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De Rham's Singular Function and Related Functions

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Abstract. For de Rham's singular function we derive new properties, in particular some formulas which express its self-similarity. Inversions and compositions of de Rham's function are considered as well as generalizations of de Rham's functional equations which have a connection to the $(3n+1)$-iteration of Collatz.

Keywords: *De Rham's singular function, inverse singular functions, compositions of such functions, functional equations, Collatz problem*

AMS subject classification: 39 B 22, 39 B 62, 26 A 30, 26 A 48

1. Introduction

It is well known that for a fixed $a \in (0,1)$ the system of functional equations

$$\left.\begin{aligned} \varphi\Big(\frac{t}{2}\Big) &= a\,\varphi(t) \\ \varphi\Big(\frac{t+1}{2}\Big) &= a + (1-a)\,\varphi(t) \end{aligned}\right\} \qquad (t \in [0,1]) \tag{1.1}$$

has a unique bounded solution. This solution φ is continuous, strictly increasing with $\varphi(0) = 0$ and it has the representation

$$\varphi\left(\sum_{j=0}^{\infty} 2^{-\gamma_j}\right) = \sum_{j=0}^{\infty} a^{\gamma_j} q^j \tag{1.2}$$

where $q = \frac{1-a}{a}$, $\gamma_j \in \mathbb{N}$ and $\gamma_0 < \gamma_1 < \gamma_2 < \dots$, in particular $\varphi(\frac{1}{2}) = a$ and $\varphi(1) = 1$. In the case of need it will be denoted more precisely by φ_a. The case $a = \frac{1}{2}$ is elementary, namely $\varphi(t) = t$. However, in the case of $a \neq \frac{1}{2}$ the solution φ has the interesting property that it is a strictly singular function, i.e. a continuous and strictly increasing function with derivative zero almost everywhere. This solution φ was first constructed by de Rham [9], so that it is called de Rham's function (cf. [3], where a detailed history of the whole context can be found). Formula (1.2) defines a continuous solution of system (1.1) also in the case of complex a with $|a| < 1$ and $|1-a| < 1$.

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In the following we derive some new properties for de Rham's singular function φ and for some similar functions which are solutions of generalized de Rham's functional equations. We consider the self-similarity of de Rham's function, show that the inverse of a singular function is also singular, and deal with compositions of the studied functions. Finally, a connection to the $(3n+1)$-iteration of Collatz is pointed out.

2. De Rham's singular function

If for $t \in [0,1]$ we introduce the dyadic representation $t = 0.d_1 d_2 \cdots$ with $d_j \in \{0,1\}$, then according to de Rham [8], the formula $\varphi(0) = 0$ and representation (1.2) can be gathered up as

$$\varphi(t) = \sum_{j=0}^{\infty} a^{j+1} d_{j+1} q^{d_1+\ldots+d_j} \tag{2.1}$$

since we have $d_{j+1} = 1$ for $j+1 = \gamma_i$ and $d_{j+1} = 0$ else, so that $d_1 + \ldots + d_j = i$. The series in (2.1) appears also in [6], however, in another context. For $t = \frac{k}{2^\ell}$ sum (2.1) terminates:

$$\varphi\Big(\frac{k}{2^\ell}\Big) = \sum_{j=0}^{\ell-1} a^{j+1} d_{j+1} q^{d_1+\ldots+d_j} \; . \tag{2.2}$$

For non-negative integers k with the dyadic representation $k = \delta_1 \delta_2 \cdots \delta_n$ $(\delta_j \in \{0,1\})$ we need the binary sum-of-digits function (cf. [1])

$$\nu(k) = \delta_1 + \ldots + \delta_n \tag{2.3}$$

which satisfies the equations

$$\left.\begin{aligned} \nu(2k) &= \nu(k) \\ \nu(2k+1) &= \nu(k) + 1 \end{aligned}\right\} . \tag{2.4}$$

Next, we shall show that the terms

$$x_n(t) = d_{n+1} q^{d_1+\ldots+d_n} \tag{2.5}$$

appearing in (2.2) are step functions with special properties, so that (2.1) is a representation of φ by a series of step functions.

