Владикавказский математический журнал
июль–сентябрь, 2007, Том 9, Выпуск 3

UDC 517.98

GENERALIZATION OF EBERLEIN'S AND SINE'S ERGODIC THEOREMS TO *LR*-NETS

E. Yu. Emel'yanov, N. Erkursun

The notion of LR-nets provides an appropriate setting for study of various ergodic theorems in Banach spaces. In the present paper, we prove Theorems 2.1, 3.1 which extend Eberlein's and Sine's ergodic theorems to LR-nets. Together with Theorem 1.1, these two theorems form the necessary background for further investigation of strongly convergent LR-nets. Theorem 2.1 is due to F. Räbiger, and was announced without a proof in [1].

Mathematics Subject Classification (2000): 47A35, 47D99, 47L07.

Key words: Banach space, operator net, LR-net, strong convergence.

1. Basic definitions and examples

1.1. In the following, let X be a Banach space, $\mathscr{L}(X)$ be the space of all bounded linear operators in X, and $I = I_X$ be the identity operator in X. Recall that a family $\Pi = (T_\upsilon)_{\upsilon\in\Upsilon} \subseteq \mathscr{L}(X)$ indexed by a directed set Υ is called an *operator net.* We say that a vector x is *fixed* under the net Π if $T_\upsilon x = x$ for every $T_\upsilon \in \Pi$, and denote by $\mathrm{Fix}(\Pi)$ the set of all fixed vectors of Π. It is easy to see that $\mathrm{Fix}(\Pi)$ is a closed subspace in X. In the following, we call $\mathrm{Fix}(\Pi)$ the *fixed space* of Π. The net Π is called *equi-continuous* if $\sup\limits_{\upsilon\in\Upsilon} \|T_\upsilon\| < \infty$. Remark that this approach to notions of a fixed space and of an equi-continuity of an operator net is not unique one. The net Π is called *strongly convergent* if the norm-limit $\lim\limits_{\upsilon\to\infty} T_\upsilon x$ exists for each x. The next definition extends the notion of an attractor, previously known for directed operator semigroups, to operator nets.

Definition 1.1. Let $\Pi = (T_\upsilon)_{\upsilon\in\Upsilon}$ be an operator net in X, and let $A \subseteq X$. The set A is called an *attractor* for Π if

$$\lim_{\upsilon\to\infty} \mathrm{dist}_{\|\cdot\|}(T_\upsilon x\ A) = 0 \quad (\forall x \in X\ \|x\| \leqslant 1).$$

1.2. The next principal concept has been introduced by F. Räbiger [1] under the name *M-net.* It was motivated by the classical notion of *$\mathscr{T}$-ergodic net* for an operator semigroup $\mathscr{T}$, and by the notion of *M-sequence* introduced by H. P. Lotz [2]. We prefer to use the term *Lotz-Räbiger net* for this notion.

Definition 1.2. An operator net $\Phi = (S_\lambda)_{\lambda\in\Lambda}$ in X is called a *Lotz-Räbiger net* (=*LR-net*) if

(LR1) Φ is equi-continuous and

(LR2) $\lim\limits_{\lambda\to\infty} S_\lambda \circ (S_\mu - I)x = 0 = \lim\limits_{\lambda\to\infty} (S_\mu - I) \circ S_\lambda x$ for all $x \in X$ and $\mu \in \Lambda$.

Remark that the adjoint net to an LR-net is not necessary an LR-net. Any Lotz-Räbiger net enjoys the following simple property.

Proposition 1.1. *Let $\Phi = (S_\lambda)_{\lambda\in\Lambda}$ be an LR-net in X and $x \in X$. Then $x \in \mathrm{Fix}(\Phi)$ if and only if there exists $\lambda(x) \in \Lambda$ satisfying $S_\lambda(x) = x$ for all $\lambda \geqslant \lambda(x)$.*

$\lhd$ The necessity is obvious. To prove sufficiency, take an element $x \in X$ satisfying $S_\lambda(x) = x$ for all $\lambda \geqslant \lambda(x)$. Let ϑ be an arbitrary element of Λ. Accordingly with the condition $(LR\,2)$, and to the continuity of $(S_\vartheta - I)$ we obtain

$$(S_\vartheta - I)\,(\lim_{\lambda\to\infty} S_\lambda x) \;=\; \lim_{\lambda\to\infty}(S_\vartheta - I)\circ S_\lambda x \;=0.$$

The equality $\lim\limits_{\lambda\to\infty} S_\lambda x = x$ holds by the assumption, since $S_\lambda x = x$ for all $\lambda \geqslant \lambda(x)$. Therefore $(S_\vartheta - I)\,x \;=\; 0$ for all $\vartheta \in \Lambda$ or, equivalently, $x \in \mathrm{Fix}(\Phi)$. $\rhd$

1.3. The following elementary result explains the relationship between the strong convergence and the fixed space of an LR-net. Remark that Propositions 1.1 and 1.2 cannot be extended to an arbitrary equi-continuous operator net.

