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BANACH–STEINHAUS TYPE THEOREM IN LOCALLY CONVEX SPACES FOR σ-LOCALLY LIPSCHITZIAN CONVEX PROCESSES

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The main purpose of this paper is to generalize the Banach–Steinhaus theorem in locally convex spaces for σ-locally Lipschitzian operators established by S. Lahrech in [1] to σ-locally Lipschitzian convex processes.

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1. Introduction

Let (X,λ) and (Y,μ) be two locally convex spaces. Assume that the locally convex topology μ is generated by the family $(q_\beta)_{\beta\in I}$ of semi norms on Y. Let $\beta(X_\lambda)$ denote the family of bounded sets in (X,λ) and let $\sigma\subset\beta(X_\lambda)$. For a linear mapping $T:\,X\to Y$, a semi norm p on Y, and $\ \in\sigma$ set $L(p,C)(T)=\sup\limits_{h\in C}p(Th)$. According to [1], T is said to be σ-locally Lipschitzian if

$$\forall\, C\in\sigma,\quad \forall\,\beta\in I:\ L(\beta,C)\equiv L(q_\beta,C)(T)<+\infty.$$

By $\mathrm{Lip}(X_\lambda,Y_\mu,\sigma)$ we denote the vector space of σ-locally Lipschitzian operators. Note that $\mathrm{Lip}(X_\lambda,Y_\mu,\sigma)$ is a locally convex space under the locally convex topology $\tau(\lambda,\mu,\sigma)$ generated by the family of semi norms $L(\beta,C)$, $\beta\in I$, $C\in\sigma$.

The operator $T:\,(X,\lambda)\to(Y,\mu)$ is said to be sequentially continuous if for every sequence (x_n) of X and every $x\in X$ such that $x_n\xrightarrow{\lambda}x$ one has $Tx_n\xrightarrow{\mu}Tx$. T is said to be bounded if T sends bounded sets in (X,λ) into bounded sets in (Y,μ). Clearly, continuous operators are sequentially continuous, sequentially continuous operators are bounded, and linear bounded operators are σ-locally Lipschitzian; but in general, converse implications fail. Let X', X^s, X^b and X^L_σ denote respectively the family of continuous linear functionals sequentially continuous linear functionals, linear bounded functionals and σ-locally Lipschitzian functionals on (X,λ). In general, the inclusions $X'\subset X^s\subset{}^b\subset{}^L_\sigma$ are strict.

Let $\theta(X,X^L_\sigma)$ denote the topology of uniform convergence on $\sigma(X^L_\sigma,X)$-Cauchy sequences of X^L_σ. Note that if $\sigma=\beta(X_\lambda)$, then $X^b=X^L_\sigma$ and consequently, $\theta(X,X^L_\sigma)=\theta(X,X^b)$.

It has been show in [1] that, if $T_n:(X,\lambda)\to(Y,\mu)$, $n\in\mathbb{N}$ is a sequence of σ-locally Lipschitzian operators admitting for each $x\in X$ a weak limit $\lim T_nx=Tx$, then the limit operator T maps $\theta(X,X^L_\sigma)$-bounded sets into bounded sets.

Our objective in this paper is to generalize the above result to σ-locally Lipschitzian convex processes.

These are multifunctions which maps every set C of σ into bounded set in (Y,μ) and whose graphs are convex cones.

Recall that a multifunction (or set-valued map) $\Phi : X \to Y$ is a map from X to the set of subsets of Y. The domain of Φ is the set

$$D(\Phi) = \{x \in X : \Phi(x) \neq \varnothing\}.$$

We say Φ has nonempty images if its domain is X. For any subset C of X we write $\Phi(C)$ for the image $\bigcup_{x\in C} \Phi(x)$ and the range of Φ is the set $R(\Phi) = \Phi(X)$. We say Φ is surjective if its range is Y. The graph of Φ is the set

$$G(\Phi) = \{(x,y) \in X \times Y : y \in \Phi(x)\},$$

and we define the inverse multifunction $\Phi^{-1} : Y \to X$ by the relationship

$$x \in \Phi^{-1}(y) \Leftrightarrow y \in \Phi(x) \text{ for } x \in X \text{ and } y \in Y.$$

A multifunction is convex, or closed if its graph is likewise. A process is a multifunction whose graph is a cone. For example, we can interpret linear closed operators as closed convex processes in the obvious way. If λ-bounded sets have μ bounded images, then we say that Φ is (λ,μ) bounded.

A convex process $\Phi : (X,\lambda) \to (Y,\mu)$ is said to be σ-locally Lipschitzian if it maps every set C of σ into bounded set in (Y,μ). Clearly, bounded convex processes are σ-locally Lipschitzian. Note also that σ-locally Lipschitzian operators can be interpreted as σ-locally Lipschitzian convex processes. Our next step is to generalize all the results established by *S.* Lahrech in [1] to σ-locally Lipschitzian convex processes. But before proving the main results, we pause to recall some terminologies and definitions which will be used later.