Proposition 2.1. *For $t \in [0,1)$ functions (2.5) satisfy the recursions*

$$\left.\begin{aligned} x_0(t) &= 0, \quad x_{n+1}(t) = x_n(2t) \qquad\quad \text{for } 0 \le t < \tfrac{1}{2} \\ x_0(t) &= 1, \quad x_{n+1}(t) = q x_n(2t-1) \; \text{for } \tfrac{1}{2} \le t < 1 \end{aligned}\right\} , \tag{2.6}$$

and if we extend $x_0(t)$ for $t \ge 1$ by

$$x_0(t) = q^{\nu(k)} x_0(t-k) \tag{2.7}$$

where $k \in \mathbb{N}$ *and* $k = [t]$, *then*

$$x_n(t) = x_0(2^n t) \tag{2.8}$$

for $0 \le t < 1$.

Proof. For the numbers d_j from the dyadic representation $t = 0.d_1 d_2 \cdots$ let us write $d_j = d_j(t)$. In the case of $0 \le t < \frac{1}{2}$ we have $d_1 = 0$ and $d_j(2t) = d_{j+1}(t)$ for $j \in \mathbb{N}$. In the case of $\frac{1}{2} \le t < 1$ we have $d_1 = 1$ and $d_j(2t-1) = d_{j+1}(t)$. Hence, (2.5) immediately implies (2.6). Solving (2.6) recursively, we find $x_n(t) = q^{\nu(k)} x_0(2^n t - k)$ for $k \le 2^n t \le k+1 \le 2^n$ and according to (2.7) finally (2.8) ∎

Proposition 2.2. *The solution* φ *of system* (1.1) *satisfies the functional equations*

$$\varphi\Big(\frac{k+\tau}{2^\ell}\Big) = \varphi\Big(\frac{k}{2^\ell}\Big) + a^\ell q^{\nu(k)} \varphi(\tau) \tag{2.9}$$

where $\ell \in \mathbb{N}$, $k = 0, 1, ..., 2^\ell - 1$, $\tau \in [0,1]$, *and for* $t = \frac{k}{2^\ell}$ *with* $k = 0, 1, ..., 2^\ell$ *it has the finite representation*

$$\varphi\Big(\frac{k}{2^\ell}\Big) = a^\ell \sum_{j=0}^{k-1} q^{\nu(j)} \ . \tag{2.10}$$

Proof. In view of (2.3) representation (2.1) can be written as

$$\varphi(t) = \sum_{j=0}^{\ell-1} a^{j+1} d_{j+1} q^{d_1 + \ldots + d_j} + a^\ell q^{\nu(k)} \sum_{j=0}^{\infty} a^{j+1} d_{\ell+j+1} q^{d_{\ell+1} + \ldots + d_{\ell+j}}$$

with $k = [2^\ell t]$. Substituting $t = \frac{k+\tau}{2^\ell}$ with $\tau \in [0,1]$, the first sum on the right-hand side is equal to $\varphi(\frac{k}{2^\ell})$ in view of (2.2). Since $\tau = 2^\ell t - k$ has the dyadic representation $\tau = 0.d_{\ell+1} d_{\ell+2} \cdots$, the last series is equal to $\varphi(\tau)$, so that (2.9) is proved. Now, in view of $\varphi(1) = 1$, representation (2.10) follows from (2.9) with $\tau = 1$ by summation ∎

Note that equations (2.2) and (2.10) are quite different in their external shape. Equation (2.9) has the following counterpart with respect to the left of $\frac{k}{2^\ell}$:

$$\varphi_a\Big(\frac{k-\tau}{2^\ell}\Big) = \varphi_a\Big(\frac{k}{2^\ell}\Big) - a^\ell q^{\nu(k-1)} \varphi_{1-a}(\tau) \tag{2.11}$$

where $k = 1, 2, ..., 2^\ell$ and $\tau \in [0,1]$, which can easily be derived from (2.9) by means of the later formula (2.12). Equations (2.9) and (2.11) express very distinctly the self-similarity of de Rham's function (with respect to the dyadic points), which is well known in the theory of fractals (cf. [5]).

Proposition 2.3. *The solution* φ_a *from system* (1.1) *with* $0 < t < 1$ *and* $0 < a < 1$ *is also strictly increasing with respect to* a. *It has the property*

$$\varphi_{1-a}(t) = 1 - \varphi_a(1-t) \ . \tag{2.12}$$

The family of all curves $y = \varphi_a(t)$ *with* $0 < a < 1$ *fills out the whole open square* $0 < t, y < 1$.

