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ON BOREL'S EXTENSION THEOREM FOR GENERAL BEURLING CLASSES OF ULTRADIFFERENTIABLE FUNCTIONS

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We obtain necessary and sufficient conditions under which general Beurling class of ultradifferentiable functions admits a version of Borel's extension theorem.

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1. Introduction

Definition 1.1. An increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ is called a weight function if

$$\log t = o(\omega(t)),\ t \to \infty;$$
$$\omega(t) = O(t);\ t \to \infty;$$
$$\varphi_\omega(x) := \omega(e^x) \text{ is convex on } [x_0, \infty).$$

A weight function ω with $\int\limits_1^\infty t^{-2}\omega(t)\,dt < \infty$ is called nonquasianalytic.

Denote by $W_\uparrow$ the set of all sequences $\Omega = \{\omega_n\}_{n=1}^\infty$ of weight functions with the folllowing property: for each $n \in \mathbb{N}$ there exists a $C_n > 0$ such that

$$\omega_n(t) + \log(t + 1) \leqslant \omega_{n+1}(t) + C_n \quad \text{for } t \geqslant 0. \tag{1}$$

By $W_\uparrow^{nq}$ denote the set of all sequences $\Omega = \{\omega_n\}_{n=1}^\infty$ of nonquasianalytic weight functions ω_n. Without loss of generality we can assume that

$$\omega_n(t) \leqslant \omega_{n+1}(t) \quad \text{for } t \geqslant 0 \text{ and } n \in \mathbb{N}.$$

The Young conjugate $\varphi_\omega^* : [0, \infty) \to [0, \infty)$ of φ_ω is defined by

$$\varphi_\omega^*(y) := \sup\{xy - \varphi_\omega(x) : x \geqslant 0\}.$$

For $A \in (0, \infty)$ we define the space

$$\mathscr{E}_\omega(\Pi_A^N) := \left\{ f \in C^\infty(\Pi_A^N) : \ |f|_{\omega,A,N} := \sup_{\alpha\in\mathbb{N}_0^N} \sup_{\|x\|\leqslant A} \frac{\left|f^{(\alpha)}(x)\right|}{e^{\varphi_\omega^*(|\alpha|)}} < \infty \right\},$$

where $\Pi_A^N := \{x \in \mathbb{R}^N : \|x\| \leqslant A\}$, $\|x\| := \max\{|x_j| : 1 \leqslant j \leqslant N\}$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $|\alpha| := \alpha_1 + \ldots + \alpha_N$ for $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$, $f^{(\alpha)} := \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$.

Next, for a weight sequence $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}$ we put

$$\mathscr{E}_{(\Omega)}(\Pi_A^N) := \bigcap_{n=1}^{\infty} \mathscr{E}_{\omega_n}(\Pi_A^N);$$

$$\mathscr{E}_{(\Omega)}(\mathbb{R}^N) := \left\{ f \in C^{\infty}(\mathbb{R}^N) \ : \ f\big|_{\Pi_A^N} \in \mathscr{E}_{(\Omega)}(\Pi_A^N) \text{ for each } A > 0 \right\}.$$

The elements of $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ are called Ω-ultradifferentiable functions of Beurling type.

Let us introduce now the corresponding spaces of sequences of complex numbers:

$$\mathscr{E}_{\omega}^N := \left\{ d = (d_\alpha)_{\alpha \in \mathbb{N}_0^N} \in \mathbb{C}^{\mathbb{N}_0^N} \ : \ |d|_{\omega,N} := \sup_{\alpha \in \mathbb{N}_0^N} \frac{|d_\alpha|}{e^{\varphi_\omega^*(|\alpha|)}} < \infty \right\}$$

and

$$\mathscr{E}_{(\Omega)}^N := \bigcap_{n=1}^{\infty} \mathscr{E}_{\omega_n}^N.$$

It is clear that the restriction operator $\rho : f \in C^{\infty}(\mathbb{R}^N) \mapsto \left(f^{(\alpha)}(0)\right)_{\alpha \in \mathbb{N}_0^N}$ acts from $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ into $\mathscr{E}_{(\Omega)}^N$. If ρ is surjective, we will say that a version of Borel's extension theorem holds for the space $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ (for the original Borel's extension theorem see [7]). For minimal Beurling class ($\omega_n = n\omega$, ω is nonquasianalytic and $\omega(2t) = O(\omega(t))$ as $t \to \infty$) Meise and Taylor [8] have shown that $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ admits a version of Borel's extension theorem if and only if ω is strong, i. e. there exists a $C > 0$ such that $\int\limits_1^{\infty} t^{-2}\omega(yt)\,dt \leqslant C\omega(y) + C$ for $y \geqslant 0$. The case of normal Beurling class, when $\Omega = \{q_n\omega\}_{n=1}^{\infty}$ with $q_n \uparrow q \in (0,\infty)$ and ω is a nonquasianalytic almost subadditive weight function, has been studied by the author in [4]. In this case, $\rho : \mathscr{E}_{(\Omega)}(\mathbb{R}^N) \to \mathscr{E}_{(\Omega)}^N$ is surjective iff ω is slowly varying, i. e. $\lim\limits_{t\to\infty} \frac{\omega(2t)}{\omega(t)} = 1$.

The main result of the present article is the following theorem.

Theorem. *Let $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}^{nq}$. Each of the following two conditions is sufficient for $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ to admit a version of Borel's extension theorem:*

(I) *for every $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\omega_n(\|\xi + t\eta\|)}{t^2+1}\,dt \leqslant \omega_m(\|\xi + i\eta\|) + C \ \ \text{for } \xi + i\eta \in \mathbb{C}^N;$$

(II) *for every $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$\omega_n(2t) \leqslant \omega_m(t) + C \ \ \text{for } t \geqslant 0$$

and

$$\frac{4}{\pi}\int\limits_1^{\infty} \frac{\omega_n(yt)}{t^2+1}\,dt \leqslant \omega_m(y) + C \ \ \text{for } y \geqslant 0.$$

Suppose additionally that for each $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $C > 0$ so that $\omega_n(x+y) \leqslant \omega_m(x) + \omega_m(y) + C$ for $x, y \geqslant 0$. If Borel's extension theorem holds for $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ for at least one $N \in \mathbb{N}$, then

(III) *for every* $n \in \mathbb{N}$ *there exist* $m \in \mathbb{N}$ *and* $C > 0$ *such that*

$$\frac{2}{\pi}\int\limits_0^\infty \frac{\omega_n(yt)}{t^2+1}\,dt \leqslant \omega_m(y) + C \quad \textit{for } y \geqslant 0\,.$$

This theorem generalizes the results of [8] and [4] mentioned above. It should be also noted that (II) implies (I).

