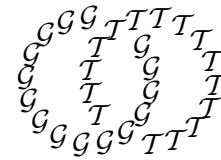


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## Heegaard splittings of exteriors of two bridge knots

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### Abstract

In this paper, we show that, for each non-trivial two bridge knot  $K$  and for each  $g \geq 3$ , every genus  $g$  Heegaard splitting of the exterior  $E(K)$  of  $K$  is reducible.

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## 1 Introduction

In this paper, we prove the following theorem.

**Theorem 1.1** *Let  $K$  be a non-trivial two bridge knot. Then, for each  $g \geq 3$ , every genus  $g$  Heegaard splitting of the exterior  $E(K)$  of  $K$  is reducible.*

We note that since  $E(K)$  is irreducible, the above theorem together with the classification of the Heegaard splittings of the 3–sphere  $S^3$  (F Waldhausen [21]) implies the next corollary.

**Corollary 1.2** *Let  $K$  be a non-trivial two bridge knot. Then, for each  $g \geq 3$ , every genus  $g$  Heegaard splitting of  $E(K)$  is stabilized.*

By H Goda, M Scharlemann, and A Thompson [6] (see also K Morimoto’s paper [15]) or [13], it is shown that, for each non-trivial two bridge knot  $K$ , every genus two Heegaard splitting of  $E(K)$  is isotopic to either one of six typical Heegaard splittings (see Figure 11). We note that Y Hagiwara [7] proved that genus three Heegaard splittings obtained by stabilizing the six Heegaard splittings are mutually isotopic. This result together with Corollary 1.2 implies the following.

**Corollary 1.3** *Let  $K$  be a non-trivial two bridge knot. Then, for each  $g \geq 3$ , the genus  $g$  Heegaard splittings of  $E(K)$  are mutually isotopic, ie, there is exactly one isotopy class of Heegaard splittings of genus  $g$ .*

We note that this result is proved for figure eight knot by D Heath [9].

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## 2 Preliminaries

Throughout this paper, we work in the differentiable category. For a submanifold  $H$  of a manifold  $M$ ,  $N(H, M)$  denotes a regular neighborhood of  $H$  in  $M$ . When  $M$  is well understood, we often abbreviate  $N(H, M)$  to  $N(H)$ . Let  $N$  be a manifold embedded in a manifold  $M$  with  $\dim N = \dim M$ . Then  $\text{Fr}_M N$  denotes the frontier of  $N$  in  $M$ . For the definitions of standard terms in 3–dimensional topology, we refer to [10] or [11].

## 2.A Heegaard splittings

A 3-manifold  $C$  is a *compression body* if there exists a compact, connected (not necessarily closed) surface  $F$  such that  $C$  is obtained from  $F \times [0, 1]$  by attaching 2-handles along mutually disjoint simple closed curves in  $F \times \{1\}$  and capping off the resulting 2-sphere boundary components which are disjoint from  $F \times \{0\}$  by 3-handles. The subsurface of  $\partial C$  corresponding to  $F \times \{0\}$  is denoted by  $\partial_+ C$ . Then  $\partial_- C$  denotes the subsurface  $\text{cl}(\partial C - (\partial F \times [0, 1] \cup \partial_+ C))$  of  $\partial C$ . A compression body  $C$  is said to be *trivial* if either  $C$  is a 3-ball with  $\partial_+ C = \partial C$ , or  $C$  is homeomorphic to  $F \times [0, 1]$  with  $\partial_- C$  corresponding to  $F \times \{0\}$ . A compression body  $C$  is called a *handlebody* if  $\partial_- C = \emptyset$ . A compressing disk  $D(\subset C)$  of  $\partial_+ C$  is called a *meridian disk* of the compression body  $C$ .