Proof. If h is a differentiable strictly increasing function of a with $0 < h < 1$ for $0 < a < 1$, then the function $a \mapsto a + (1-a)h(a)$ is strictly increasing. Since $\varphi_a(\frac{1}{2}) = a$ is a strictly increasing polynomial, the specialization of system (1.1)

$$\begin{aligned} \varphi_a\Big(\frac{k}{2^{\ell+1}}\Big) &= a\,\varphi_a\Big(\frac{k}{2^\ell}\Big) \\ \varphi_a\Big(\frac{2^\ell + k}{2^{\ell+1}}\Big) &= a + (1-a)\,\varphi_a\Big(\frac{k}{2^\ell}\Big) \end{aligned}$$

with $0 < k < 2^\ell$ shows by induction that all functions $\varphi_a(\frac{k}{2^\ell})$ are also strictly increasing polynomials in a. Hence, at arbitrarily fixed $t \in (0,1)$, the function $a \mapsto \varphi_a$ is at least (improper) increasing, and we have to exclude intervals of constancy. In order to do this we show that, for $|a| < 1$ and $|1-a| < 1$, the function φ_a is holomorphic. Namely, choosing $|a| < 1$ and $|1-a| \le 1-\varepsilon < 1$ in representation (1.2) with $t = \sum_{j=0}^{\infty} 2^{-\gamma_j}$ we obtain the estimate

$$|\varphi_a(t)| \le \sum_{j=0}^{\infty} |a|^{\gamma_j - j} |1-a|^j \le \sum_{j=0}^{\infty} (1-\varepsilon)^j = \frac{1}{\varepsilon}$$

in view of $j < \gamma_j$. This implies that series (1.2) of polynomials is uniformly convergent in every compact subset of the domain $\{a : (|a| < 1) \cap (|1-a| < 1)\}$. Consequently, in this domain φ_a is holomorphic. If it would by constant in a certain real interval, then it would be constant everywhere. But this is impossible since in view of $j < \gamma_j$ representation (1.2) implies $\lim_{a\to 0} \varphi_a(t) = 0$ and $\lim_{a \to 1} \varphi_a(t) = 1$ for $0 < t < 1$. Moreover, the both last relations imply in connection with the continuity that the curves fill out the whole open unit square.

If in system (1.1) we replace the constant a by $1-a$ and t by $1-t$, we obtain

$$\begin{aligned} \varphi_{1-a}\Big(\frac{1-t}{2}\Big) &= (1-a)\,\varphi_{1-a}(1-t) \\ \varphi_{1-a}\Big(1 - \frac{t}{2}\Big) &= 1 - a + a\,\varphi_{1-a}(1-t), \end{aligned}$$

and if we further replace $\varphi_{1-a}(1-t) = 1 - \varphi(t)$, we again obtain system (1.1), only with interchanged equations. Since in the space of continuous functions system (1.1) is uniquely solvable, the proposition is proved ∎

Figure 1: The graphs of de Rham's function for $a = 0.1(0.1)0.9$

Proposition 2.3 is illustrated by means of Figure 1, which shows de Rham's function for different parameters a (cf. also [6]).

Let us mention a connection to a functional equation, which was studied by Klemmt [4], and which gives us a new possibility to prove that φ is a singular function in the case of $a \neq \frac{1}{2}$. The equations in system (1.1) easily imply for $0 < t < 1$

$$\varphi'\Big(\frac{t}{2}\Big) + \varphi'\Big(\frac{t+1}{2}\Big) = 2\varphi'(t)$$

almost everywhere. According to [4], φ' must be constant almost everywhere: $\varphi'(t) = c$ with $c \geq 0$. Hence $ct \leq \varphi(t)$, and in view of $\varphi(\frac{1}{2^n}) = a^n$ for $n \in \mathbb{N}$ we obtain $0 \leq c \leq (2a)^n$ and therefore $c = 0$ in the case of $0 < a < \frac{1}{2}$. The case $\frac{1}{2} < a < 1$ can be transferred to the foregoing one by means of (2.12).

3. Related functions

Since de Rham's function φ is continuous and strictly increasing in t, its inverse φ^{-1} exist and we can deal with it.