The paper has five sections. In Section 2 we get a criterion of surjectivity of ρ in terms of entire functions. In Section 3, using the method of Meise and Taylor [8], we obtain sufficient conditions on $\Omega \in W_\uparrow^{nq}$ under which $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ admits a version of Borel's extension theorem. Necessary conditions are derived in Section 4 by the method of [8] and [1]. The last section consists of two new examples of Beurling classes. We show that Borel's extension theorem holds for the first class and does not hold for the second one.

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2. Criterion in terms of entire functions

Let $\Omega = \{\omega_n\}_{n=1}^\infty$ be in $W_\uparrow$, and let the topology of $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ (resp. $\mathscr{E}_{(\Omega)}^N$) be given by the system of seminorms $\big(|\cdot|_{\omega_n,n,N}\big)_{n\in\mathbb{N}}$ (resp. by the normsystem $\big(|\cdot|_{\omega_n,N}\big)_{n\in\mathbb{N}}$).

For a weight function ω and a number $A \in (0,\infty)$ we define the following space of entire functions

$$H_{\omega,A}(\mathbb{C}^N) := \left\{ f \in H(\mathbb{C}^N) \;:\; \|f\|_{\omega,A,N} := \sup_{z\in\mathbb{C}^N} \frac{|f(z)|}{e^{A\|Im\, z\| + \omega(\|z\|)}} < \infty \right\},$$

where $\|z\| = \max\{|z_j| \;:\; 1 \leqslant j \leqslant N\}$ for $z = (z_1,\dots,z_N) \in \mathbb{C}^N$. Obviously, $H_{\omega,A}(\mathbb{C}^N)$ is a Banach space with the norm $\|\cdot\|_{\omega,A,N}$. Next, for a weight sequence $\Omega = \{\omega_n\}_{n=1}^\infty \in W_\uparrow$ we put

$$H_{(\Omega)}(\mathbb{C}^N) := \bigcup_{n=1}^\infty H_{\omega_n,n}(\mathbb{C}^N), \quad H_{(\Omega)}^N := \bigcup_{n=1}^\infty H_{\omega_n,0}(\mathbb{C}^N)\,.$$

Let $H_{(\Omega)}(\mathbb{C}^N)$ (resp. $H_{(\Omega)}^N$) be equipped with the topology of $\operatorname*{ind}_{n\in\mathbb{N}} H_{\omega_n,n}(\mathbb{C}^N)$ (resp. $\operatorname*{ind}_{n\in\mathbb{N}} H_{\omega_n,0}(\mathbb{C}^N)$). Note that $H_{(\Omega)}^N$ and $H_{(\Omega)}(\mathbb{R}^N)$ are (DFS)-spaces.

By theorem 1 of [3], the Fourier–Laplace transform

$$\widetilde{F} : \mu \mapsto \widehat{\mu}(z) = \mu_x\left(e^{-i\langle x,z\rangle}\right)$$

is a topological isomorphism from $\big(\mathscr{E}_{(\Omega)}(\mathbb{R}^N)\big)_b'$ onto $H_{(\Omega)}(\mathbb{C}^N)$. As usual, we denote by E_b' the strong dual of a local convex space E.

A description of $\big(\mathscr{E}_{(\Omega)}^N\big)_b'$ is given by

Proposition 2.1. *Let* e_α, $\alpha \in \mathbb{N}_0^N$, *be unit vectors in* $\mathbb{R}^{\mathbb{N}_0^N}$, *and* $\Omega = \{\omega_n\}_{n=1}^\infty$ *be in* $W_\uparrow$. *Then the Fourier–Laplace transform*

$$F : \mu \mapsto \widehat{\mu}(z) = \sum_{\alpha\in\mathbb{N}_0^N} \mu(e_\alpha)(-iz)^\alpha$$

is a topological isomorphism from $\big(\mathscr{E}_{(\Omega)}^N\big)_b'$ *onto* $H_{(\Omega)}^N$.

Here $z^\alpha = z_1^{\alpha_1}\dots z_N^{\alpha_N}$ for $z = (z_1,\dots,z_N) \in \mathbb{C}^N$ and $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}_0^N$.

◁ Since the proof is very similar to that of [4], we omit it. It should be only pointed out that the proof is based on the following property of $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}$ derived in Lemma 1 of [3]: if B_n is determined by the condition

$$\omega_n(t) \leqslant B_n t \quad \text{for } t \geqslant 1, \tag{2}$$

and C_n is determined by (1), then

$$\varphi^*_{\omega_{n+1}}(s) - \varphi^*_{\omega_n}(s) \leqslant -\log s + \log(B_n e^{C_n+1}) \quad \text{for all } s > 0. \ \triangleright$$

Now we have a commutative diagram

$$\begin{tikzcd} \left(\mathscr{E}_{(\Omega)}(\mathbb{R}^N)\right)'_b \arrow[d, "\widetilde{F}"'] & \left(\mathscr{E}^N_{(\Omega)}\right)'_b \arrow[l, "\rho'"'] \arrow[d, "F"] \\ H_{(\Omega)}(\mathbb{C}^N) & H^N_{(\Omega)} \arrow[l, "id"'] \end{tikzcd}$$

where ρ' is the conjugate operator of ρ. It is easily checked that $\widetilde{F} \circ \rho' \circ F^{-1}$ is the identity mapping acting from $H^N_{(\Omega)}$ into $H_{(\Omega)}(\mathbb{C}^N)$.

