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AN ALMOST F-ALGEBRA MULTIPLICATION EXTENDS FROM A MAJORIZING SUBLATTICE[1]

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It is proved that an almost f-algebra multiplication defined on a majorizing sublattice of a Dedekind complete vector lattice can be extended to the whole vector lattice.

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A lattice ordered algebra $(E, \bullet)$ is called an *almost f-algebra* if $x \wedge y = 0$ implies $x \bullet y = 0$ for all $x, y \in E$ or equivalently $|x| \bullet |x| = x \bullet x$ for every $x \in E$ (cp. [2]). C. B. Huijsmans in [6] posed the question of whether the multiplication of an almost f-algebra can be extended to its Dedekind completion. G. Buskes and A. van Rooij in [4, Theorem 10] answered in the affirmative to the question and this result raises naturally another question: Can an almost f-algebra multiplication given on a majorizing vector sublattice of a Dedekind complete vector lattice be extended to an almost f-algebra multiplication on the ambient vector lattice? A positive answer was announced in [8, Corollary 7] by the author:

Theorem. *Let E be a majorizing sublattice of a Dedekind complete vector lattice $\widehat{E}$ and simultaneously an almost f-algebra. Then $\widehat{E}$ can be endowed with an almost $f$algebra multiplication that extends the multiplication on E.*

The aim of this note is to present the proof. Our reasoning is along the same lines as in [4] and rely upon a general structure theorem for almost f-algebras saying that they are actually distorted f-algebras as was shown in [4, Theorem 2]. All vector lattices and lattice ordered algebras are assumed to be Archimedean.

Recall that an *f-algebra* is a lattice-ordered algebra whenever $x \wedge y = 0$ implies $(a \bullet x) \wedge y = 0$ and $(x \bullet a) \wedge y = 0$ (or equivalently $(x \bullet y) \wedge a = 0$ provided that $x \wedge a = 0$ or $y \wedge a = 0$) for all $x, y \in A$ and $a \in A_+$. It is well known that an f-algebra multiplication is commutative [2] and order continuous [9].

For an arbitrary vector lattice E there exists a (essentially unique) pair $(E^{\odot}, \odot)$ such that $E^{\odot}$ is a vector lattice, $\odot$ is a symmetric lattice bimorphism from $E \times E$ to $E^{\odot}$ and the following universal property holds: for every symmetric lattice bimorphism b from $E \times E$ to some vector lattice F there exists a unique lattice homomorphism $\Phi_b : E^{\odot} \to F$ with $b = \Phi_b \odot$. This notion was introduced by G. Buskes and A. van Rooij, see [5] and [3]. The said universal property remains valid if we replace b and Φ_b by a positive orthosymmetric ($\equiv x \wedge y = 0 \ \Rightarrow\ b(x, y) = 0$) bilinear operator and a positive linear operator provided that F is uniformly complete, see [5, Theorem 9] and [3, Theorem 3.1].

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We now present the needed structure result from [4, Theorem 2]. Let E be an arbitrary vector lattice and h a lattice homomorphism from E into a Dedekind complete semiprime f-algebra G with a multiplication $\circ$. Then $F := h(E)$ is a sublattice of G and in view of [3, Proposition 2.5] $F^{\odot}$ can be considered also as a sublattice of G with $(x, y) \mapsto x \circ y$ instead of $\odot$. Denote by $F^{(2)}$ the linear hull of $\{x \circ y : u, v \in F\}$. Then $F^{\odot}$ is the sublattice of G generated by $F^{(2)}$ and $F^{(2)}$ is uniformly closed in $F^{\odot}$.

Let a positive linear operator Φ from $F^{(2)}$ to E and an element $\omega \in G$ are such that $h\Phi(u) = \omega \circ u$ for all $u \in F^{(2)}$. Of course, one can consider Φ as a positive operator from $F^{\odot}$ to E^{ru}, the uniform completion of E. Put $x \bullet y := \Phi(hx \circ hy)$ $(x, y \in E)$. Then $(E, \bullet)$ is an almost f-algebra. Indeed, $(x, y) \mapsto x \bullet y$ can be taken as an almost f-algebra multiplication, since evidently $x \wedge y = 0$ implies $hx \circ hy = 0$, whence $x \bullet y = 0$ and its associativity is also easily seen:

$$(x \bullet y) \bullet z = \Phi(\omega \circ hx \circ hy \circ hz) = x \bullet (y \bullet z).$$

It is proved in [4, Theorem 2] that every Archimedean almost f-algebra arises in this way.

$\lhd$ Proof of the Theorem. Let E be a majorizing sublattice of a Dedekind complete vector lattice $\widehat{E}$ and also an almost f-algebra under a multiplication $\bullet$. According to [4, Theorem 2] one can choose F, G, $\circ$, h, and Φ as above. By [1, Theorem 7.17] or [7, Theorem 3.3.11 (2)] there exists a lattice homomorphism $\widehat{h}$ from $\widehat{E}$ onto a sublattice $\widehat{F} \subset G$ extending h. Moreover, F is a majorizing and order dense sublattice of $\widehat{F}$. By [3, Proposition 2.7] $F^{\odot}$ is also a majorizing and order dense sublattice of $(\widehat{F})^{\odot}$. According to [1, Theorem 2.8] or [7, Theorem 3.1.7] the positive operator Φ from $F^{\odot}$ to $E^{\mathrm{ru}} \subset \widehat{E}$ has a positive extension $\widehat{\Phi}$ from$(\widehat{F})^{\odot}$ to $\widehat{E}$. Now, $\widehat{h}\widehat{\Phi}$ is obviously an extension of $h\Phi$ and it remains to ensure that $\widehat{h}\widehat{\Phi}(u) = \omega \circ u$ $(u \in (\widehat{F})^{\odot})$, since in this event an almost f-algebra multiplication on $\widehat{E}$ can be defined by $x \bullet y := \widehat{\Phi}(\widehat{h}(x) \circ \widehat{h}(y))$ $(x, y \in \widehat{E})$ as was observed above. For a fixed $u \in (\widehat{F})^{\odot}$ take arbitrarily $u', u'' \in F^{\odot}$ such that $u' \leqslant u \leqslant u''$. Then $\omega \circ u' = \widehat{h}\widehat{\Phi}(u') \leqslant \widehat{h}\widehat{\Phi}(u) \leqslant \widehat{h}\widehat{\Phi}(u'') = \omega \circ u''$ and thus, by order continuity of f-algebra multiplication, $\sup\{\omega \circ u'\} = \widehat{h}\widehat{\Phi}(u) = \inf\{\omega \circ u''\} = \omega \circ u$. $\rhd$

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