Proposition 3.1. *If f is a strictly singular function, then the inverse $g = f^{-1}$ is also strictly singular.*

Proof. Since g is strictly increasing, g is differentiable almost everywhere with $g'(\tau) \geq 0$. For arbitrary $0 < \alpha < \beta$ let $E_{\alpha,\beta}$ be the set of all τ such that $g'(\tau)$ exists and $\alpha \leq g'(\tau) \leq \beta$. Denote by $|E_{\alpha,\beta}|$ the Lebesgue measure of the measurable set $E_{\alpha,\beta}$. According to $f'(g(\tau)) = \frac{1}{g'(\tau)}$ we have $f'(t) \geq \frac{1}{\beta}$ for all $t \in g(E_{\alpha,\beta})$, which implies that $|g(E_{\alpha,\beta})| = 0$ since f is singular. In view of $g'(\tau) \geq \alpha$ for $\tau \in E_{\alpha,\beta}$ we have the estimate $\alpha|E_{\alpha,\beta}| \leq |g(E_{\alpha,\beta})|$ (cf. [7: p. 234]). Consequently, $|E_{\alpha,\beta}| = 0$ for $0 < \alpha < \beta$. Since the set E of all τ with $g'(\tau) > 0$ is representable as countable union of such sets, we obtain $|E| = 0$. Hence $g' = 0$ almost everywhere ∎

There is another possibility to prove Proposition 3.1 by means of measure theory. Namely, if f is an increasing singular function, then it generates a Stieltjes measure which is singular with respect to the Lebesgue measure. If, moreover, $x = f(t)$ is continuous and strictly increasing, then the inverse function $t = f^{-1}(x)$ generates automatically also a measure singular to the Lebesgue measure, i.e. f^{-1} is also a singular function.

In particular, for fixed $a \neq \frac{1}{2}$ the inverse φ^{-1} of de Rham's function is also strictly singular with respect to t. System (1.1) implies that

$$\left.\begin{aligned} \varphi^{-1}(at) &= \tfrac{1}{2}\varphi^{-1}(t) \\ \varphi^{-1}(a+(1-a)t) &= \tfrac{1}{2} + \tfrac{1}{2}\varphi^{-1}(t) \end{aligned}\right\} \qquad (0 \leq t \leq 1) \tag{3.1}$$

(cf. [2]). Moreover, from (2.12) we obtain that

$$\varphi_{1-a}^{-1}(t) = 1 - \varphi_a^{-1}(1-t) \tag{3.2}$$

for $0 \le t \le 1$.

Systems (1.1) and (3.1) can be generalized by

$$\left.\begin{aligned} \varphi(ct) &= a\,\varphi(t) \\ \varphi(c+(1-c)t) &= a+(1-a)\,\varphi(t) \end{aligned}\right\} \qquad (t \in [0,1]) \tag{3.3}$$

with fixed $0 < a, c < 1$.

Proposition 3.2. *The following assertions are valid:*

(i) *The composition $\varphi(t) = \varphi_a(\varphi_c^{-1}(t))$ is the unique bounded solution of the functional equations* (3.3).

(ii) *This solution is continuous, strictly increasing and maps $[0,1]$ onto $[0,1]$.*

Proof. It can easily be checked that the composition $\varphi_a\varphi_c^{-1}$ satisfies equations (3.3):

$$\varphi_a(\varphi_c^{-1}(ct)) = \varphi_a\big(\tfrac{1}{2}\varphi_c^{-1}(t)\big) = a\,\varphi_a(\varphi_c^{-1}(t))$$

and

$$\varphi_a\big(\varphi_c^{-1}(c+(1-c)t)\big) = \varphi_a\big(\tfrac{1}{2}+\tfrac{1}{2}\varphi_c^{-1}(t)\big) = a+(1-a)\,\varphi_a(\varphi_c^{-1}(t))\ .$$

Moreover, $\varphi = \varphi_a\varphi_c^{-1}$ has in fact properties (ii).

