students. Details and concrete examples are missing here.

The last part analyses in detail written tasks, tests, exams, as well as classwork in small groups between 1991 and 1994. A first aim is to give a precise idea of difficulties and errors on specific tasks concerning subspaces and the rank of a set of vectors, and of the evolution during instruction of the procedures used by the students. This part is not very well written. Typographical errors in an equation (p. 126) and a poor lay-out make that the reader sometimes has to puzzle to understand. Moreover the printing of subscripts and superscripts in small font size is very poor and makes some mathematics almost unreadable. The general message is clear however. A second aim is to investigate the impact of weaknesses in logic and set theory (inclusion, implication, parameters, equations as constraints) on the performance in linear algebra. As this impact seems considerable, the authors suggest that instruction of naive set theory and elementary logic be integrated in the linear algebra course. In particular, instruction may benefit from the logical and set-theoretical links between equations and geometric objects.

In Chapter 2 Aline Robert introduces "levels of conceptualisation" to characterise different levels in the organisation of mathematical knowledge about a certain concept. It is the teacher's task to help smoothen the transition from one level to the next one. She gives examples of topics in secondary school in France, where students are prepared for a higher level of conceptualisation, but where this level is not attained as formalisation is not completed. One example is linear algebra: students learn to manipulate vectors in the plane, but vector spaces are not formally introduced. Taking into account the difficulties that university students encounter with the formalism of linear algebra (see Chapter 1) and the poor results after the introduction of vector spaces in secondary schools in my country, it is my opinion that giving the secondary school students only "the beginning of the story", as the author states it, may be the better approach anyway.

In Chapter 3 Marc Rogalski describes and discusses an experimental teaching project, which is based on earlier findings (Chapter 1) and has been implemented since 1991. This teaching takes into account the nature of linear algebra as a discipline that formalises, unifies, generalises and simplifies. It focuses on making students understand and accept theoretical and generalising detours. To realise this, a long preparation, with multiple changes in framework and points of view, precedes the introduction of the first concepts in linear algebra. Reflections at "meta" level play an important role. Prerequisites are important as well and some, such as elementary set theory, logic and Cartesian geometry, are taken care of within the linear algebra course. The instruction starts with Gaussian elimination, is centered around the concept of rank, gives equal importance to equations and parametrisation, presents the equation $T(x) = b$ as an important tool in modelling and emphasises the choice of an appropriate basis when solving a problem.

In Chapter 4 J. L. Dorier, A. Robert, J. Robinet and M. Rogalski explain what they call a "meta level". It is a recourse to information on knowledge about mathematics (general methods, structures, organisation and rules), which may lead to reflection about ones own activities, the way to learn or to do mathematics and the nature itself of mathematics. The distinction between mathematics and meta mathematics is not absolute, it depends on the person involved. The introduction of meta components is motivated and illustrated by three experiments.

The first experiment is an introduction to the notion of structure (of vector spaces) and uses already acquired knowledge and skills in solving equations. In this artificial context students arrive at formulating general properties needed to solve equations and are able to enrich their knowledge on algebraic structures at meta level. Ideas of generalisation and simplification are also present in the second experiment about polynomial interpolation (formula of Gregory). As the number of data increases, students recognise the limit of their competence with regard to interpolation and realise that a new framework (choice and comparison of general formulas) has to be chosen. This creates a fertile ground for reflections at meta level. In fact students are explicitly told that solving the given mathematical problem is not sufficient. They also have to reflect on and state their opinion about the validity of different methods for similar problems in different contexts. The last experiment is about classification of problems in different domains in terms of their solvability by the linear model $T(x) = b$. It shows that a spontaneous meta approach by the students does not take place without explicit intervention by the teacher.

Finally the authors discuss the difficulties encountered in terms of evaluation of the effects of reflections at meta level and present new perspectives for interventions and analyses.

Guershon Harel, the author of Chapter 5, was a member of the Linear Algebra Curriculum Study Group (LACSG) in the USA, which published a set of recommendations for the beginning linear algebra course at university level (Carlson et al. 1993). In the first part G. Harel presents the course content proposed by LACSG and personal interpretations of the LACSG recommendations. They concern proofs in the mathematics curriculum, the introduction in secondary school of elementary notions of linear algebra related to geometry, and the incorporation of MATLAB (or a similar package) in the linear algebra course.

Next he describes a theoretical framework in terms of principles of both teaching and learning, namely the principles of concretisation, of necessity and of generalisability. The principle of concretisation states that a student can only make abstraction of the mathematical structure of a given model on condition that the elements of the model are conceptual entities for the student. Two examples are given to illustrate the meaning of conceptual entity as defined by Greeno. In order to realise this principle students have to construct their understanding of concepts in concrete contexts. This preliminary condition is essential in experiments that have been carried out by Harel in Israel, both in secondary and in higher education. In these experiments geometry plays a central role in constructing new concepts. The principle of necessity refers to the intellectual need of the student for what he is being taught, including justification by proofs. The principle of generalisability concerns the abstraction of concepts that have been introduced in a particular context. Didactic situations must therefore contain appropriate constraints that allow for abstraction of mathematical concepts while at the same time keeping control in a realistic context. Finally Harel explains how these principles support his personal elaboration on the LACSG recommendations.

In Chapter 6 Joel Hillel explains how the difficulties of American and Canadian students in their first linear algebra course at university level are partly due to their first encounter with proofs and with a formal general theory. But he agrees that there must be other reasons as well, because French students, with a much more formal preparation in mathematics, also experience difficulties. He therefore wants to focus on conceptual difficulties inherent to linear algebra.