Our main result in this section is

Theorem 2.2. *Let $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}$. Then the following assertions are equivalent:*

(i) *a version of Borel's extension theorem holds for $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$;*

(ii) *for each set $B \subset H^N_{(\Omega)}$ contained and bounded in $H_{\omega_n,n}(\mathbb{C}^N)$ for some $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that B is contained and bounded in $H_{\omega_m,0}(\mathbb{C}^N)$;*

(iii) *for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$\sup_{z \in \mathbb{C}^N} \frac{|f(z)|}{e^{\omega_m(\|z\|)}} \leqslant C \sup_{z \in \mathbb{C}^N} \frac{|f(z)|}{e^{n\|Im\, z\| + \omega_n(\|z\|)}} \quad \text{for all } f \in H^N_{(\Omega)}; \tag{3}$$

(iv) *for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ so that*

$$|f(z)| \leqslant e^{n\|Im\, z\| + \omega_n(\|z\|)} \quad \text{for all } z \in \mathbb{C}^N, \quad f \in H^N_{(\Omega)} \tag{4}$$

imply

$$|f(z)| \leqslant C\, e^{\omega_m(\|z\|)} \quad \text{for all } z \in \mathbb{C}^N. \tag{5}$$

◁ (i) ⇔ (ii): By the Surjectivity criterion 26.1 of [9], ρ maps $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ onto $\mathscr{E}^N_{(\Omega)}$ if and only if for each bounded set A in $\left(\mathscr{E}_{(\Omega)}(\mathbb{R}^N)\right)'_b$ the set $(\rho')^{-1}(A)$ is bounded in $\left(\mathscr{E}^N_{(\Omega)}\right)'_b$. With the commutative diagram the first part of the theorem is proved.

(ii) ⇒ (iii): Fix any $n \in \mathbb{N}$ and set $B_n := \{f \in H(\mathbb{C}^N) : \|f\|_{\omega_n,n,N} \leqslant 1\}$. Using (ii) with $B = H^N_{(\Omega)} \cap B_n$, we deduce that there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\|g\|_{\omega_m,0,N} \leqslant C \quad \text{for all } g \in H^N_{(\Omega)} \cap B_n. \tag{6}$$

Let $f \in H^N_{(\Omega)}$ be fixed. If $\|f\|_{\omega_n,n,N} = 0$ or $\|f\|_{\omega_n,n,N} = \infty$, then (3) is trivially true. In case $0 < \|f\|_{\omega_n,n,N} < \infty$ we use (6) with $g = \frac{f}{\|f\|_{\omega_n,n,N}}$. Then we have

$$\|f\|_{\omega_m,0,N} \leqslant C\, \|f\|_{\omega_n,n,N}.$$

This means that (3) holds.

Implications (iii) ⇒ (iv) ⇒ (ii) are easily checked. ▷

3. Sufficient conditions

Throughout last three sections we suppose that $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}^{nq}$. We start by

Proposition 3.1. *Let $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}^{nq}$. Then the space $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ is nonquasianalytic. That is, there is a function $f \in \mathscr{E}_{(\Omega)}(\mathbb{R}^N)\backslash\{0\}$ such that*

$$f^{(\alpha)}(0) = 0 \quad \textit{for all } \alpha \in \mathbb{N}_0^N .$$

$\lhd$ First we choose $0 = t_0 < t_1 < \ldots$ satisfying

$$\int\limits_{t_n}^{\infty} \frac{\omega_n(t)}{t^2}\, dt < \frac{1}{n^3} \quad \text{for each } n \in \mathbb{N} .$$

Then we introduce a function

$$\omega(t) = \begin{cases} 0 & \text{for } t \in [0, t_1), \\ n\omega_n(t) & \text{for } t \in [t_n, t_{n+1}) . \end{cases}$$

Since we assume that $\omega_n(t) \leqslant \omega_{n+1}(t)$ for $t \geqslant 0$, it follows that ω is nondecreasing on $[0, \infty)$. Next, $\omega_n(t) = o(\omega(t))$ as $t \to \infty$ and

$$\int\limits_{1}^{\infty} \frac{\omega(t)}{t^2}\, dt = \sum_{n=1}^{\infty} \int\limits_{t_n}^{t_{n+1}} \frac{n\omega_n(t)}{t^2}\, dt \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By Lemma 3.2 of [2] we find a cotinuous nondecreasing function $\sigma : [0, \infty) \to [0, \infty)$ such that

$$\omega(t) = o(\sigma(t)), \quad t \to \infty; \quad \sigma(2t) \leqslant 4\sigma(t) \text{ for all } t \geqslant 0;$$
$$\int\limits_{1}^{\infty} \frac{\sigma(t)}{t^2}\, dt < \infty.$$

Using Proposition 2.3 of [5] for the σ and compact set $K = \{0\}$, we construct a function $\varphi \in C^{\infty}(\mathbb{R}^N)$ with the following properties:

$$\varphi(x) = 1 \text{ for } x \in [-\varepsilon, \varepsilon]^N; \quad \operatorname{supp} \varphi \subset [-3\varepsilon, 3\varepsilon]^N;$$
$$A_{\varphi,\sigma} := \int\limits_{\mathbb{R}^N} |\widehat{\varphi}(t)|\, e^{\sigma(\|t\|)}\, dt < \infty.$$

Here $\widehat{\varphi}(t) = \int\limits_{\mathbb{R}^N} \varphi(x)\, e^{-i\langle t,x\rangle}\, dx$ is a Fourier transformation of φ.

