Владикавказский математический журнал
Октябрь–декабрь, 2005, Том 7, Выпуск 4

УДК **517.98**

ONE GENERAL METHOD IN OPERATOR THEORY

S. S. Kutateladze

An order bounded operator with target a Dedekind complete vector lattice is determined up to an orthomorphism from the kernels of its strata. Some applications to 2-disjoint operators are briefly discussed.

A linear functional on a vector space is determined up to a scalar from its zero hyperplane. In contrast, a linear operator is recovered from its kernel up to a simple multiplier on a rather special occasion. Fortunately, Boolean valued analysis [1] prompts us that some operator analog of the functional case is valid for each operator with target a Kantorovich space, a Dedekind complete vector lattice. The present expository talk addresses some opportunities that are opened up along the lines of this rather promising approach.

Let X be a Riesz space, and let Y be a Kantorovich (or Dedekind complete Riesz) space Y with base a complete Boolean algebra B. Without loss of generality, we may assume that Y is a nonzero space embedded as an order dense ideal in the universally complete Kantorovich space $\mathcal{R}{\downarrow}$ which is the descent of the reals $\mathcal{R}$ inside the separated Boolean valued universe $\mathbb{V}^{(B)}$ over B (cp. [1, Theorem 5.2.4]).

We further let $X^{\wedge}$ stand for the standard name of X in $\mathbb{V}^{(B)}$. Clearly, $X^{\wedge}$ is a Riesz space over $\mathbb{R}^{\wedge}$ inside $\mathbb{V}^{(B)}$. Denote by $l := T{\uparrow}$ the ascent of T to $\mathbb{V}^{(B)}$. Clearly, l acts from $X^{\wedge}$ to the ascent $Y{\uparrow}$ of Y in the sense of the Boolean valued universe $\mathbb{V}^{(B)}$. Therefore,

$$l(x^{\wedge}) = Tx$$

inside $\mathbb{V}^{(B)}$ for all $x \in X$, which means in terms of truth values that

$$[\![l : X^{\wedge} \to \mathcal{R}]\!] = \mathbb{1}, \quad (\forall x \in X)\ [\![l(x^{\wedge}) = Tx]\!] = \mathbb{1}.$$

Since l is defined up to a scalar from $\ker(l)$, we infer the following analog of the Sard theorem.

Theorem 1. *Let S and T be linear operators from X to Y. Then $\ker(bS) \supset \ker(bT)$ for all $b \in B$ if and only if there is an orthomorphism α on Y such that $S = \alpha T$.*

We see that a linear operator T is in sense determined up to an orthomorphism from the family of the kernels of the *strata* bT of T. This remark opens a possibility of studying some properties of T in terms of the kernels of the strata of T.

Clearly, T is a Riesz homomorphism if and only if so is its ascent $l = T{\uparrow}$. Since the ascent of the sum is the sum of the ascents of the summands, we reduce the proof of Theorem 2 to the case of the functionals.

From now on we will consider an order bounded operator $T : X \to Y$. Straightforward calculations of truth values show that $T_+\!\uparrow = l_+$ and $T_-\!\uparrow = l_-$ inside $\mathbb{V}^{(B)}$. Moreover, $[\![\ker(l)$ is a Riesz subspace of $X^\wedge]\!] = \mathbb{1}$ whenever so are $\ker(bT)$ for all $b \in B$. Indeed, given $x, y \in X$, put

$$b := [\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!].$$

This means that $x, y \in \ker(bT)$. Hence, we see by condition that $bT(x \vee y) = 0$. In other words,

$$[\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!] \leqslant [\![T(x \vee y) = 0^\wedge]\!].$$

Whence

$$[\![\ker(l) \text{ is a Riesz subspace of } X^\wedge]\!]$$
$$= [\![(\forall x, y \in X^\wedge)(l(x) = 0^\wedge \wedge l(y) = 0^\wedge \to l(x \vee y) = 0^\wedge)]\!]$$
$$= \bigwedge_{x,y\in X} [\![l(x^\wedge) = 0^\wedge \wedge l(y^\wedge) = 0^\wedge \to l((x \vee y)^\wedge) = 0^\wedge]\!] = \mathbb{1}.$$