**Remark 2.1** The following properties are known for compression bodies.

- (1) Compression bodies are irreducible.
- (2) By extending the cores of the 2-handles in the definition of the compression body  $C$  vertically to  $F \times [0, 1]$ , we obtain a union of mutually disjoint meridian disks  $\mathcal{D}$  of  $C$  such that the manifold obtained from  $C$  by cutting along  $\mathcal{D}$  is homeomorphic to a union of  $\partial_- C \times [0, 1]$  and some (possibly empty) 3-balls. This gives a dual description of compression bodies. That is, a connected 3-manifold  $C$  is a compression body if there exists a compact (not necessarily connected) surface  $\mathcal{F}$  without 2-sphere components and a union of (possibly empty) 3-balls  $\mathcal{B}$  such that  $C$  is obtained from  $\mathcal{F} \times [0, 1] \cup \mathcal{B}$  by attaching 1-handles to  $\mathcal{F} \times \{0\} \cup \partial \mathcal{B}$ . We note that  $\partial_- C$  is the surface corresponding to  $\mathcal{F} \times \{1\}$ .
- (3) Let  $\mathcal{D}$  be a union of mutually disjoint meridian disks of a compression body  $C$ , and  $C'$  a component of the manifold obtained from  $C$  by cutting along  $\mathcal{D}$ . Then, by using the above fact 2, we can show that  $C'$  inherits a compression body structure from  $C$ , ie,  $C'$  is a compression body such that  $\partial_- C' = \partial_- C \cap C'$  and  $\partial_+ C' = (\partial_+ C \cap C') \cup \text{Fr}_C C'$ .
- (4) Let  $S$  be an incompressible surface in  $C$  such that  $\partial S \subset \partial_+ C$ . If  $S$  is not a meridian disk, then, by using the above fact 2, we can show that  $S$  is  $\partial$ -compressible into  $\partial_+ C$ , ie, there exists a disk  $\Delta$  such that  $\Delta \cap S = \partial \Delta \cap S = a$  is an essential arc in  $S$ , and  $\Delta \cap \partial C = \text{cl}(\partial \Delta - a)$  with  $\Delta \cap \partial C \subset \partial_+ C$ .

Let  $N$  be a cobordism rel  $\partial$  between two surfaces  $F_1, F_2$  (possibly  $F_1 = \emptyset$  or  $F_2 = \emptyset$ ), ie,  $F_1$  and  $F_2$  are mutually disjoint surfaces in  $\partial N$  with  $\partial F_1 \cong \partial F_2$  such that  $\partial N = F_1 \cup F_2 \cup (\partial F_1 \times [0, 1])$ .

**Definition 2.2** We say that  $C_1 \cup_P C_2$  (or  $C_1 \cup C_2$ ) is a *Heegaard splitting* of  $(N, F_1, F_2)$  (or simply,  $N$ ) if it satisfies the following conditions.

- (1)  $C_i$  ( $i = 1, 2$ ) is a compression body in  $N$  such that  $\partial_- C_i = F_i$ ,
- (2)  $C_1 \cup C_2 = N$ , and
- (3)  $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$ .

The surface  $P$  is called a *Heegaard surface* of  $(N, F_1, F_2)$  (or,  $N$ ). In particular, if  $P$  is a closed surface, then the genus of  $P$  is called the *genus* of the Heegaard splitting.

**Definition 2.3**

- (1) A Heegaard splitting  $C_1 \cup_P C_2$  is *reducible* if there exist meridian disks  $D_1, D_2$  of the compression bodies  $C_1, C_2$  respectively such that  $\partial D_1 = \partial D_2$
- (2) A Heegaard splitting  $C_1 \cup_P C_2$  is *weakly reducible* if there exist meridian disks  $D_1, D_2$  of the compression bodies  $C_1, C_2$  respectively such that  $\partial D_1 \cap \partial D_2 = \emptyset$ . If  $C_1 \cup_P C_2$  is not weakly reducible, then it is called *strongly irreducible*.
- (3) A Heegaard splitting  $C_1 \cup_P C_2$  is *stabilized* if there exists another Heegaard splitting  $C'_1 \cup_{P'} C'_2$  such that the pair  $(N, P)$  is isotopic to a connected sum of pairs  $(N, P') \# (S^3, T)$ , where  $T$  is a genus one Heegaard surface of the 3–sphere  $S^3$ .
- (4) A Heegaard splitting  $C_1 \cup_P C_2$  is *trivial* if either  $C_1$  or  $C_2$  is a trivial compression body.

**Remark 2.4**

- (1) We note that  $C_1 \cup_P C_2$  is stabilized if and only if there exist meridian disks  $D_1, D_2$  of  $C_1, C_2$  respectively such that  $\partial D_1$  and  $\partial D_2$  intersect transversely in one point.
- (2) If  $C_1 \cup_P C_2$  is stabilized and not a genus one Heegaard splitting of  $S^3$ , then  $C_1 \cup_P C_2$  is reducible.

## 2.B Orbifold version of Heegaard splittings

Throughout this subsection, let  $N$  be a compact, orientable 3–manifold,  $\gamma$  a 1–manifold properly embedded in  $N$ , and  $F, F_1, F_2, D, S$ , connected surfaces embedded in  $N$ , which are in general position with respect to  $\gamma$ .