We wish now to show that $\varphi \in \mathscr{E}_{(\Omega)}(\mathbb{R}^N)$. By the Fourier inversion formula we have

$$\begin{aligned} |\varphi^{(\alpha)}(x)| &\leqslant \left| \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} \widehat{\varphi}(t)\, (it)^{\alpha}\, e^{i\langle t,x\rangle}\, dt \right| \leqslant \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} |\widehat{\varphi}(t)|\, \|t\|^{|\alpha|}\, dt \leqslant \\ &\leqslant \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} |\widehat{\varphi}(t)|\, e^{\omega_n(\|t\|)}\, dt \cdot \exp \sup_{t \in \mathbb{R}^N\backslash\{0\}} \big(|\alpha| \log \|t\| - \omega_n(\|t\|)\big) \leqslant \\ &\leqslant \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} |\widehat{\varphi}(t)|\, e^{\omega_n(\|t\|)}\, dt \cdot e^{\varphi^*_{\omega_n}(|\alpha|)} . \end{aligned}$$

Since $\omega_n(t) = o(\omega(t)) = o(o(\sigma(t))) = o(\sigma(t))$ as $t \to \infty$, there is an $M_n > 0$ such that

$$\int\limits_{\mathbb{R}^N} |\widehat{\varphi}(t)|\, e^{\omega_n(\|t\|)}\, dt \leqslant \int\limits_{\mathbb{R}^N} |\widehat{\varphi}(t)|\, e^{\sigma(\|t\|)+M_n}\, dt = e^{M_n}\, A_{\varphi,\sigma}\,.$$

Hence,

$$|\varphi^{(\alpha)}(x)| \leqslant \frac{e^{M_n}\, A_{\varphi,\sigma}}{(2\pi)^N}\, e^{\varphi^*_{\omega_n}(|\alpha|)} \ \text{ for all } \alpha \in \mathbb{N}_0^N.$$

This means that $\varphi \in \mathscr{E}_{(\Omega)}(\mathbb{R}^N)$. Setting $f(x) = \varphi(x) - 1$ we finally obtain the result. $\rhd$

To formulate the main result of the section we need some notation. For a nonquasianalytic function ω we define the function $P_\omega : \mathbb{C}^N \to \mathbb{R}$ as follows:

$$P_\omega(x+iy) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\omega(\|x+ty\|)}{t^2+1}\, dt \ \text{ for } x, y \in \mathbb{R}^N.$$

It should be noted that in earlier papers dealt with extension theorems (for instance, in [8] and [4]) the function P_ω was considered for $N=1$ only. As is well known, in case $N=1$, P_ω is harmonic in the open upper and lower half plane and it is continuous and subharmonic in the whole plane $\mathbb{C}$. Moreover, $\omega(|z|) \leqslant P_\omega(z)$ for $z \in \mathbb{C}$.

Theorem 3.2. *Let $\Omega = \{\omega_n\}_{n=1}^\infty \in W_\uparrow^{nq}$. Suppose that*
(I) *for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$P_{\omega_n}(z) \leqslant \omega_m(\|z\|) + C \ \text{ for all } z \in \mathbb{C}^N.$$

Then the operator $\rho : \mathscr{E}_{(\Omega)}(\mathbb{R}^N) \to \mathscr{E}^N_{(\Omega)}$ is surjective.

$\lhd$ Let us show that condition (iv) of Theorem 2.2 holds. Fix any $n \in \mathbb{N}$. Assume that $f \in H^N_{(\Omega)}$ satisfies (4). Then there are $n_f \in \mathbb{N}$ and $D_f > 0$ such that

$$|f(z)| \leqslant D_f\, e^{\omega_{n_f}(\|z\|)} \ \text{ for } z \in \mathbb{C}^N. \tag{7}$$

Given $z = x+iy \in \mathbb{C}^N$ with $y \neq 0$, we define the entire function

$$F : \mathbb{C} \to \mathbb{C},\ F(w) := f\left(x + w\,\frac{y}{\|y\|}\right) \ \text{ for } w \in \mathbb{C}.$$

We can rewrite (7) as

$$|F(w)| \leqslant D_f\, e^{\omega_{n_f}(\|x\|+|w|)} \ \text{ for all } w \in \mathbb{C}. \tag{8}$$

Next, since

$$\left\|\mathrm{Im}\left(x + w\,\frac{y}{\|y\|}\right)\right\| = |\,\mathrm{Im}\ w|,$$

(4) implies that

$$|F(w)| \leqslant \exp\left(n|\,\mathrm{Im}\ w| + \omega_n\left(\left\|x + w\frac{y}{\|y\|}\right\|\right)\right) \ \text{ for } w \in \mathbb{C}. \tag{9}$$

By the Phragmèn–Lindelöf principle (Theorem 6.5.4 in [6]) we find that for $u \in \mathbb{R}$, $v \in \mathbb{R}\backslash\{0\}$

$$\log|F(u+iv)| \leqslant \frac{|v|}{\pi} \int\limits_{-\infty}^{\infty} \frac{\log|F(t)|}{(u-t)^2+v^2}\,dt + |v|d, \tag{10}$$

where

$$d = \limsup_{r\to\infty} \frac{2}{\pi}\frac{1}{r}\int\limits_0^\pi \log\left|F(re^{i\theta})\right| \sin\theta\,d\theta\,.$$

Using (8) we have

$$\frac{2}{\pi}\frac{1}{r}\int\limits_0^\pi \log\left|F(re^{i\theta})\right|\sin\theta\,d\theta \leqslant \frac{2}{\pi}\frac{1}{r}\int\limits_0^\pi \big(\log D_f + \omega_{n_f}(\|x\|+r)\big)\sin\theta\,d\theta =$$
$$= \frac{4}{\pi}\Big(\frac{\log D_f}{r} + \frac{\omega_{n_f}(\|x\|+r)}{r}\Big) \quad \text{for all } r \geqslant 0.$$

Nonquasianalyticity of ω_{n_f} gives us that

$$\frac{\omega_{n_f}(r+\|x\|)}{r+\|x\|} = \int\limits_{r+\|x\|}^{\infty} \frac{\omega_{n_f}(r+\|x\|)}{s^2}\,ds \leqslant \int\limits_{r+\|x\|}^{\infty} \frac{\omega_{n_f}(s)}{s^2}\,ds \to 0 \text{ as } r\to\infty\,.$$