Proposition 1.2. *Let $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ be an LR-net in X. Then Θ is strongly convergent if and only if*

$$X = \mathrm{Fix}(\Theta) \oplus \overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}. \tag{1}$$

Moreover, in this case, the strong limit P of Θ is a projection onto $\mathrm{Fix}(\Theta)$.

$\lhd$ The sufficiency. Assume $X = \mathrm{Fix}(\Theta) \oplus \overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}$. The net Θ is equi-continuous. Therefore, to show that Θ converges strongly, it is enough to prove that the norm-limit $\lim\limits_{\lambda\to\infty} T_\lambda x$ exists for all $x \in (I - T_\mu)X$, where μ is an arbitrary element of Λ. Fix a $\mu \in \Lambda$ and an $x \in (I - T_\mu)X$. Thus $x = (T_\mu - I)v$ for some $v \in X$. Now, the norm-convergence of the net $(T_\lambda\, x)_{\lambda\in\Lambda}$ is provided by $(LR\,2)$, since

$$\lim_{\lambda\to\infty} T_\lambda x = \lim_{\lambda\to\infty} T_\lambda \circ (T_\mu - I)v = 0. \tag{2}$$

The necessity. Assume Θ is strongly convergent. Then the strong limit P of Θ is a continuous operator. Take an $x \in X$. In view of $(LR\,2)$, we have

$$T_\mu(Px) - Px = (T_\mu - I)\lim_{\lambda\to\infty} T_\lambda x = \lim_{\lambda\to\infty}(T_\mu - I)\circ T_\lambda x = 0$$

for every $\mu \in \Lambda$. Hence, $Px \in \mathrm{Fix}(\Theta)$ and $P^2x = Px$. By arbitrariness of x, P is a continuous projection onto $\mathrm{Fix}(\Theta)$, henceforth $X = P(X) \oplus \ker P$ and $P(X) = \mathrm{Fix}(\Theta)$. Thus, to show $X = \mathrm{Fix}(\Theta) \oplus \overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}$ it is enough to prove

$$\overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X} = \ker P. \tag{3}$$

The inclusion $\overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X} \subseteq \ker P$ follows from (2). Assume $x \in \ker P$. Then $\lim\limits_{\lambda\to\infty} T_\lambda x = Px = 0$ and

$$x = \lim_{\lambda\to\infty}(I - T_\lambda)x \in \overline{\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X} \subseteq \overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}.$$

This proves the equality (3) and therefore the required decomposition $X = \mathrm{Fix}(\Theta) \oplus \overline{\mathrm{span}\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}$. $\rhd$

1.4. Many concrete examples of LR-nets appear naturally in the investigation of operator semigroups. Every equi-continuous strongly convergent directed semigroup $\mathscr{T} \subseteq \mathscr{L}(X)$ is an LR-net. We refer for examples of LR-nets to the papers [2] and [1] of Lotz and Räbiger, and to the Krengel book [3]. Here we mention only a few of them.

Example 1.1. Let $T \in \mathscr{L}(X)$ be an operator in X satisfying $n^{-1}T^n \to 0$ strongly, with the uniformly bounded sequence of *Cesàro averages* $\mathscr{A}_n^T := \frac{1}{n}\sum\limits_{k=0}^{n-1} T^k$. It is well known that $(\mathscr{A}_n^T)_{n=1}^{\infty}$ satisfies $(LR2)$ and therefore becomes an LR-net. The continuous version of this LR-net is obvious.

Example 1.2. Every *$\mathscr{T}$-ergodic net* for a given operator semigroup $\mathscr{T} \subseteq \mathscr{L}(X)$ (see [3, p. 75] for the definition) is an LR-net. This extends the previous example.

It can be easily shown [3, p. 75] that every equi-continuous Abelian operator semigroup $\mathscr{T}$ admits a $\mathscr{T}$-ergodic net. We emphasize that not every LR-net can be represented as a $\mathscr{T}$-ergodic net for some semigroup $\mathscr{T}$.