Now, let φ be a further solution of equations (3.3). For $0 \le t \le 1$ we put $d(t) = |\varphi(t) - \varphi_a(\varphi_c^{-1}(t))|$. Assume that there exists a point $t_0 \in [0,1]$ with $d(t_0) > 0$. If $t_0 \le c$, then for $t_1 = \frac{1}{c}t_0$ we have $t_1 \in [0,1]$ and the first equation of (3.3) implies that $d(t_1) = \frac{1}{a}d(t_0)$. In the case of $c < t_0 \le 1$ the point $t_1 = \frac{t_0-c}{1-c}$ lies in [0,1] and from the second equation of (3.3) we obtain that $d(t_1) = \frac{1}{1-a}d(t_0)$. Putting $m = \min\{\frac{1}{a}, \frac{1}{1-a}\}$ and

$$t_{n+1} = \begin{cases} \frac{1}{c}t_n & \text{for } 0 \le t_n \le c \\ \frac{t_n-c}{1-c} & \text{for } c < t_n \le 1 \end{cases}$$

where $n \in \mathbb{N}$, we get $d(t_n) \ge m^n d(t_0)$. However, in view of $m > 1$ this is a contradiction to the boundedness of φ ∎

Proposition 3.3. *The solution $\varphi = \varphi_a\varphi_c^{-1}$ of system* (3.3) *has the representation*

$$\varphi\left(\sum_{j=0}^{\infty} c^{\gamma_j} q_c{}^j\right) = \sum_{j=0}^{\infty} a^{\gamma_j} q_a{}^j \tag{3.4}$$

where $q_a = \frac{1-a}{a}$, $q_c = \frac{1-c}{c}$, $\gamma_j \in \mathbb{N}$ and $\gamma_0 < \gamma_1 < \gamma_2 < \dots$. Moreover, for $\ell \in \mathbb{N}$ and $k = 0, 1, \dots, 2^\ell - 1$ we have

$$\varphi\Big(\varphi_c\Big(\frac{k}{2^\ell}\Big) + c^\ell q_c^{\nu(k)}\varphi_c(\tau)\Big) = \varphi_a\Big(\frac{k}{2^\ell}\Big) + a^\ell q_a^{\nu(k)}\varphi_a(\tau) \tag{3.5}$$

for $0 \le \tau \le 1$. In the case of $a \ne c$ the solution φ of system (3.3) *is strictly singular and its derivative is* 0 *whenever it exists.*

Proof. Representation (3.4) follows from (1.2) in view of $\varphi(\varphi_c(t)) = \varphi_a(t)$ with $t = \sum_{j=0}^{\infty} 2^{-\gamma_j}$. From $\varphi(\varphi_c(t)) = \varphi_a(t)$ with $t = \frac{k+\tau}{2^\ell}$ we also get (3.5) by twofold use of (2.9), but once with c instead of a.

Let be $x \in [0,1]$ such that $\varphi'(x)$ exists. For $n \in \mathbb{N}$ choose integers k_n with $0 \le k_n \le 2^n - 1$ and $x_n = \varphi_c\left(\frac{k_n}{2^n}\right)$, $y_n = \varphi_c\left(\frac{k_n+1}{2^n}\right)$ so that $x_n \le x \le y_n$. From (2.9) and (3.5) with $\tau = 0$ respectively $\tau = 1$ we obtain

$$D_n = \frac{\varphi(y_n) - \varphi(x_n)}{y_n - x_n} = \frac{a^n q_a^{\nu(k_n)}}{c^n q_c^{\nu(k_n)}} \to \varphi'(x) \qquad (n \to \infty)$$

owing to $\varphi(0) = 0$ and $\varphi(1) = 1$. Now, putting $z_n = x_n + c(y_n - x_n)$, there are two possibilities, either $x_n \le x \le z_n$ or $z_n < x \le y_n$. From (2.9) and (3.5) with $\tau = 0$, $\tau = \frac{1}{2}$ respectively $\tau = 1$ we get in view of $\varphi(c) = a$ that

$$\frac{\varphi(z_n) - \varphi(x_n)}{z_n - x_n} = \frac{a}{c} D_n \qquad \text{and} \qquad \frac{\varphi(y_n) - \varphi(z_n)}{y_n - z_n} = \frac{1-a}{1-c} D_n \,.$$

At least one of both possibilities mentioned before occurs infinitely many times. Consequently, $\varphi'(x) = \frac{a}{c}\varphi'(x)$ or $\varphi'(x) = \frac{1-a}{1-c}\varphi'(x)$. Hence $\varphi'(x) = 0$ in view of $a \ne c$ ∎

Denoting the solution of system (3.3) by $\varphi_{a,c}$, we easily see the validity of the relations

1. $\varphi_{a,b}(\varphi_{b,c}(t)) = \varphi_{a,c}(t)$
2. $\varphi_{a,c}^{-1}(t) = \varphi_{c,a}(t)$
3. $\varphi_{a,c}(1-t) = 1 - \varphi_{1-a,1-c}(t)$

for arbitrary $0 < a, b, c < 1$.