By the above this means that $d \leqslant 0$. Now, using (9) in (10) and (I), we have

$$\log|F(u+iv)| \leqslant \frac{|v|}{\pi}\int\limits_{-\infty}^{\infty} \frac{\omega_n(\|x + t\,\frac{y}{\|y\|}\|)}{(u-t)^2+v^2}\,dt =$$
$$= \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\omega_n(\|x+(u+vt)\,\frac{y}{\|y\|}\|)}{t^2+1}\,dt = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\omega_n(\|(x+u\,\frac{y}{\|y\|})+tv\,\frac{y}{\|y\|}\|)}{t^2+1}\,dt =$$
$$= P_{\omega_n}\Big(\big(x+u\,\frac{y}{\|y\|}\big)+iv\frac{y}{\|y\|}\Big) \leqslant \omega_m\Big(\Big\|\big(x+u\,\frac{y}{\|y\|}\big)+iv\frac{y}{\|y\|}\Big\|\Big)+C\,.$$

Setting $u=0$, $v=\|y\|$ we get

$$\log|f(x+iy)| = \log|F(i\|y\|)| \leqslant \omega_m(\|x+iy\|)+C.$$

Thus,

$$|f(z)| \leqslant e^C\,e^{\omega_m(\|z\|)} \text{ for all } z\in\mathbb{C}^N \text{ with } \operatorname{Im} z\neq 0\,.$$

By continuity this inequality holds for all $z\in\mathbb{C}^N$. Theorem 3.2 is thus completely proved. $\triangleright$

Corollary 3.3. *Suppose that $\Omega = \{\omega_n\}_{n=1}^\infty \in W_\uparrow^{nq}$ satisfies*

$$(\mathrm{II})\begin{cases} (\mathrm{II}_1) \text{ for each } n\in\mathbb{N} \text{ there exist } m\in\mathbb{N} \text{ and } C>0 \text{ such that} \\ \qquad \omega_n(2t)\leqslant\omega_m(t)+C \text{ for all } t\geqslant 0\,; \\ (\mathrm{II}_2) \text{ for each } n\in\mathbb{N} \text{ there exist } m\in\mathbb{N} \text{ and } C>0 \text{ such that} \\ \qquad \dfrac{4}{\pi}\displaystyle\int\limits_1^\infty \frac{\omega_n(yt)}{t^2+1}\,dt \leqslant \omega_m(y)+C \text{ for all } y\geqslant 0\,. \end{cases}$$

Then $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ admits a version of Borel's extension theorem for all $N\in\mathbb{N}$.

$\lhd$ Fix any $n \in \mathbb{N}$ and find $m_1 \geqslant n$ and $D_1 > 0$ such that

$$\frac{4}{\pi}\int_1^\infty \frac{\omega_n(yt)}{t^2+1}\,dt \leqslant \omega_{m_1}(y) + D_1 \text{ for } y \geqslant 0\,. \tag{11}$$

Next, for the m_1 there exist $m \in \mathbb{N}$ and $D_2 > 0$ such that

$$\omega_{m_1}(2t) \leqslant \omega_m(t) + D_2 \text{ for } t \geqslant 0\,. \tag{12}$$

Combining (11) and (12), we have for all $z = x + iy \in \mathbb{C}^N$

$$\begin{aligned} P_{\omega_n}(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega_n(\|x+ty\|)}{t^2+1}\,dt \leqslant \frac{2}{\pi}\int_0^\infty \frac{\omega_n(\|x\|+t\|y\|)}{t^2+1}\,dt \leqslant \\ &\leqslant \frac{1}{2}\omega_n(\|x\|+\|y\|) + \frac{2}{\pi}\int_1^\infty \frac{\omega_n\big((\|x\|+\|y\|)t\big)}{t^2+1}\,dt \leqslant \\ &\leqslant \frac{1}{2}\omega_n(\|x\|+\|y\|) + \frac{1}{2}\omega_{m_1}(\|x\|+\|y\|) + D_1 \leqslant \\ &\leqslant \omega_{m_1}(\|x\|+\|y\|) + D_1 \leqslant \omega_{m_1}(2\max\{\|x\|,\|y\|\}) + D_1 \leqslant \\ &\leqslant \omega_m(\max\{\|x\|,\|y\|\}) + D_1 + D_2 \leqslant \omega_m(\|z\|) + C, \end{aligned}$$

where $C := D_1 + D_2$. This means that condition (I) of Theorem 3.2 holds. So ρ maps $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ onto $\mathscr{E}^N_{(\Omega)}$. $\rhd$

Remark. Let us explain how results of [8] and [4] for spaces of ultradifferentiable functions (UDF) of minimal and normal type can be derived from the present results. First recall that condition

(I′) *for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$P_{\omega_n}(z) \leqslant \omega_m(|z|) + C \quad \text{for all } z \in \mathbb{C}$$

provides that Borel's extension theorem holds for the corresponding class $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ of minimal or normal type independently of the number N of variables.

In case of spaces of minimal type (see [8]), (II$_1$) means that $\omega(2t) = O(\omega(t))$, $t \to \infty$. That was the general assumption of [8]. It is not hard to see that (II$_2$) $\Leftrightarrow$ (I′) in this situation. Indeed, in the proof of Corollary 3.3 we just have proved that under assumption (II$_1$), (II$_2$) implies (I), and so (I′). Implication (I′) $\Rightarrow$ (II$_2$) follows from

$$\begin{aligned} \frac{4}{\pi}\int_1^\infty \frac{\omega_n(yt)}{t^2+1}\,dt &\leqslant \frac{4}{\pi}\int_0^\infty \frac{\omega_n(yt)}{t^2+1}\,dt = 2P_{\omega_n}(iy) \leqslant 2\omega_m(y) + 2C = \\ &= 2m\omega(y) + 2C = \omega_{2m}(y) + 2C \text{ for } y \geqslant 0\,. \end{aligned}$$

In case of spaces of normal type, in Lemmas 2.5 and 2.7 of [4] it was shown that (II$_1$) $\Rightarrow$ (II$_2$) and (II$_1$) $\Leftrightarrow$ (I′). So (II) $\Leftrightarrow$ (II$_1$) $\Leftrightarrow$ (I′).