He starts by an analysis of the different levels of representation that are used in linear algebra: the language of the general theory of vector spaces, the language of the more specific theory of real n-tupels and the geometric language of 2 or 3 dimensional space. He discusses the use of these languages, the way they interact and the difficulties encountered in passing from one to another. He focuses in particular on the representation of vectors and linear operators at the three different levels: abstract, algebraic and geometrical. Students have difficulties in identifying a vector with its representations in different bases. He illustrates how things get even worse when linear operators are represented either by their standard matrix or by matrices in other bases. He explains that starting linear algebra on n-tupels, as is done in North-America, is in a certain way an obstacle in understanding the general theory and accepting objects like polynomials and matrices as vectors. The fact that an n-tupel can be a representation of any vector in finite dimension is difficult to accept for many students. Class discussions at meta level may help them to understand better.

Chapter 7, written by Anna Sierpinska, Astrid Defence, Tsolaire Khatcherian and Luis Saldanha, discusses their latest research project on the analysis of three ways of reasoning in linear algebra: synthetic-geometric, analytic-arithmetic and analytic-structural. These three ways of reasoning are linked to the three levels of representation introduced by J. Hillel. The paper characterises the three ways of reasoning, gives several examples and draws a parallel with two movements in the historical development of linear algebra: first the "arithmetisation" of space, namely the transition from synthetic to analytic geometry, then the structuralisation, due to which space becomes an algebraic system, closed under certain operations.

Spontaneously students use different ways of reasoning, but not always in their pure form, in fact they often use a mixture. As an illustration several protocols of tutoring sessions at college level are discussed. The authors suggest to try to use the creativity and preferences of students for improving linear algebra teaching. A possibility could be to use the preference for computations to direct analytic reasoning into numerical thinking.

In Chapter 8 Jean-Luc Dorier comments on four recent research projects in France.

The first research, by Kallia Pavlopoulou, is an application and verification in linear algebra of the theory of R. Duval on registers of semiotic representation. K. Pavlopoulou distinguishes three such registers: graphics (arrows), tables (columns of coordinates) and symbols (axiomatic theory). So here again there is a link to the three levels of representation introduced by J. Hillel. A detailed description is given of activities involved in converting statements from one register into another and of specific difficulties for different types of conversion. An analysis of several tests shows that students do not succeed in converting without some specific teaching. A teaching experiment centered explicitly on the conversion of registers shows positive results.

The second research by Marlene Dias is related to the previous one, as well as to work done by Rogalski, Hillel and Sierpinska. Cognitive flexibility between Cartesian and parametric representations is presented as an essential component in the learning process. It can not be reduced to just a technical ability in semiotic representation, but it involves also more global conceptual elements. As in Rogalski's work the research is centered on the notion of rank.

The third research by Philippe Bardy, Denis Le Bellac, Roger Le Roux, Jean Memin and Danielle Saby is based on two tests. It does not present a deep didactic analysis, but gives a precise description of difficulties in linear algebra after one course at tertiary level. The tests focus on the notions of kernel, span, dimension and linear independence. Difficulties result from the extensiveness of the new terminology introduced, from the new types of proof and from the nature of the universal "model" of linear algebra. Remedial proposals are formulated.

The last research by Ahmed Behaj is on structuralisation of knowledge. His theoretical work interacts with an experimental part based on interviews of teachers and students about the links between the notions of subspace, span, linear dependence, basis and dimension. Structuralisation is influenced, not only by the organisation of the course, but also by the worked examples and exercises, and by the perception of the utility and importance of the different notions.

In the conclusion J. L. Dorier and A. Robert want to give an overview of common ideas in the different research projects and to point at some possible common research tracks. But they focus again on one of the French research projects.

4. References

The book contains an extensive list of references (19 pages), organised into 3 parts. The historical bibliography contains references for the first part of the book. Next there is a list with specialised works on the teaching of linear algebra. The general bibliography contains all other references.

5. Final comments

The book gives a clear view on the difficulties encountered by first year university students in learning linear algebra. The international perspective shows that in essence these difficulties are not due to a lack of preparation. Throughout the text the coordinator J. L. Dorier tries to explain the difficulties as being the result of the unifying and generalising nature of linear algebra, as described in his historical analysis.

The research projects on the teaching and learning of linear algebra present general ideas and illustrate the effect of some global choices. Improvement is shown in subdomains for which a specific training was organised. But in experimental teaching, as Dorier states himself (p. 275), students always are more receptive and show profit in the domain linked to this teaching. So the book does not give a miracle solution under the form of a model course that can solve all problems. There probably is not such a solution. The reader may benefit however from an awareness of the nature of certain difficulties and the knowledge of some partial remedies.

The essential benefit is in the analysis of the genesis of linear algebra, showing clearly that linear algebra is merely a language, a way to see things, a tool to organise mathematical knowledge, more than subject matter that is mathematically important in itself.

As is the case in most books that are a collection of separate papers, there is not always uniformity in the presentation, nor in the writing style. Emphasis is more on French research and teaching than on American. But the coordinator tries to link papers as much as possible, especially by referring to the unifying and generalising nature of linear algebra.

It is unfortunate that the print of subscripts and superscripts is so poor all over the text, making some mathematics almost unreadable. Printing on better quality paper may solve this problem.

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