Example 1.3. Let $\Lambda \subseteq \mathbb{C}$ be a directed set and let $(R_\lambda)_{\lambda\in\Lambda} \subseteq \mathscr{L}(X)$ be a *pseudoresolvent*, i.e. satisfies the Hilbert identity $R_\lambda - R_\mu = (\mu-\lambda)R_\lambda \circ R_\mu$ for all $\lambda, \mu \in \Lambda$. Let the net $(\lambda R_\lambda)_{\lambda\in\Lambda}$ be equi-continuous.

(a) If $\lim\limits_{\lambda\to\infty} \lambda = a \in \mathbb{C}$ then $((\lambda - c)R_\lambda)_{\lambda\in\Lambda}$ is an LR-net.

(b) If $\lim\limits_{\lambda\to\infty} |\lambda| = \infty$ then $(I - \lambda R_\lambda)_{\lambda\in\Lambda}$ is an LR-net.

2. Extension of Eberlein's theorem to LR-net

2.1. A lot of results about concrete LR-nets, mentioned in subsection 1.4, belongs to the classical ergodic theory. However, only very few facts about general LR-nets, like Propositions 1.1, 1.2, are known. For instance, Theorem 2.1 below is well know more than sixty years for LR-nets of Cesàro averages as the *Mean Ergodic Theorem*, and for $\mathscr{T}$-ergodic nets it is known as the *Eberlein Theorem*. Theorem 2.1 had been proved for M-sequences by Lotz in [2, Theorem 3]. The general form of Theorem 2.1 had been stated, without a proof, by Räbiger in [1, Proposition 2.3]. In this section, we present the complete proof of this result.

Theorem 2.1 (Räbiger). *Let $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ be an LR-net in X. Then the following assertions are equivalent*:

(i) *Θ is strongly convergent.*

(ii) *The net $(T_\lambda x)_{\lambda\in\Lambda}$ has a weak cluster point for every $x \in X$.*

◁ The implication $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$: Let $x \in X$ be an arbitrary element, and let y be a weak cluster point of the set $(T_\lambda x)_{\lambda\in\Lambda}$. By the Mazur theorem,

$$y \in \mathrm{co}(T_\lambda x)_{\lambda\in\Lambda}. \tag{4}$$

(I) Firstly, we show that $y \in \mathrm{Fix}(\Theta)$. It is sufficient to prove that, for arbitrary $\mu \in \Lambda$ and $\varepsilon > 0$,

$$|\langle T_\mu y, h\rangle - \langle y, h\rangle| \leqslant \varepsilon \tag{5}$$

holds. Fix a $\mu \in \Lambda$ and an $\varepsilon > 0$. Since y is a weak cluster point of the LR-net $(T_\lambda x)_{\lambda\in\Lambda}$, there exists a $\zeta \in \Lambda$ satisfying the following three formulas:

$$|\langle T_\mu\, y, h\rangle - \langle T_\mu \circ T_\zeta\, x, h\rangle| \leqslant \frac{\varepsilon}{3}\,, \tag{6}$$

$$|\langle T_\mu \circ T_\zeta\, x, h\rangle - \langle T_\zeta\, x, h\rangle| \leqslant \frac{\varepsilon}{3}\,, \tag{7}$$

$$|\langle T_\zeta\, x, h\rangle - \langle y, h\rangle| \leqslant \frac{\varepsilon}{3}. \tag{8}$$

The summation of (6), (7), and (8), gives (5). Therefore $y \in \mathrm{Fix}(\Theta)$.

(II) Now we prove that the net $(T_\lambda x)_{\lambda\in\Lambda}$ converges in the norm to y. The supremum $M := \sup_{\lambda\in\Lambda} \|T_\lambda x\|$ is finite since the net Θ is equi-continuous. Fix an $\varepsilon > 0$. By (4), there exists an $S \in \mathrm{co}(T_\lambda)_{\lambda\in\Lambda}$ satisfying

$$\|y - Sx\| \leqslant \varepsilon. \tag{9}$$

By $(LR\,2)$, we have

$$\lim_{\lambda\to\infty} T_\lambda \circ (S - I)\, x = 0. \tag{10}$$

In view of (10), there exists a $\lambda_0 \in \Lambda$ (which depends on x and ε) satisfying

$$\|T_\lambda \circ S\, x - T_\lambda\, x\| \leqslant \varepsilon \quad (\forall \lambda \geqslant \lambda_0). \tag{11}$$

The combining of (9) and (11) with $y \in \mathrm{Fix}(\Theta)$ (this was proved in **(I)**), gives us

$$\|y - T_\lambda x\| = \|y - T_\lambda \circ S\, x + T_\lambda \circ S\, x - T_\lambda x\| \leqslant$$

$$\|T_\lambda(y - S\, x)\| + \|T_\lambda \circ S\, x - T_\lambda x\| \leqslant M\varepsilon + \varepsilon \quad (\forall \lambda \geqslant \lambda_0). \tag{12}$$