Next, we consider the generalization of system (1.1)

$$\left.\begin{aligned} g\Big(\frac{t}{2}\Big) &= a\,g(t) \qquad && (0 \le t \le 1) \\ g\Big(\frac{t+1}{2}\Big) &= a + c\,g(t) \qquad && (0 < t \le 1) \end{aligned}\right\} \tag{3.6}$$

with $|a| < 1$ and $|c| < 1$. A bounded solution of system (3.6) must have the particular values $g(0) = 0$, $g(1) = \frac{a}{1-c}$ and $g(\frac{1}{2}) = \frac{a^2}{1-c}$, where also $g(+0) = 0$. However, in the case of $a + c \ne 1$ it cannot be right-continuous in all points, since $g(\frac{1}{2} + 0) = a \ne g(\frac{1}{2})$. However, system (3.6) possesses the left-continuous solution

$$g\left(\sum_{j=0}^{\infty} 2^{-\gamma_j}\right) = \sum_{j=0}^{\infty} a^{\gamma_j} q^j \tag{3.7}$$

with $1 \le \gamma_j < \gamma_{j+1}$ and $q = \frac{c}{a}$ (cf. (1.2)). On the other side, for

$$t_n = \sum_{j=0}^{n} 2^{-\gamma_j} = \sum_{j=0}^{n-1} 2^{-\gamma_j} + \sum_{j=0}^{\infty} 2^{-\gamma_n - j - 1} \tag{3.8}$$

with $n \geq 0$ we have

$$g(t_n) = \sum_{j=0}^{n-1} a^{\gamma_j} q^j + \sum_{j=0}^{\infty} a^{\gamma_n+j+1} q^{n+j} \qquad \text{and} \qquad g(t_n+0) = \sum_{j=0}^{n} a^{\gamma_j} q^j \ ,$$

and in view of

$$\sum_{j=0}^{\infty} a^{\gamma_n+j+1} q^{n+j} = \frac{a^{\gamma_n+1} q^n}{1-aq} = \frac{a^{\gamma_n+1} q^n}{1-c} \tag{3.9}$$

consequently

$$g(t_n+0) - g(t_n) = \frac{a^{\gamma_n} q^n}{1-c}(1-a-c) \ . \tag{3.10}$$

Hence, the solution g is discontinuous at all dyadic points, so far as $a \neq 1-c$. However, it is bounded and Lebesgue integrable as limit of uniformly converging step functions. According to (3.10) it is not increasing for $a+c > 1$.

Proposition 3.4. *In the case of* $0 < a, c$ *and* $a + c < 1$ *the solution* g *of system* (3.6) *is strictly increasing and continuous except in the dyadic points* $t = t_n$ *from* (3.8) *where*

$$g(t_n - 0) = g(t_n) < g(t_n + 0) \tag{3.11}$$

with jumps (3.10). *Moreover,* $g' = 0$ *almost everywhere.*

Proof. Assuming that $t, t' \in (0,1]$ have the representations

$$t = \sum_{j=0}^{\infty} 2^{-\gamma_j} \qquad \text{and} \qquad t' = \sum_{j=0}^{\infty} 2^{-\gamma'_j}$$

with γ_j as before respectively γ'_j, then $t > t'$ if and only if there exists an integer m such that $\gamma_j = \gamma'_j$ for $j = 0, ..., m-1$ and $\gamma'_m \geq \gamma_m + 1$. Owing to (3.7) we have

$$g(t) = \sum_{j=0}^{\infty} a^{\gamma_j} q^j \geq \sum_{j=0}^{m-1} a^{\gamma_j} q^j + a^{\gamma_m} q^m$$

since $q = \frac{c}{a} > 0$. Moreover, $\gamma'_m \geq \gamma_m + 1$ implies that $\gamma'_{m+j} \geq \gamma_m + 1 + j$ for all $j \geq 0$ so that in view of $0 < aq = c < 1$ we get

$$g(t') = \sum_{j=0}^{m-1} a^{\gamma'_j} q^j + \sum_{j=m}^{\infty} a^{\gamma'_j} q^j \leq \sum_{j=0}^{m-1} a^{\gamma_j} q^j + \sum_{j=0}^{\infty} a^{\gamma_m+1+j} q^{m+j} \ .$$

Hence, according to (3.9) and $a < 1-c$ we obtain $g(t) > g(t')$, i.e. g is strictly increasing.