4. Necessary conditions

Assume that $\Omega = \{\omega_n\}_{n=1}^\infty \in W_\uparrow^{nq}$ satisfies the additional condition

(A) *for each $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $A_n > 0$ so that*

$$\omega_n(x+y) \leqslant \omega_m(x) + \omega_m(y) + A_n \text{ for } x, y \geqslant 0\,. \tag{13}$$

Note that analogous assumption was also made in [4] for spaces of normal type (see Definition 6.1 of [4]). In case of spaces of minimal type, (13) is a simple consequence of the general assumption $\omega(2t) = O(\omega(t))$, $t \to \infty$.

We start by several lemmas.

Lemma 4.1. *Suppose that $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}^{nq}$ has property (A). For each $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that for every $\varepsilon > 0$ we can find a $C > 0$ with*

$$P_{\omega_n}(z) \leqslant \varepsilon|\operatorname{Im} z| + \omega_m(|\operatorname{Re} z|) + C \text{ for } z \in \mathbb{C}. \tag{14}$$

$\lhd$ Fix any $n \in \mathbb{N}$ and find $m \in \mathbb{N}$ and $A_n > 0$ so that (13) holds. Next, for an arbitrary $\varepsilon > 0$ choose $r > 0$ with

$$\frac{2}{\pi}\int_r^\infty \frac{\omega_m(t)}{t^2+1}\,dt < \varepsilon\,.$$

We have for $x \geqslant 0$ and $y > 0$

$$P_{\omega_n}(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\omega_n(|x+t|)}{t^2+y^2}\,dt \leqslant \frac{2y}{\pi}\int_0^\infty \frac{\omega_m(x)+\omega_m(t)+A_n}{t^2+y^2}\,dt = \omega_m(x) + I_m(y) + A_n,$$

where $I_m(y) := \dfrac{2y}{\pi}\displaystyle\int_0^\infty \frac{\omega_m(t)}{t^2+y^2}\,dt$. If $y \leqslant 1$, then

$$I_m(y) = \frac{2}{\pi}\int_0^\infty \frac{\omega_m(yt)}{t^2+1}\,dt \leqslant \frac{2}{\pi}\int_0^\infty \frac{\omega_m(t)}{t^2+1}\,dt =: D_m < \infty\,.$$

If $y > 1$, then

$$I_m(y) = \frac{2y}{\pi}\int_0^r \frac{\omega_m(t)}{t^2+y^2}\,dt + \frac{2y}{\pi}\int_r^\infty \frac{\omega_m(t)}{t^2+y^2}\,dt \leqslant \frac{2}{\pi}\int_0^{r/y} \frac{\omega_m(yt)}{t^2+1}\,dt + \frac{2y}{\pi}\int_r^\infty \frac{\omega_m(t)}{t^2+1}\,dt \leqslant \omega_m(r) + \varepsilon y\,.$$

Hence,

$$I_m(y) \leqslant \varepsilon y + \omega_m(r) + D_m \text{ for all } y > 0,$$

and so,

$$P_{\omega_n}(x+iy) \leqslant \omega_m(x) + \varepsilon y + C \text{ for } x \geqslant 0,\ y > 0,$$

where $C := \omega_m(r) + D_m + A_n$. By continuity the preceding inequality holds also for $y = 0$. Since

$$P_{\omega_n}(x+iy) = P_{\omega_n}(x-iy) = P_{\omega_n}(-x+iy),$$

we finally obtain the result. $\rhd$

Lemma 4.2. *If $\Omega = \{\omega_n\}_{n=1}^{\infty} \in W_{\uparrow}^{nq}$ has property (A), then for each $n \in \mathbb{N}$ we can find an $R_n > 0$ such that for every $R \geqslant R_n$ there exists an $r > 0$ for which*

$$P_{\omega_n}(iR) \leqslant \frac{2R}{\pi}\int_0^r \frac{\omega_{n+1}(t)}{t^2+R^2}\,dt\,. \tag{15}$$

◁ Use of (1) gives that

$$P_{\omega_n}(iR) \leqslant \frac{2R}{\pi}\int\limits_0^\infty \frac{\omega_{n+1}(t)}{t^2+R^2}\,dt - \log R + C_n = \frac{2R}{\pi}\int\limits_0^r \frac{\omega_{n+1}(t)}{t^2+R^2}\,dt + L_{r,R}, \tag{16}$$

where

$$L_{r,R} := \frac{2R}{\pi}\int\limits_r^\infty \frac{\omega_{n+1}(t)}{t^2+R^2}\,dt - \log R + C_n\,.$$

Put $R_n := e^{C_n+1}$. Given $R \geqslant R_n$ choose $r > 0$ such that

$$\frac{2R}{\pi}\int\limits_r^\infty \frac{\omega_{n+1}(t)}{t^2+R^2}\,dt < 1\,.$$

Then $L_{r,R} < 1 - \log R_n + C_n = 0$. Use of this in (16) gives (15). ▷

In the next lemma we construct a special family of polynomials. Let us first introduce two new functions. For $k \in \mathbb{N}$ and $r > 0$ we put

$$\omega_k^r(t) := \begin{cases} \omega_k(t) & \text{for } t \in [0,r] \\ r(\omega_k)'_-(r)\log\frac{t}{r} + \omega_k(r) & \text{for } t \in (r,\infty) \end{cases}$$

and

$$P_{\omega_k^r}(x+iy) := \begin{cases} \dfrac{|y|}{\pi}\displaystyle\int\limits_{-\infty}^{\infty} \frac{\omega_k^r(|t|)}{(t-x)^2+y^2}\,dt & \text{for } y \neq 0 \\ \omega_k^r(|x|) & \text{for } y = 0 \end{cases} \qquad (x+iy \in \mathbb{C})$$

$P_{\omega_k^r}$ has the same properties that P_ω, i. e. $P_{\omega_k^r}$ is harmonic in the open upper and lower half plane and it is continuous and subharmonic in the whole plane.

