Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 4, 1117–1122

On the Hilbert Inequality

Gao Mingzhe

Abstract. It is shown that the Hilbert inequality for double series can be improved by introducing the positive real number $\frac{1}{\pi^2}\big(\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2}\big)$ where $s(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}$ and $\|x\|^2 = \sum_{n=1}^{\infty} x_n^2$ $(x = a, b)$. The coefficient π of the classical Hilbert inequality is proved not to be the best possible if $\|a\|$ or $\|b\|$ is finite. A similar result for the Hilbert integral inequality is also proved.

Keywords: *Hilbert inequality, binary quadratic form, exponential integral, inner product*

AMS subject classification: 26 D, 46 C

1. Introduction

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be arbitrary real sequences. Then the *Hilbert inequality* for double series can be written as

$$\left(\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}\right)^2 \leq \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right). \tag{1}$$

Additionally,

$$\left(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{a_m b_n}{m-n}\right)^2 \leq \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right) \tag{2}$$

is also called *Hilbert inequality.* Furthermore, if $f, g \in L^2(\mathbb{R}_+)$ where $\mathbb{R}_+ = (0, \infty)$, then the inequality analogous to (1)

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\, dsdt\right)^2 \leq \pi^2 \left(\int_{\mathbb{R}_+} f^2(t)\, dt\right)\left(\int_{\mathbb{R}_+} g^2(t)\, dt\right) \tag{3}$$

is called the *Hilbert integral inequality.* The constant π contained in these inequalities, especially in (1), was proved to be the best possible (see [3]). However, if $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ or $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we can select a number $r > 0$ such that the right-hand side of (1) can be replaced by

$$\pi^2(1-r)\left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right),$$

Gao Mingzhe: Xiangxi Educ. Coll., Dept. Math., Jishou, Hunan 416000, P.R. China

ISSN 0232-2064 / $ 2.50

i.e. an improvement of (1) will be obtained. Similarly, an improvement of (3) will be established. Namely, the right-hand side of (3) can be written as

$$\pi^2(1-R)\Big(\int_{\mathbb{R}_+} f^2(t)\,dt\Big)\Big(\int_{\mathbb{R}_+} g^2(t)\,dt\Big)$$

with a number $R > 0$. The main purpose of the present paper is to prove the existence of such numbers r and R and to find expressions for them.

We first introduce some notations and functions.

If α and β are elements of an inner product space E, then its inner product is denoted by (α, β) and the norm of α is given by $\|\alpha\| = \sqrt{(\alpha,\alpha)}$. Further, if $a = (a_n)_{n\ge 1}$ and $b = (b_n)_{n\ge 1}$ are two real sequences, then its inner product (a, b) and the norm $\|a\|$ of a are defined by

$$(a,b) = \sum_{n=1}^{\infty} a_n b_n \qquad \text{and} \qquad \|a\| = \sqrt{(a,a)}. \tag{4}$$

Analogously, for functions $f, g \in L^2(a, b)$ its inner product (f, g) and the norm $\|f\|$ of f are defined by

$$(f,g) = \int_a^b f(t)g(t)\,dt \qquad \text{and} \qquad \|f\| = \Big(\int_a^b f^2(t)\,dt\Big)^{\frac{1}{2}}. \tag{5}$$

We next introduce a binary quadratic form $F(\cdot,\cdot)$ defined by

$$F(x,y) = \|\alpha\|^2 x^2 - 2(\alpha,\beta)xy + \|\beta\|^2 y^2 \tag{6}$$

where $x = (\beta, \gamma)$ and $y = (\alpha, \gamma)$ for $\gamma \in E$. We further denote

$$G(\alpha,\beta,\gamma) = F((\beta,\gamma),(\alpha,\gamma)). \tag{7}$$

The results involve $G(\alpha, \beta, \gamma)$ with α and β specified beforehand, and γ to be chosen for maximum felicity. It is obvious that if γ is orthogonal to both α and β, then $G(\alpha, \beta, \gamma) = 0$. It will turn out that if $(\alpha, \gamma)^2 + (\beta, \gamma)^2 > 0$ (see Lemma 1). Therefore, it is shrewd in every case to choose γ not orthogonal to both α and β.