Recall that a subspace H of a Riesz space X is a *G-space* or *Grothendieck subspace* (cp. [2, 3]) provided that H enjoys the following property:

$$(\forall x, y \in H)\ (x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

By analogous calculations of truth values we infer that

$$[\![\ker(l) \text{ is a Grothendieck subspace of } X^\wedge]\!]$$
$$= [\![(\forall x, y \in X^\wedge)(l(x) = 0^\wedge \wedge l(y) = 0^\wedge \to l(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge)]\!]$$
$$= \bigwedge_{x,y\in X} [\![l(x^\wedge) = 0^\wedge \wedge l(y^\wedge) = 0^\wedge \to l((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge]\!].$$

Assuming that the kernel of each stratum bT is a Grothendieck subspace, take $x, y \in X$ and put

$$b := [\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!].$$

This means that $x, y \in \ker(bT)$. By hypothesis $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$. In other words,

$$[\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!] \leqslant [\![T(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge]\!].$$

It follows now that

$$[\![\ker(l) \text{ is a Grothendieck subspace of } X^\wedge]\!] = \mathbb{1}.$$

By way of example, we may now assert that the following theorems appear as the descents of their scalar analogs.

Theorem 2. *An order bounded operator T from X to Y may be presented as the difference of some Riesz homomorphisms and only if the kernel of each stratum bT of T is a Riesz subspace of X for all $b \in B$.*

Theorem 3. *The modulus of an order bounded operator $T : X \to Y$ is the sum of some pair of Riesz homomorphisms if and only if the kernel of each stratum bT of T with $b \in B$ is a Grothendieck subspace of the ambient Riesz space X.*

To prove the relevant scalar claims, we use one of the formulas of subdifferential calculus:

Theorem 4 (of decomposition). *Assume that $H_1, \dots, H_N$ are cones in a Riesz space X. Assume further that f and g are positive functionals on X. The inequality*

$$f(h_1 \vee \dots \vee h_N) \geqslant g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \dots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \dots + f_N$ such that

$$f_k(h_k) \geqslant g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, N).$$

Remark 1. The complete proofs of Theorems 2 and 3 are given in [4, 5]. Theorem 4 appeared in this form in [6].

Remark 2. Note that the sums of Riesz homomorphisms were first described by S. J. Bernau, C. B. Huijsmans, and B. de Pagter in terms of n-disjoint operators in [7]. A survey of some conceptually close results on n-disjoint operators is given in [8, § 5.6].

References

1. *Kusraev A. G., Kutateladze S. S.* Boolean Valued Analysis.—Dordrecht: Kluwer Academic Publishers, 1999.—322 p.
2. *Grothendieck A.* Une caractérisation vectorielle-métrique des espaces L^1 // Canad. J. Math.—1955.—V. 4.—P. 552–561.
3. *Lindenstrauss J., Wulbert D. E.* On the classification of the Banach spaces whose duals are L_1-spaces // J. Funct. Anal.—1969.—V. 4, № 3.—P. 32–349.
4. *Kutateladze S. S.* On differences of Riesz homomorphisms // Sibirsk. Mat. Zh.—2005.—V. 46, № 2.—P. 390–393.
5. *Kutateladze S. S.* On Grothendieck subspaces // Sibirsk. Mat. Zh.—2005.—V. 46, № 3.—P. 620–624.
6. *Kutateladze S. S.* Choquet boundaries in K-spaces // Russ. Math. Surveys.—1975.—V. 30, № 4.—P. 115–155.
7. *Bernau S. J., Huijsmans C. B., de Pagter B.* Sums of lattice homomorphisms // Proc. Amer. Math. Soc.—1992.—V. 115, № 1.—P. 151–156.
8. *Kusraev A. G.* Dominated Operators.—Dordrecht: Kluwer Academic Publishers.—2000.—446 p.

Received by the editors May 4, 2005.

Prof. Kutateladze S. S.
Novosibirsk, Sobolev Institute of Mathematics
E-mail: `sskut@member.ams.org`