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Generalized Dedekind sums

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Abstract Classical Dedekind sums are connected to the modular group through the construction of a (Dedekind) symbol on the cusp set of the modular group. In this paper we study generalizations of Dedekind symbols and sums that can be associated to certain Fuchsian groups uniformizing 1-punctured tori.

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Dedicated to Andrew Casson on the occasion of his 60th birthday

1 Introduction

A classical and important construction which arises in many contexts is that of the *Dedekind sum* which is defined for coprime integers a and c by

$$s(a,b) = \sum_{k=1}^{|b|-1} \left(\left(\frac{k}{b}\right)\right) \left(\left(\frac{ka}{b}\right)\right)$$

where $((x)) = x - [x] - 1/2$. Dedekind sums arise naturally in various topological settings, one of the most famous being Hirzebuch's description of $4s(b,a)$ as the signature defect of the Lens space $L(a,b)$ coming from Rademacher's cotangent formula

$$s(a,b) = \frac{1}{4|b|} \sum_{k=1}^{|b|-1} \cot\left(\frac{k\pi}{b}\right) \left(\left(\frac{ka\pi}{b}\right)\right),$$

as well as in Walker's formula for the generalized Casson invariant.

From the point of view of this note, it is the beautiful construction in [1] of Dedekind sums based upon the classical modular group $\mathrm{PSL}(2,\mathbb{Z})$ that is of interest. We describe some of this briefly, as it is useful in the development of what follows. It is shown in [1] that there exists a 2–cocycle $\epsilon\colon \mathrm{PSL}(2,\mathbb{Z}) \times$

$\mathrm{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$ and a function $\phi\colon\ \mathrm{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$ (the Rademacher ϕ–function) which satisfy $\delta\phi = 3\epsilon$ (where δ is the coboundary operator). Furthermore, it is shown in [1] that the function ϕ is closely related to the Dedekind sums mentioned above. Namely, in [1] the authors define a Dedekind symbol S on $\mathbb{Q} \cup \infty$ which maps ∞ to ∞ and otherwise, $S(\frac{a}{c}) = \phi(M) + \chi(M)$ where $M \in \mathrm{PSL}(2,\mathbb{Z})$ satisfies $M(\infty) = \frac{a}{c}$ and χ is a function depending on the entries of M (see section 2.2). As pointed out in [1, section 0.8], the relationship between S and the Dedekind sum s above is $S(\frac{a}{c}) = 12\,\mathrm{sign}(c) s(a,c)$.

For us, since $\mathbb{Q}\cup\infty$ coincides with the cusp set (that is the set of all parabolic fixed points) of $\mathrm{PSL}(2,\mathbb{Z})$, S can be viewed as a function defined on the cusp set of $\mathrm{PSL}(2,\mathbb{Z})$. In [2] it was shown that there exist finite coarea Fuchsian groups not commensurable with the modular group but whose cusp set is precisely $\mathbb{Q}\cup\infty$. The purpose of this note is to show that these groups give rise to very natural generalizations of Dedekind sums.

We begin by recalling briefly the construction of [2]. The starting point of that paper was to take the two generator group $\Delta(u^2, 2t)$ generated by elements g_1 and g_2 as below

$$g_1 = \begin{pmatrix} (-1+t)/\sqrt{-1+t-u^2} & u^2/\sqrt{-1+t-u^2} \\ 1/\sqrt{-1+t-u^2} & 1/\sqrt{-1+t-u^2} \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} u/\sqrt{-1+t-u^2} & u/\sqrt{-1+t-u^2} \\ 1/(u\sqrt{-1+t-u^2}) & (t-u^2)/u\sqrt{-1+t-u^2} \end{pmatrix}$$

where the parameters u^2 and t are real and satisfy $t > u^2+1$.

One sees easily that in the hyperbolic plane, g_1 maps the directed edge $\{-1,0\}$ to the directed edge $\{\infty, u^2\}$ and g_2 mapping $\{\infty,-1\}$ to $\{u^2,0\}$, and moreover the commutator

$$g_1 g_2^{-1} g_1^{-1} g_2 = \begin{pmatrix} -1 & -2t \\ 0 & -1 \end{pmatrix}$$

is parabolic and generates the stabiliser of infinity. It follows that $\mathbf{H}^2/\Delta(u^2,2t)$ is a complete finite-area once-punctured torus. This family includes a modular torus as $\Delta(1,6)$, as well as other arithmetic once-punctured tori, and if u^2 and t are chosen to be rational the set of cusps of these groups must be a subset of $\mathbb{Q}\cup\infty$. In the arithmetic cases, the cusp set is precisely $\mathbb{Q}\cup\infty$, although this is not always the case for rational pairs $(u^2, 2t)$. (See [2]).