For convenience, we introduce yet the notations

$$u(a,b) = \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}, \quad v(a,b) = \sum_{\substack{m,n=1\\ m\ne n}}^{\infty} \frac{a_m b_n}{m-n}, \qquad s(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

We shall frequently use these notations below.

2. Lemmas

To prove our theorems, we need the following results.

Lemma 1. *Let $G(\alpha,\beta,\gamma)$ be defined as in (7). If $\alpha,\beta \in E$ are linearly independent and $(\alpha,\gamma)^2+(\beta,\gamma)^2>0$, then $G(\alpha,\beta,\gamma)>0$.*

Lemma 2. *Let $G(\alpha,\beta,\gamma)$ be defined as defined in (7). If $\alpha,\beta \in E$ are linearly dependent, then $G(\alpha,\beta,\gamma)=0$.*

Lemma 3. *Let $G(\alpha,\beta,\gamma)$ be defined as defined in (7). If $\alpha,\beta \in E$ are arbitrary and $\gamma \in E$ with $\|\gamma\|=1$, then*

$$(\alpha,\beta)^2 \le \|\alpha\|^2\|\beta\|^2 - G(\alpha,\beta,\gamma), \tag{8}$$

and equality holds in (8) if and only if α,β,γ are linearly dependent.

The proofs of Lemmas 1 and 2 have been given in our previous paper [1]. Lemma 3 is actually a sharpening of the Cauchy-Schwarz inequality. This result has been given also in the paper [1], and in [5]. Hence the proofs of all lemmas are omitted.

Using the inner product defined by (5) and Lemma 3, we have the following result.

Corollary 1. *If $f,g \in L^2(a,b)$, then*

$$(f,g)^2 \le \|f\|^2\|g\|^2 - F(x,y) \tag{9}$$

where $F(x,y)=\|f\|^2x^2-2(f,g)xy+\|g\|^2y^2$ with $x=(g,\gamma)$ and $y=(f,\gamma)$, $\gamma \in L^2(a,b)$ with $\|\gamma\|=1$.

3. Main results

In this section we will combine the two forms (1) and (2) of the Hilbert inequality into one similar form, and make inequalities (1) - (3) relaize significant improvements. The following theorems are the main results in this paper.

Theorem 1. *If $a=(a_n)$ and $b=(b_n)$ are real sequences with non-negative terms, with $0<\|a\|<\infty$ or $0<\|b\|<\infty$, then*

$$u^2(a,b)+v^2(a,b) < \pi^2(1-r)\|a\|^2\|b\|^2 \tag{10}$$

where $r=\frac{1}{\pi^2}\left(\frac{s^2(a)}{\|a\|^2}+\frac{s^2(b)}{\|b\|^2}\right)$.

Proof. Let us define two real functions $f,g:(0,2\pi)\to\mathbb{R}$ by

$$f(t)=\sum_{n=1}^{\infty} a_n\sqrt{t}\sin(nt) \qquad \text{and} \qquad g(t)=\sum_{n=1}^{\infty} b_n\sqrt{t}\cos(nt).$$

It is easily to deduce that, with the notations of the space $L^2(0,2\pi)$,

$$|u(a,b)+v(a,b)| = \tfrac{1}{\pi}|(f,g)|. \tag{11}$$

According to (5) and (6) we have $(f,g)^2 \le \|f\|^2\|g\|^2 - F(x,y)$ where $\|f\|^2 = \pi^2\|a\|^2$, $\|g\|^2 = \pi^2\|b\|^2$ and

$$F(x,y) = \|f\|^2x^2 - 2(f,g)xy + \|g\|^2y^2 \ge (\|f\|x - \|g\|y)^2 = \pi^2(\|a\|x - \|b\|y)^2.$$