It follows that the intervals $(g(t_n), g(t_n+0))$ are disjoint. Since the set of all dyadic points is dense in [0,1] the union

$$G = \bigcup_{\ell=0}^{\infty} \bigcup_{k=0}^{2^\ell-1} \left(g\Big(\frac{2k+1}{2^{\ell+1}}\Big), g\Big(\frac{2k+1}{2^{\ell+1}} + 0\Big) \right)$$

is an open Cantor set with Lebesgue measure

$$|G| = \sum_{t_n}^{\infty} \frac{a^{\gamma_n} q^n}{1-c}(1-a-c)$$

(cf. (3.10) where we have to sum over all dyadic t_n of $(0,1)$). Since there are $\binom{k}{n}$ possibilities for γ_n to be equal to $k+1$ we find that

$$\sum_{t_n}^{\infty} a^{\gamma_n} q^n = \sum_{n=0}^{\infty} q^n \sum_{k=n}^{\infty} \binom{k}{n} a^{k+1} = \sum_{k=0}^{\infty} a^{k+1}(1+q)^k = \frac{a}{1-a-c}$$

in view of $aq = c$. Therefore we obtain that $|G| = \frac{a}{1-c} = g(1)$. Consequently, the increasing function g cannot have further jumps.

For the set $M = [0,1] \setminus \cup\{t_n\}$ we have $|M| = 1$ and $|g(M)| = 0$ which implies that $g' = 0$ almost everywhere (cf. [7: p. 234]). Hence, the proposition is proved ∎

Remarks.

1. $P = \left[0, \frac{a}{1-c}\right] \setminus G$ is a perfect Cantor set with measure zero.

2. Note that the boundary points $t = 0$ and $t = 1$ do not belong to the points (3.8).

3. The results can easily be transferred to the case that the first equation in system (3.6) is valid for $0 \le t < 1$ and the second equation for $0 \le t \le 1$, where the solution is determined by $g(1) = \frac{a}{1-c}$, $g\left(\sum_{j=0}^{n} 2^{-\gamma_j}\right) = \sum_{j=0}^{n} a^{\gamma_j} q^j$ and right continuity with q and γ_j as before.

Supplement. Finally, we consider the generalization of systems (1.1) and (3.6)

$$\left.\begin{aligned} f\left(\frac{t}{2}\right) &= a\,f(t) \\ f\left(\frac{t+1}{2}\right) &= b + c\,f(t) \end{aligned}\right\} \tag{3.12}$$

where we admit that the solution is not defined for all $t \in (0,1)$.

Proposition 3.5. *For $a \neq 0$, $|a| < 1$, $|c| < 1$ and $0 < t \le 1$, system (3.12) has the left-continuous solution*

$$f\left(\sum_{j=0}^{\infty} 2^{-\gamma_j}\right) = \frac{b}{a} \sum_{j=0}^{\infty} a^{\gamma_j} q^j \tag{3.13}$$

with $\gamma_j \in \mathbb{N}$, $\gamma_j < \gamma_{j+1}$ and $q = \frac{c}{a}$. If $1 < a$, $0 < b$ and $0 < c < 1$, then every $y > f(1) = \frac{b}{1-c}$ has infinitely many inverse images under f.

Proof. If g is the solution (3.7) of system (3.6), then $f = \frac{b}{a} g$ is the solution of system (3.12). If $|q| < 1$, but $|a| > 1$, then the right-hand side of (3.13) can diverge, and f remains undefined at the corresponding points of (0,1]. However, for $c \neq 1$ the solution of system (3.12) always possesses the value $f(1) = \frac{b}{1-c}$. Now, let $1 < a$, $0 < b$, $0 < c < 1$ and $y > f(1)$. We look for a sequence γ_j such that

$$\frac{b}{a}\left(\sum_{j=0}^{k-1} a^{\gamma_j} q^j + \frac{a^{\gamma_k} q^k}{1-c}\right) < y \le \frac{b}{a}\left(\sum_{j=0}^{k-1} a^{\gamma_j} q^j + \frac{a^{\gamma_k+1} q^k}{1-c}\right) \tag{3.14}$$