2. Banach–Steinhaus theorem for σ-locally Lipschitzian convex processes in locally convex spaces

Let (X,λ) and Y,μ be two locally convex spaces. If $\Phi : (X,\lambda) \to R$ is a σ-locally Lipschitzian convex process, then we say that Φ is a real σ-locally Lipschitzian convex process with respect to the topology λ.

Denote by $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$ the class of σ-locally Lipschitzian convex processes acting from X,λ into $\mathbb{R}$.

Let (Φ_n) be a sequence of multifunctions acting from X into Y such that $\bigcap_n D(\Phi_n) \neq \varnothing$.

We say that (Φ_n) is a Cauchy sequence along the topology μ, if $\forall\, x \in X$ satisfying $x \in \bigcap\limits_n D(\Phi_n)$ $\exists\,(\Phi_{n_k})$ a subsequence of (Φ_n), $\exists\, x_k \in \Phi_{n_k}(x)$, $k = 1,2,\dots$ such that (x_{n_k}) is a Cauchy sequence for the weak topology $\sigma(Y,Y') \equiv \sigma(Y,(Y,\mu)')$. Denote by $\mathrm{Lip}_c((X,\sigma,\lambda),(Y,\mu))$ the class of σ-locally Lipschitzian convex processes acting from (X,λ) into (Y,μ).

Let $B \subset X$, $B \neq \varnothing$. We say that B is bounded along the class $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$, if for every Cauchy sequence Φ_n along the topology μ satisfying: $\Phi_n \in \mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$, $B \subset \bigcap D(\Phi_n)$, and for every sequence $(x_k)_k$ of elements of B and for every double sequence $y_n^k \in \Phi_n(x_k)$, $k = 1,2,\dots$, $n = 1,2,\dots$, the double sequence $(y_n^k)_{n,k}$ is bounded in R.

Let (Φ_n) be a sequence of multifunctions acting from X into Y such that $\bigcap_n D(\Phi_n) \neq \varnothing$.

We define the upper limit $(\limsup \Phi_n)$ of Φ_n with respect to the topology μ by:

$$\forall\, x \in X \quad \limsup \Phi_n(x) = \big\{y \in Y : \exists\,(\Phi_{n_k}) \text{ a subsequence of } (\Phi_n),\ \exists\, y_{n_k} \in \Phi_{n_k}(x) \quad (\forall\, k) \\ \text{such that } y_{n_k} \to y \text{ for the weak topology } \sigma(Y,(Y,\mu)')\big\}.$$

Let $\Phi: X \to Y$ be a multifunction. We say that Φ_n converges to Φ along the topology μ, if the following conditions holds:

1) $\limsup \Phi_n = \Phi$,

2) $\forall x \in X$, $\bigcup_n \Phi_n(x)$ is conditionally sequentially compact in $(Y, \sigma(Y, (Y,\mu)'))$. In this case, we set $\lim \Phi_n = \limsup \Phi_n = \Phi$.

Proposition 1. *Assume that (Φ_n) is a sequence of convex processes converging to a some multifunction Φ along the topology μ. Then:*

1) $\bigcap_n D(\Phi_n) \subset D(\Phi)$,

2) *Φ is a convex process.*

$\lhd$ Let $x \in \bigcap_n D(\Phi_n)$. Then, there is a sequence $x_n \in \Phi_n(x)$ $(n \in \mathbb{N})$. On the other hand, $\bigcup_n \Phi_n(x)$ is conditionally sequentially compact in $(Y, \sigma(Y, Y'))$. Hence, there exists a subsequence $(x_{n_k}) \in \Phi_{n_k}(x)$ of (x_n) converging to some y for the weak topology $\sigma(Y, Y')$. Consequently, $y \in \lim \Phi_n(x) = \Phi(x)$. This implies that $\Phi(x) \neq \varnothing$. Thus, $x \in D(\Phi)$, and the desired inclusion follows.

Let now $(x, y) \in G(\Phi)$ and $\lambda \geqslant 0$. Then, there is a subsequence (Φ_{n_k}) of (Φ_n) and $y_{n_k} \in \Phi_{n_k}(x)$ such that $y_{n_k} \overset{\sigma(Y,Y')}{\longrightarrow} y$. Therefore, $\lambda y \in \Phi(\lambda x)$. Hence, $\lambda(x, y) \in G(\Phi)$. So, using the same argument, we prove that $G(\Phi)$ is convex. Thus, we achieve the proof. $\rhd$

It follows from the above proposition that if $\lim \Phi_n$ exists, then (Φ) is a Cauchy sequence.

Let us remark also that if the topology μ, is separated and if (A_n) is a sequence of linear operators converging to some operator A acting from X into Y at each $x \in X$ for the weak topology $\sigma(Y, Y')$, then (Φ_n) converge to Φ in our sense, where (Φ_n) and Φ are the convex processes defined by $\Phi_n(x) = \{A_n x\}$, and $\Phi(x) = \{Ax\}$.