Lemma 4.3. *Suppose that $\Omega = \{\omega_n\}_{n=1}^\infty \in W_\uparrow^{nq}$ satisfies (A). There exists a family of polynomials $\{g_{R,n}(\zeta) : n \in \mathbb{N},\ R \in [R_n,\infty)\}$ of one variable $\zeta \in \mathbb{C}$ with the following properties:*

1) *for each $n \in \mathbb{N}$ and each $R \geqslant R_n$*

$$g_{R,n}(iR) \geqslant \exp P_{\omega_n}(iR); \tag{17}$$

2) *for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$|g_{R,n}(\zeta)| \leqslant C\exp\big(m|\operatorname{Im}\zeta| + \omega_m(|\zeta|)\big) \quad \text{for all } R \in [R_n,\infty) \text{ and } \zeta \in \mathbb{C}\,. \tag{18}$$

◁ Given $n \in \mathbb{N}$ we find an $R_n > 0$ according to Lemma 4.2. Next we fix any $R \geqslant R_n$ and find $r > 0$ such that (15) holds. Then

$$P_{\omega_n}(iR) \leqslant \frac{2R}{\pi}\int\limits_0^r \frac{\omega_{n+1}(t)}{t^2+R^2}\,dt = \frac{2R}{\pi}\int\limits_0^r \frac{\omega_{n+1}^r(t)}{t^2+R^2}\,dt \leqslant P_{\omega_{n+1}^r}(iR)\,. \tag{19}$$

Applying Lemma 1 of [1] to the subharmonic function $P_{\omega_{n+1}^r}(z)$ and the point $\xi = iR$, we construct an entire function $g_{R,n}(\zeta)$, $\zeta \in \mathbb{C}$, for which

$$g_{R,n}(iR) = \exp P_{\omega_{n+1}^r}(iR), \tag{20}$$

$$|g_{R,n}(\zeta)| \leqslant A(1+|\zeta|^2)^2 \exp P^1_{\omega^r_{n+1}}(\zeta) \quad \text{for all } \zeta \in \mathbb{C}. \tag{21}$$

Here A is an absolute constant and $P^1_{\omega^r_{n+1}}(\zeta) := \sup\left\{P_{\omega^r_{n+1}}(\zeta+w) : |w| \leqslant 1\right\}$. Combining (20) and (19), we immediately derive (17).

In order to show that (18) holds, we estimate $P^1_{\omega^r_{n+1}}(\zeta)$. Since $\omega^r_{n+1}(t) \leqslant \omega_{n+1}(t)$ it follows that $P_{\omega^r_{n+1}}(\zeta) \leqslant P_{\omega_{n+1}}(\zeta)$, and so, $P^1_{\omega^r_{n+1}}(\zeta) \leqslant P^1_{\omega_{n+1}}(\zeta)$ for all $\zeta \in \mathbb{C}$.

Next, by Lemma 4.1, there are $k \in \mathbb{N}$ and $D_1 > 0$ such that

$$P_{\omega_{n+1}}(z) \leqslant k|\operatorname{Im} z| + \omega_k(|z|) + D_1 \quad \text{for } z \in \mathbb{C}\,.$$

In [3] it was proved (see inequality (5) of [3]) that

$$\omega_k(t+1) \leqslant \omega_k(t) + B_k e^2 \quad \text{for all } t \geqslant 0,$$

where B_k is determined by (2). Thus,

$$P^1_{\omega^r_{n+1}}(\zeta) \leqslant P^1_{\omega_{n+1}}(\zeta) \leqslant k|\operatorname{Im}\zeta| + \omega_k(|\zeta|+1) + k + D_1 \leqslant k|\operatorname{Im}\zeta| + \omega_k(|\zeta|) + D,$$

where $D := B_k e^2 + k + D_1$.

It is easily checked that

$$(1+|\zeta|^2)^2 \leqslant e^{4\log(|\zeta|+1)} \quad \text{for } \zeta \in \mathbb{C}\,.$$

Now we can continue (21) as follows

$$|g_{R,n}(\zeta)| \leqslant A\exp\left(k|\operatorname{Im}\zeta| + 4\log(|\zeta|+1) + \omega_k(|\zeta|) + D\right).$$

Setting $m = k+4$ and $C = A\exp(C_k + C_{k+1} + C_{k+2} + C_{k+3} + D)$, we finally obtain (18).

Proof of the fact that $g_{R,n}$ are polynomials is the same as in Lemma 2 of [1], so we omit it. Our result is thus completely proved. $\rhd$

The main result of the section is

Theorem 4.4. *Let $\Omega = \{\omega_n\}_{n=1}^\infty$ be a weight sequence in $W^{nq}_\uparrow$ with property (A). If $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ admits a version of Borel's extension theorem for at least one $N \in \mathbb{N}$, then*

(III) *for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$ so that*

$$P_{\omega_n}(iy) \leqslant \omega_m(y) + C \quad \textit{for all } y \geqslant 0\,. \tag{22}$$

$\lhd$ First note that if $\rho : \mathscr{E}_{(\Omega)}(\mathbb{R}^N) \to \mathscr{E}^N_{(\Omega)}$ is surjective for some $N \geqslant 2$, then it is also surjective for $N = 1$.

Assume that there is an $n_0 \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ and each $k \in \mathbb{N}$ there exists an $R_{m,k} > R_{n_0}$ such that

$$P_{\omega_{n_0}}(iR_{m,k}) > \omega_m(R_{m,k}) + k\,. \tag{23}$$

Here R_{n_0} is determined by Lemma 4.2.