Hence

$$(f,g)^2 \le \pi^4\|a\|^2\|b\|^2 - \pi^2(\|a\|x - \|b\|y)^2 \tag{12}$$

where $x = (g,\gamma)$ and $y = (f,\gamma)$, $\gamma \in L^2(0,2\pi)$ with $\|\gamma\| = 1$. We can choose $\gamma = \frac{1}{2\pi}\sqrt{2t}$. Then $x = 0$ and $y = -\sqrt{2}\sum_{n=1}^{\infty}\frac{a_n}{n} = -\sqrt{2}\,s(a)$. Hence

$$(\|a\|x - \|b\|y)^2 = 2\|b\|^2s^2(a). \tag{13}$$

In virtue of (11) - (13) we obtain

$$|u(a,b) + v(a,b)|^2 \le \pi^2\|a\|^2\|b\|^2 - 2\|b\|^2s^2(a). \tag{14}$$

Since the vectors f, g, γ are linearly independent, by Lemma 3, it is impossible to take equality in (14). Hence we have

$$|u(a,b) + v(a,b)|^2 < \pi^2\|a\|^2\|b\|^2 - 2\|b\|^2s^2(a). \tag{15}$$

Notice that $u(b,a) = u(a,b)$ and $v(b,a) = -v(a,b)$. Interchanging a and b in (11), similarly we obtain

$$|u(a,b) - v(a,b)|^2 < \pi^2\|a\|^2\|b\|^2 - 2\|a\|^2s^2(b). \tag{16}$$

Adding (15) and (16), inequality (10) is yielded after some simplifications. Thus the proof of the theorem is completed ∎

Remark. Since $a = (a_n)$ and $b = (b_n)$ are real sequences with non-negative terms, with $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$, it follows that $r > 0$. Hence inequality (10) is a significant refinement of the paper [4].

Corollary 2. *If $a = (a_n)$ is a real sequence with non-negative terms and $0 < \|a\| < \infty$, then*

$$u^2(a,a) + v^2(a,a) < \pi^2(1-\tilde{r})\|a\|^4 \tag{17}$$

where $\tilde{r} = \frac{2}{\pi^2}\frac{s^2(a)}{\|a\|^2}$.

If $v^2(a,b)$ in (10) is replaced by 0, then we have the following

Corollary 3. *With the assumptions of Theorem 1, then*

$$u^2(a,b) < \pi^2(1-r)\|a\|^2\|b\|^2 \tag{18}$$

where $r = \frac{1}{\pi^2}\left(\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2}\right)$.

We see from the above Remark that inequality (18) is a significant improvement of (1). According to Corollary 2 we obtain at once the following

Corollary 4. *If $a = (a_n)$ is a real sequence with non-negative terms and $0 < \|a\| < \infty$, then*

$$u^2(a,a) < \pi^2(1-\tilde{r})\|a\|^4 \tag{19}$$

where $\tilde{r} = \frac{2}{\pi^2}\frac{s^2(a)}{\|a\|^2}$.

Similarly, we can establish an improvement of the Hilbert integral inequality. For this we need the integral

$$e(t) = \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t}\, ds \qquad (t \in \mathbb{R}_+)$$

called *exponential integral with parameter t.*

Theorem 2. *Let $f, g \in L^2(\mathbb{R}_+)$ be positive. Then*

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\, dsdt\right)^2 < \pi^2(1-R)\|f\|^2\|g\|^2 \tag{20}$$

where $R = \frac{1}{\pi}\big(\frac{x}{\|g\|} - \frac{y}{\|f\|}\big)^2$ with $x = \big(\frac{2}{\pi}\big)^{\frac{1}{2}}(g,e)$ and $y = (2\pi)^{\frac{1}{2}}(f, e^{-s})$, e being the exponential integral with parameter.