Since $x \in X$ and $\varepsilon > 0$ were chosen arbitrary, the formula (12) implies that the net Θ converges strongly. $\triangleright$

2.2. It follows immediately from Theorem 2.1 and Definition 1.1, that every *LR*-net possessing a weakly compact attractor is strongly convergent. This property explains a big difference between the strong convergence of semigroups and the strong convergence of *LR*-nets. A principal source of this difference consists in the fact that *not every* bounded operator semigroup is an *LR*-net. Another important difference consists in the fact that the convex hull of any semigroup $\mathscr{T} \subseteq \mathscr{L}(X)$ is always a semigroup, but the convex hull of an *LR*-net is not always an *LR*-net.

3. Extension of Sine's theorem to *LR*-nets

3.1. In this section we present an extension of Sine's ergodic theorem to *LR*-nets. This theorem was discovered by R. Sine in the special case when an *LR*-net is a net of Cesàro averages of a single operator (cf. Krengel's book [3, Theorem 2.1.4]). It was extended to arbitrary $\mathscr{T}$-ergodic nets, by J.J. Koliha, R. Nagel, and R. Sato [3, Theorem 2.1.9]. For "small" *LR*-nets (=M-sequences) it is due to H.P. Lotz [2, Theorem 3].

Theorem 3.1. *An LR-net $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ in X is strongly convergent if and only if its fixed space* $\mathrm{Fix}(\Theta)$ *separates the fixed space* $\mathrm{Fix}(\Theta^*)$ *of the adjoint operator net $\Theta^* = (T^*_\lambda)_{\lambda\in\Lambda}$ in X^*.*

◁ Assume that $\mathrm{Fix}(\Theta)$ separates $\mathrm{Fix}(\Theta^*)$. In view of Proposition 1.2, to show the strong convergence of Θ, it suffices to prove (1). If (1) is failed then, by the Hahn – Banach theorem, there exists an $h \in X^*$, $h \neq 0$, with $\langle x, h\rangle = 0$ for all

$$x \in \mathrm{Fix}(\Theta) \oplus \overline{\mathrm{span} \bigcup_{\lambda\in\Lambda} (I - T_\lambda)X}.$$

Show that $h \in \mathrm{Fix}(\Theta^*)$. Since

$$(y - T_\mu y) \in \overline{\mathrm{span} \bigcup_{\lambda\in\Lambda} (I - T_\lambda)X} \quad (\forall y \in X\,, \forall \mu \in \Lambda)\,,$$

we have

$$\langle y, h\rangle = \langle T_\mu y, h\rangle = \langle y, T^*_\mu h\rangle = 0 \quad (\forall y \in X\,, \forall \mu \in \Lambda). \tag{13}$$

It follows from (13) that $T^*_\mu h = h$ for all $\mu \in \Lambda$ and therefore $h \in \mathrm{Fix}(\Theta^*)$. Thus, h is a nonzero fixed point of Θ^* such that $\langle x, h\rangle = 0$ for all $x \in \mathrm{Fix}(\Theta)$. This contradicts the assumption.

Assume the net Θ converges strongly, and denote its limit in the strong operator topology by P. Take an $h \in \mathrm{Fix}(\Theta^*)$, $h \neq 0$. In view of $h \neq 0$, there exists an $x \in X$ with $\langle x, h\rangle \neq 0$. Consequently

$$\langle Px, h\rangle = \lim_{\lambda\to\infty} \langle T_\lambda x, h\rangle = \lim_{\lambda\to\infty} \langle x, T^*_\lambda h\rangle = \langle x, h\rangle \neq 0. \tag{14}$$

Since $Px \in \mathrm{Fix}(\Theta)$, the formula (14) shows that $\mathrm{Fix}(\Theta)$ separates $\mathrm{Fix}(\Theta^*)$. ▷

References

1. *Räbiger F.* Stability and ergodicity of dominated semigroups: II. The strong case // Math. Ann.—1993.—V. 297.—P. 103–116.
2. *Lotz H. P.* Tauberian theorems for operators on Banach spaces // Semesterbericht Functionalanalysis, Tübingen, WS—1983/84.—P. 1–15.
3. *Krengel U.* Ergodic Theorems.—Berlin– New York: De Gruyter, 1985.

Received 5 November, 2006

Eduard Emel'yanov
Middle East Technical University
Ankara, 06531, TURKEY
E-mail: `emelanov@math.metu.edu.tr`

Nazife Erkusan
Middle East Technical University
Ankara, 06531, TURKEY