Despite the apparently complicated nature of the entries in these matrices because of the presence of square roots, an easy computation shows that if one considers $G = \ker\{\Delta \to \mathbb{Z}/2 \oplus \mathbb{Z}/2\}$, then the trace-field of G, and hence the invariant trace-field of $\Delta(u^2, 2t)$ is the field $\mathbb{Q}(u^2, t)$. In fact all the entries of the matrix representatives for G lie in the field $\mathbb{Q}(u^2, t)$. This real field will be called the *invariant field of definition* of $\Delta(u^2, 2t)$ as it is the most germane field for our considerations. In particular, the cusp set of $\Delta(u^2, 2t)$ can clearly be no larger than the field $\mathbb{Q}(u^2, t) \cup \infty$

The main result of [2] is that there are rational choices of parameters $(u^2, 2t)$ which give rise to nonarithmetic groups whose cusp sets are precisely the rationals. Such groups we call *pseudomodular.* There is a good deal of evidence that such groups exist for fields more general than the rationals, that is to say, their cusp sets are equal to their invariant field of definition - such groups we will describe as *maximally cusped.* It is these groups which we will use to construct Dedekind sums; since our family includes the modular group, it will include a construction of the classical Dedekind sum. In this note we will show

Theorem 1.1 *Suppose that Δ as above has invariant field of definition K and is maximally cusped. Then associated to Δ is a function*

$$S_\Delta\colon\ K \cup \infty \to K \cup \infty$$

Such functions we say are *generalized Dedekind sums.*

2 The construction

Following [1], we first construct an analogue of the Rademacher ϕ–function. Fix one of the groups $\Delta(u^2, 2t)$ of [2]; (at this stage it is not necessary that the group be pseudomodular) and suppose that its invariant field of definition is K.

All once-punctured tori are hyperelliptic so we can adjoin to this group the orientation-preserving involution τ which conjugates the generators to their inverses, to form a new discrete group Γ. The surface $F = \mathbf{H}^2/\Gamma$ is a sphere with three cone points of angle π and a cusp. Note that as an element of $\mathrm{GL}(2,\mathbb{R})$, τ is represented by the matrix $\begin{pmatrix} 0 & 2u \\ -2/u & 0 \end{pmatrix}$, so $\tau(\infty) = 0$.

Following [1], we define an area 2–cocycle

$$\epsilon\colon\ \Gamma \times \Gamma \to \mathbb{Z}$$

by setting $\epsilon(A,B) = \mathrm{area}(\infty, A\infty, AB\infty)/\pi$ where this area is to be regarded as oriented, it follows that ϵ takes on the values $0, \pm 1$.

Equivalently, one can usefully think of $\epsilon(A,B)$ as the sign of $AB\infty - A\infty$, where this is to be interpreted as zero if either term of the difference is infinite.

Notice that ϵ is a cocycle, because the coboundary

$$\delta\epsilon(A,B,C) = \epsilon(B,C) - \epsilon(AB,C) + \epsilon(A,BC) - \epsilon(A,B)$$

involves four triangular areas and the first has vertices $(\infty, B\infty, BC\infty)$ which has same oriented area as $(A\infty, AB\infty, ABC\infty)$, so that taken together with other three this forms a tetrahedron, and hence the total area is 0.

Lemma 2.1 *There is a unique K–valued 1–cochain $\Gamma \to K$ with coboundary ϵ.*

Proof Note that $\Gamma \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ so that

$$H^1(\Gamma; \mathbb{Z}) \cong 0$$

and

$$H^2(\Gamma; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

since the integral homology of $\mathbb{Z}/2$ is zero in odd dimensions and $\mathbb{Z}/2$ in even dimensions. For our purposes, we need only use that $H^2(\Gamma; K) = H^1(\Gamma; K) = 0$. The fact that $H^2(\Gamma; K) = 0$ implies immediately the existence of a K–valued 1–cochain with coboundary ϵ.

We prove uniqueness as follows. If $\delta(\phi_1) = \epsilon = \delta(\phi_2)$, then ϕ_i are both cocycles and hence since $H^1(\Gamma; K) = 0$, both are coboundaries. It follows that there is a 0–cochain β with $\delta(\beta) = \phi_1 - \phi_2$. We are computing group cohomology with trivial coefficients, so that this coboundary map is zero and $\phi_1 = \phi_2$ as required. □

Definition We shall denote this K–valued 1–cochain by ϕ.

2.1 Computation of ϕ

It will be useful to have a computation of the cochain ϕ. A consequence of Lemma 2.1 is that there is a function $\phi\colon \Gamma \to K$ which satisfies

$$\phi(AB) - \phi(A) - \phi(B) = -\lambda \,\mathrm{sign}(AB\infty - A\infty) \qquad (*)$$

for some $\lambda \in K$ which will be determined.

Taking $A = B = I$ we see that $\phi(I) = 0$. Taking $A = B = -I$, we also get $\phi(-I) = 0$. Taking $A = -I$ and $B = g$, we deduce from $(*)$ that $\phi(g) = \phi(-g)$ for every $g \in \Gamma$.

More generally, if A and B both stabilise ∞, then the relation says

$$\phi(AB) - \phi(A) - \phi(B) = 0$$

that is to say, ϕ is a homomorphism on $\mathrm{stab}(\infty)$.