For a multifunction $\Phi : X \to Y$ and $y' \in Y' \equiv (Y, \mu)'$, we define $y' \circ \Phi$ to be the multifunction $\Phi_1 : X \to \mathbb{R}$ defined by $\Phi_1(x) = y'(\Phi(x))$.

Now we are ready to prove the main results of our paper.

Theorem 2 (Banach–Steinhaus theorem for σ-locally Lipschitzian convex processes). *Let $\Phi_n : (X, \lambda) \to (Y, \mu)$, $n = 1, 2, \ldots$ be a sequence of σ-locally Lipschitzian convex processes converging along the topology μ to some multifunction $\Phi : X \to Y$. Then the limit convex process Φ is $(\mathrm{Lip}_c((X, \sigma, \lambda), \mathbb{R}), \mu)$-bounded. That is for any $B \subset X$ such that B is bounded along the class $\mathrm{Lip}_c((X, \sigma, \lambda), \mathbb{R})$, $\Phi(B)$ is bounded in (Y, μ).*

$\lhd$ Let $y' \in Y' \equiv (Y, \mu)'$. Then, $y' \circ \Phi_n$ converge to y' along the topology μ. Therefore, $(y' \circ \Phi_n)$ is a Cauchy sequence along the topology μ. Assume that B is a bounded set along the class $\mathrm{Lip}_c((X, \sigma, \lambda), \mathbb{R})$. Let (x_k) be a sequence of elements of B and let $y_k \in \Phi(x_k)$, $k = 1, 2, \ldots$

Since $y_k \in \Phi(x_k)$, then without loss of generality we can assume that there is a sequence $y_k^n \in \Phi_n(x_k)$, $k, n \in \mathbb{N}$ such that, for any k, y_k^n converges to y_k with respect to the weak topology $\sigma(Y, Y')$. On the other hand, B is bounded along the class $\mathrm{Lip}_c((X, \sigma, \lambda), \mathbb{R})$. Consequently, the sequence $(\langle y', y_k^n \rangle)_{n,k}$ is bounded in $\mathbb{R}$. Therefore, $\lim\limits_{k,s \to +\infty} \frac{1}{s} \langle y', y_k^n \rangle = 0$ uniformly in $n \in \mathbb{N}$.

Now fix $\varepsilon > 0$. Then, there is an integer k_0 such that $|\langle y', y_k^n \rangle| < \frac{\varepsilon}{2} k_0$ for all $n \in \mathbb{N}$ and all $k \geqslant k_0$. Fix a $k \geqslant k_0$. Since $y_k^n \to y_k$ as $n \to +\infty$, then there is an $n_0 \in \mathbb{N}$ such that $|\langle y', y_k^{n_0} \rangle - \langle y', y_k \rangle| < \frac{\varepsilon}{2}$. Therefore, $|\langle y', y_k \rangle| \leqslant |\langle y', y_k \rangle - \langle y', y_k^{n_0} \rangle| + |\langle y', y_k^{n_0} \rangle| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} k_0$. This shows that the set $\{\langle y', y \rangle : x \in B,\ y \in \Phi(x)\}$ is bounded. Since $y' \in (Y, \mu)'$ is arbitrary, $\Phi(B)$ is μ-bounded by the classical Mackey theorem. Thus, we achieve the proof. $\rhd$

The next result give us a sufficient conditions to guarantee that the limit convex process Φ in the above theorem is σ-locally Lipschitzian.

We say that $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$ is sequentially complete, if every Cauchy sequence in $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$ converges in $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$.

Theorem 3. *Let $\Phi_n : (X,\lambda) \to (Y,\mu)$, $n = 1, 2, \dots$ be a sequence of σ-locally Lipschitzian convex processes converging along the topology μ to some multifunction $\Phi : X \to Y$. Assume that $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$ is sequentially complete. Then the limit convex process Φ is σ-locally Lipschitzian from (X,λ) into (Y,μ).*

$\lhd$ Let $C \in \sigma$. We must prove that the set $\Phi(C)$ is bounded in (Y,μ). Let $y' \in (Y,\mu)'$. Then $y' \circ \Phi_n \to y' \circ \Phi$ along the topology μ. Consequently, $(y' \circ \Phi_n)$ is a Cauchy sequence in $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$.

On the other hand, $\mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$ is sequentially complete. Therefore, $y' \circ \Phi \in \mathrm{Lip}_c((X,\sigma,\lambda),\mathbb{R})$. Thus, $y'(\Phi(C))$ is bounded. Hence, $\Phi(C)$ is μ-bounded by the classical Mackey theorem. $\rhd$

Remark 4. Let us remark that if $A_n : X \to Y$ is a sequence of linear σ-locally Lipschitzian operators converging to A at each $x \in X$ with respect to the weak topology $\sigma(Y,Y')$, and if moreover, the topology μ is separated, then the multifunction $\Phi_n : X \to Y$ defined by $\Phi_n(x) = \{A_n x\}$ is σ-locally Lipschitzian convex process converging to the convex process Φ defined by $\Phi(x) = \{Ax\}$. Therefore, we recapture all the results given by S. Lahrech in [1] using our theorems.

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