Let $\{g_{R,n}(\zeta) : n \in \mathbb{N}, R \in [R_n, \infty)\}$ be a family of polynomials with properties of Lemma 4.3. Then there are $n_1 \in \mathbb{N}$ and $D_1 > 0$ such that

$$|g_{r,n_0}(\zeta)| \leqslant D_1 \exp\left(n_1|\operatorname{Im}\zeta| + \omega_{n_1}(|\zeta|)\right) \quad \text{for all } \zeta \in \mathbb{C}\,.$$

Setting $f_{m,k} := \dfrac{1}{D_1}\, g_{R_{m,k},n_0}$, we can rewrite the previous inequality as

$$|f_{m,k}(\zeta)| \leqslant \exp\left(n_1|\operatorname{Im}\zeta| + \omega_{n_1}(|\zeta|)\right) \quad \text{for all } \zeta \in \mathbb{C}\,. \tag{24}$$

Next, from (17) with $n = n_0$ and (23) we have for all $m, k \in \mathbb{N}$

$$f_{m,k}(iR_{m,k}) = \frac{1}{D_1}\, g_{R_{m,k},n_0}(iR_{m,k}) \geqslant \frac{1}{D_1} \exp P_{\omega_{n_0}}(iR_{m,k}) > \frac{k}{D_1} \exp \omega_m(R_{m,k})\,. \tag{25}$$

Being polynomials $f_{m,k}$ are in $H^1_{(\Omega)}$. From (24) and (25) it then follows that assertion (iv) of Theorem 2.2 is false. Thus, the operator $\rho : \mathscr{E}_{(\Omega)}(\mathbb{R}) \to \mathscr{E}^1_{(\Omega)}$ is not surjective. This contradiction proves the theorem. ▷

Remark. It should be noted that Theorem 4.4 gives us the corresponding necessary conditions of [8] and [4] for spaces of UDF of minimal and normal type. Moreover, in these two cases condition (II) of Corollary 3.3 is equivalent to condition (III) of the previuos theorem (see [4, 8]). This means that whole criteria of [8] and [4] can be derived from our present results.

5. Examples

One of the classical nonquasianalytic weight functions is $\omega(t) = t^\alpha$, $0 < \alpha < 1$. It was shown in [8] that space of UDF of minimal type defined by this function admits the analog of Borel's extension theorem. In contrast, the corresponding space of normal type does not admit this analog (see [4]). Let us consider

Example 5.1. $\omega_n(t) = t^{\alpha_n}$, $0 < \alpha_n \uparrow \alpha \leqslant 1$.

We should like to verify condition (II) of Corollary 3.3. It is easily seen that (II$_1$) holds. Next, for $n \in \mathbb{N}$ and $y > 0$ we have

$$\frac{4}{\pi}\int_1^\infty \frac{\omega_n(yt)}{t^2+1}\,dt = y^{\alpha_n}\,\frac{4}{\pi}\int_1^\infty \frac{t^{\alpha_n}}{t^2+1}\,dt \leqslant y^{\alpha_n}\,\frac{4}{\pi}\int_1^\infty t^{\alpha_n-2}\,dt = y^{\alpha_n}\,\frac{4}{\pi(1-\alpha_n)}.$$

Obviuosly, we can find a $C > 0$ such that

$$y^{\alpha_n}\,\frac{4}{\pi(1-\alpha_n)} \leqslant y^{\alpha_{n+1}} + C \quad \text{for all } y \geqslant 0\,.$$

This means that (II$_2$) also holds. Thus the space $\mathscr{E}_{(\Omega)}(\mathbb{R}^N)$ with $\Omega = \{t^{\alpha_n}\}_{n=1}^\infty$ admits a version of Borel's theorem for all $N \in \mathbb{N}$.

It is of particular interest that α_n could tend to 1, whereas $\omega(t) = t$ is not a nonquasianalytic weight function.

Another well-known weight function is $\omega(t) = \frac{t}{\log^\beta(e+t)}$, where $\beta > 1$. Recall (see [8] and [4]) that Borel's theorem does not hold for the corresponding spaces of UDF, both of minimal and normal type. Now we wish to consider a sequence of such functions.

Example 5.2. $\omega_n(t) = \frac{t}{\log^{\beta_n}(e+t)}$, $\beta_n \downarrow \beta \geqslant 1$.

Without loss of generality we can assume that $\beta_1 < \beta + 1$. First note that $\Omega = \{\omega_n\}_{n=1}^\infty$ satisfies condition (II$_1$), and so, condition (A). Now, let us show that condition (III) of Theorem 4.4 does not hold. For $y > 0$ we have

$$P_{\omega_n}(iy) = \frac{2}{\pi}\int_0^\infty \frac{\omega_n(yt)}{t^2+1}\,dt = \frac{y}{\pi}\int_0^\infty \frac{d(\log(t^2+1))}{\log^{\beta_n}(e+yt)} = y^2\,\frac{\beta_n}{\pi}\int_0^\infty \frac{\log(t^2+1)}{\log^{\beta_n+1}(e+yt)}\,\frac{dt}{e+yt}.$$

Put

$$t_y := \frac{y + \sqrt{y^2 + 4(e-1)}}{2}\,.$$

Then $t^2+1 \geqslant e+yt$ for $t \geqslant t_y$. Hence, for an arbitrary $m \in \mathbb{N}$ we can write

$$P_{\omega_n}(iy) \geqslant y^2 \frac{\beta_n}{\pi} \int\limits_{t_y}^{\infty} \frac{1}{\log^{\beta_n}(e+yt)} \frac{dt}{e+yt} = \frac{\beta_n}{\pi(\beta_n-1)} \frac{y}{\log^{\beta_n-1}(e+yt_y)} =$$

$$= \omega_m(y) \frac{\beta_n}{\pi(\beta_n-1)} \frac{\log^{\beta_m}(e+y)}{\log^{\beta_n-1}(e+yt_y)}.$$

Since $\beta_m > \beta > \beta_n - 1$, the quotient $\dfrac{\log^{\beta_m}(e+y)}{\log^{\beta_n-1}(e+yt_y)}$ tends to ∞ as $y \to \infty$, and so, (III) does not hold. By theorem 4.4 we derive that $\rho : \mathscr{E}_{(\Omega)}(\mathbb{R}^N) \to \mathscr{E}^N_{(\Omega)}$ is not surjective.

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