Note that in the group Γ, we have that $g_1 g_2^{-1}\tau$ stabilises infinity and one checks easily that this is the generating matrix for the parabolic subgroup and is given by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

By scaling by an appropriate element of K, we may assume that λ is chosen so that ϕ maps this generating parabolic matrix to t, so that ϕ is now determined on the parabolic subgroup.

It also follows from $(*)$ that

$$\phi(\alpha^{-1}) = -\phi(\alpha) = -\phi(-\alpha)$$

for any element α, in particular, if ξ is any projective involution in Γ, (that is to say $\xi^2 = \pm I$) we deduce that $\phi(\xi) = 0$.

Now in the notation introduced above we have

$$\phi(g_1\tau) - \phi(g_1) - \phi(\tau) = -\lambda\,\mathrm{sign}(u^2 - t + 1)$$

Since $g_1\tau$ and τ are both projective involutions and recalling that the groups in question are required to have $0 > 1 + u^2 - t$ we get

$$\phi(g_1) = -\lambda$$

By considering τg_2, a similar computation also shows $\phi(g_2) = -\lambda$.

Now for any $k \in K$, for which the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ lies in Γ, we have that

$$\phi\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\tau\right) - \phi\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right) - 0 = 0$$

In the special case that $k = t$, the leftmost term is the product $(g_1 g_2^{-1}\tau)\tau = -g_1 g_2^{-1}$, so we deduce from the properties described above that $\phi(g_1 g_2^{-1}) = t$.

Since $\begin{pmatrix} t/u & -u \\ 1/u & 0 \end{pmatrix} = g_1 g_2^{-1}$, (or from purely geometric considerations) we see that $g_2 g_1^{-1}\infty = 0$. Finally, noting that $g_2\infty = u^2 > 0$ together with the relation

$$\phi(g_2 g_1^{-1}) - \phi(g_2) + \phi(g_1) = -\lambda \operatorname{sign}(0 - u^2)$$

it follows that $\lambda = -t$, since the leftmost term is $-t$ by the previous calculation and the inverse rule.

To sum up, we now have a complete inductive description of ϕ on the group Γ, namely it satisfies

$$\phi(AB) - \phi(A) - \phi(B) = t \operatorname{sign}(AB\infty - A\infty)$$

and

$$\phi(g_1) = \phi(g_2) = \phi(g_2 g_1^{-1}) = t$$

Remark This is in keeping with the computations of [1] which are for the modular group and have $\lambda = -3$.

2.2 Generalized Dedekind sums

Now fix some maximally-cusped $\Delta = \Delta(u^2, 2t)$ defined over the field K.

For any $M \in \Delta$, by applying the cocycle condition we have

$$\phi\left(M\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right) - \phi(M) - k = t \cdot \operatorname{sign}(M\infty - M\infty) = 0$$

from which it follows that

$$\phi\left(M\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right) = \phi(M) + k \qquad (**)$$

For $M \in \Delta \setminus \operatorname{stab}(\infty)$, set

$$\chi(M) = (M_{1,1} + M_{2,2})/M_{2,1}.$$

Since $M_{2,1} \neq 0$, the value $\chi(M)$ is an element of the field K, since the groups Δ consist of matrices of the shape $\sqrt{r}X$ for a matrix $X \in \mathrm{GL}(2, K)$ and $r \in K$. Now a matrix computation shows that

$$\chi(M.\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) = \chi(M) + k$$

so that by taking the difference between this and $(**)$ we get

$$\phi\left(M\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right) - \chi\left(M\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\right) = \phi(M) - \chi(M)$$

which is to say the function

$$S(M) = \phi(M) - \chi(M)$$

is invariant under right muliplication by the parabolic subgroup.

These observations are independent of whether Δ is maximally-cusped or not. If we now assume that it is, we can define a *generalized Dedekind sum* as follows.

Given any element $\kappa \in K$, since Δ is maximally cusped, there is an element $M \in \Delta$ with $M(\infty) = \kappa$ and we may set

$$S_\Delta(\kappa) = S(M)$$

The ambiguity in such $M \in \Delta$ is accounted for by right multiplication by elements of the parabolic subgroup stab(∞) so that this function depends only on κ. We will define $S_\Delta(\infty) = \infty$, and this defines the advertised function in Theorem 1.1.

Remark This construction gives a scalar multiple of the classical Dedekind sum when $(u^2, 2t) = (1, 6)$ (see [1, section 0.8]).

Examples It is proved in [2] that the group $\Delta(3/5, 4)$ is pseudomodular, so provides an example of a generalized Dedekind sum of this type. It is not difficult to write a computer program which computes its values based upon the iterative procedure outlined above. A table of the groups currently proven to be pseudomodular (and some conjectural examples) is provided in [2].

In subsequent work, the authors have extended this table of conjectural examples to groups which are maximally cusped for real quadratic number fields, for example $\Delta(1, 2((1+\sqrt{13})/2))$ appears to be maximally cusped. Questions about whether there are analogues of, for example, Dedekind reciprocity and formulae of the classical type seem interesting and appear worthy of further investigation.

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