Proof. Define functions F and G by

$$F(s,t) = \frac{f(s)}{(s+t)^{\frac{1}{2}}}\Big(\frac{s}{t}\Big)^{\frac{1}{4}} \qquad \text{and} \qquad G(s,t) = \frac{g(t)}{(s+t)^{\frac{1}{2}}}\Big(\frac{t}{s}\Big)^{\frac{1}{4}}.$$

Using inequality (9) we have in $L^2(\mathbb{R}_+^2)$

$$\begin{aligned} \left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\, dsdt\right)^2 &= (F,G)^2 \\ &\le \|F\|^2\|G\|^2 - F(x,y) \\ &\le \|F\|^2\|G\|^2 - (\|F\|x - \|G\|y)^2 \end{aligned} \tag{21}$$

where $x = (G,\gamma)$ and $y = (F,\gamma)$, $\gamma \in L^2(\mathbb{R}_+^2)$ with $\|\gamma\| = 1$. We can choose

$$\gamma(s,t) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}}\Big(\frac{s}{t}\Big)^{\frac{1}{4}}.$$

Hence we get

$$x = \big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}(g,e) \qquad \text{and} \qquad y = (2\pi)^{\frac{1}{2}}(f,e^{-s}). \tag{22}$$

It is easy to deduce that

$$\|F\|^2 = \pi\|f\|^2 \qquad \text{and} \qquad \|G\|^2 = \pi\|g\|^2. \tag{23}$$

Substituting (22) and (23) into (21) we obtain

$$(F,G)^2 \le \pi^2\|f\|^2\|g\|^2 - \pi(\|f\|x - \|g\|y)^2. \tag{24}$$

Since F, G, γ are linearly independent, it is impossible to have equality in (24). Consequently, inequality (20) is obtained from (24) after some simplifications. Thus the theorem is proved ∎

Corollary 5. *If $f \in L^2(\mathbb{R}_+)$ is positive, then*

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)f(t)}{s+t}\, dsdt\right)^2 < \pi^2(1-\tilde{R})\|f\|^4$$

where $\tilde{R} = \frac{1}{\pi}\frac{(x-y)^2}{\|f\|^2}$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(f,e)$ and $y = (2\pi)^{\frac{1}{2}}(f,e^{-s})$, e being the exponential integral with parameter.

Obviously, this is an immediate consequence of Theorem 2.

4. Conclusions

Some classical reasults concerning the Hilbert inequality show that the constant π in (1) is the best possible (see, i.e., [1, 2, 5, 6]). We see from (18) that inequality in (1) can be obtained only if $r = 0$. However, to change r into 0, it is necessary to take both $\|a\|$ and $\|b\|$ infinite. Therefore, generally, the constant π in (1) is not the best possible because the constant r contained in (18) is not equal to 0 if $\|a\|$ or $\|b\|$ is finite. In other words, the factor π in (1) can be decreased if $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$.

Similarly, we see from (20) that strong inequality in (3) can be obtained only if $R = 0$. In other words, the factor π in (3) is also not the best possible if $\|f\|$ or $\|g\|$ is finite.

Acknowledgement. The author is indebted to the referees for many valuable suggestions in this subject.

References

[1] Gao Mingzhe, Tan Li and L. Debnath: *Some improvements on Hilbert's integral inequality.* J. Math. Anal. Appl. 229 (1999), 682 – 689.

[2] Gao Mingzhe and Yang Bichen: *On the extended Hilbert's inequality.* Proc. Amer. Math. Soc. 126 (1998), 751 – 759.

[3] Hardy, G. H., Littlewood, J. E. and G. Polya: *Inequalities.* Cambridge: Univ. Press 1952.

[4] Hu Ke: *On Hilbert's inequality.* Chin. Ann. Math. (Ser. B) 13 (1992), 35 – 39.

[5] Mitrinovic, D. S.: *Analytic Inequalities.* New York: Springer-Verlag 1970.

[6] Oleszkiewicz, K.: *An elementary proof on Hilbert's inequality.* Amer. Math. Monthly 100 (1993), 276 – 280.

Received 04.01.1999