for infinitely many k. For $k = 0$ this inequality means

$$\frac{ba^{\gamma_0 - 1}}{1-c} < y \leq \frac{ba^{\gamma_0}}{1-c}$$

and determines γ_0 uniquely in view of $a > 1$. If $\gamma_0, ..., \gamma_n$ are already determined, we choose an arbitrary integer $k > n$ depending on n, define $\gamma_j = \gamma_n + j - n$ for $j = n+1, ..., k-1$, and determine γ_k out of (3.14). The last step is uniquely possible, since in view of

$$\frac{a^{\gamma_n} q^n}{1-c} = \sum_{j=n}^{k-1} a^{\gamma_j} q^j + \frac{a^{\gamma_{k-1}+1} q^k}{1-c}$$

there always exits such a $\gamma_k \geq \gamma_{k-1} + 1$. In this way we find infinitely many sequences γ_j such that the right-hand side of (3.13) is equal to y ∎

Let $|a| > 1$. If we define $\ell = \overline{\lim} \frac{\gamma_j}{j}$, the series at the right-hand side of (3.13) is convergent for $\ell < 1 - \frac{\ln |c|}{\ln |a|}$ and divergent for $\ell > 1 - \frac{\ln |c|}{\ln |a|}$ in view of the root test. In the "periodic" case $\gamma_{pj+k} = rj + \varrho_k$ for sufficiently great j and $k = 0, 1, ..., p-1$, where $t = \sum_{j=0}^{\infty} 2^{-\gamma_j}$ is rational, we have $\ell = \frac{r}{p}$.

From system (3.12) we can derive further functional equations. Namely, for $k = 0, 1, ..., 2^\ell - 1$ with $\ell \in \mathbb{N}$, the dyadic representation $k = d_1 d_2 \cdots d_n$, $d_j \in \{0, 1\}$, where $d_1 = 0$ is allowed, and $0 < t \leq 1$ we find

$$f\Big(\frac{t+k}{2^\ell}\Big) = b \sum_{j=0}^{\ell-1} a^j d_{j+1} q^{d_1 + \ldots + d_j} + a^\ell q^{\nu(k)} f(t)$$

(cf. (2.9) and (2.2)). For $|a| > 1$ this formula shows that f is unbounded in every subinterval of $(0, 1]$, since $f(\frac{t}{2^n}) = a^n f(t)$.

Let us mention a curious connection to the $(3n+1)$-problem of L. Collatz, which for negative n is equivalent to the $(3n-1)$-problem, i.e. to the iteration of the function

$$t(n) = \begin{cases} \frac{1}{2} n & \text{for } n \text{ even} \\ \frac{1}{2}(3n-1) & \text{for } n \text{ odd.} \end{cases} \tag{3.15}$$

The iterates of $n \in \mathbb{N}$ under t have the fixed point 1 and the two cycles (5, 7, 10) as well as (17, 25, 37, 55, 82, 41, 61, 91, 136, 68, 34), and one conjectures that all t-trajectories eventually end in one of these three sets (cf. [10: p. 13]). It suffices to restrict ourselves to odd n and to replace $t(n)$ for such n by $T(n) = 2^{-p_n}(3n-1)$ if $2^{p_n} | (3n-1)$ but $2^{p_n+1} \not| (3n-1)$, $p_n \in \mathbb{N}$. The equation for T can be inverted by

$$n = \tfrac{1}{3}(1 + 2^{p_n} T(n)) \; . \tag{3.16}$$

We denote the iterates of T by $T_k(n) = T(T_{k-1}(n))$ with $T_0(n) = n$, and for a fixed n we introduce the notations $\gamma_0 = 1$ and $\gamma_k = 1 + p_{T_0(n)} + ... + p_{T_{k-1}(n)}$ for $k \geq 1$. Then (3.16) implies the representation

$$n = \frac{1}{6}\Big(2^{\gamma_0} + \frac{1}{3} 2^{\gamma_1} + ... + \frac{1}{3^{k-1}} 2^{\gamma_{k-1}} + \frac{1}{3^{k-1}} 2^{\gamma_k} T_k(n)\Big)$$

for every odd n, and for $k \to \infty$ the right-hand side converges to the right-hand side of (3.13) with $a = 2$, $b = \frac{1}{3}$, $c = \frac{2}{3}$ and therefore $q = \frac{1}{3}$.

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