**Definition 2.5** We say that  $D$  is a  $\gamma$ -disk, if (1)  $D$  is a disk, and (2) either  $D \cap \gamma = \emptyset$ , or  $D$  intersects  $\gamma$  transversely in one point.

Let  $\ell(\subset F)$  be a simple closed curve such that  $\ell \cap \gamma = \emptyset$ .

**Definition 2.6** We say that  $\ell$  is  $\gamma$ -inessential if  $\ell$  bounds a  $\gamma$ -disk in  $F$ . We say that  $\ell$  is  $\gamma$ -essential if it is not  $\gamma$ -inessential.

**Definition 2.7** We say that  $D$  is a  $\gamma$ -compressing disk for  $F$  if  $D$  is a  $\gamma$ -disk,  $D \cap F = \partial D$ , and  $\partial D$  is a  $\gamma$ -essential simple closed curve in  $F$ . The surface  $F$  is  $\gamma$ -compressible if it admits a  $\gamma$ -compressing disk, and  $F$  is  $\gamma$ -incompressible if it is not  $\gamma$ -compressible. We note that if  $D$  is a  $\gamma$ -compressing disk for  $F$ , then we can perform a  $\gamma$ -compression on  $F$  along  $D$  (Figure 1).

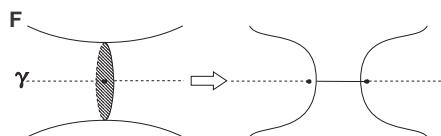


Figure 1

**Definition 2.8** Suppose that  $\partial F_1 = \partial F_2$ , or  $\partial F_1 \cap \partial F_2 = \emptyset$ . We say that  $F_1$  and  $F_2$  are  $\gamma$ -parallel, if there is a submanifold  $R$  in  $N$  such that  $(R, R \cap \gamma)$  is homeomorphic to  $(F_1 \times [0, 1], \mathcal{P} \times [0, 1])$  as a pair, where (1)  $\mathcal{P}$  is a union of points in  $\text{Int} F_1$ , and (2)  $\partial F_1 = \partial F_2$  and  $F_1$  ( $F_2$  respectively) is the subsurface of  $\partial R$  corresponding to the closure of the component of  $\partial(F_1 \times [0, 1]) - (\partial F_1 \times \{1/2\})$  containing  $F_1 \times \{0\}$  ( $F_1 \times \{1\}$  respectively), or  $\partial F_1 \cap \partial F_2 = \emptyset$  and  $F_1$  ( $F_2$  respectively) is the subsurface of  $\partial R$  corresponding to  $F_1 \times \{0\}$  ( $F_1 \times \{1\}$  respectively). The submanifold  $R$  is called a  $\gamma$ -parallelism between  $F_1$  and  $F_2$ .

We say that  $F$  is  $\gamma$ -boundary parallel if there is a subsurface  $F'$  in  $\partial N$  such that  $F$  and  $F'$  are  $\gamma$ -parallel.

**Definition 2.9** We say that  $S$  is a  $\gamma$ -sphere if (1)  $S$  is a sphere, and (2) either  $S \cap \gamma = \emptyset$ , or  $S$  intersects  $\gamma$  transversely in two points. We say that a 3-ball  $B^3$  in  $N$  is a  $\gamma$ -ball if either  $B^3 \cap \gamma = \emptyset$ , or  $B^3 \cap \gamma$  is an unknotted arc properly embedded in  $B^3$ . A  $\gamma$ -sphere  $S$  is  $\gamma$ -compressible if there exists a  $\gamma$ -ball  $B^3$  in  $N$  such that  $\partial B^3 = S$ . A  $\gamma$ -sphere  $S$  is  $\gamma$ -incompressible if it is not  $\gamma$ -compressible. We say that  $N$  is  $\gamma$ -reducible if  $N$  contains a  $\gamma$ -incompressible 2-sphere. The manifold  $N$  is  $\gamma$ -irreducible if it is not  $\gamma$ -reducible.

**Definition 2.10** We say that  $F$  is  $\gamma$ -essential if  $F$  is  $\gamma$ -incompressible, and not  $\gamma$ -boundary parallel.

Let  $a$  be an arc properly embedded in  $F$  with  $a \cap \gamma = \emptyset$ .

**Definition 2.11** We say that  $a$  is  $\gamma$ -inessential if there is a subarc  $b$  of  $\partial F$  such that  $\partial b = \partial a$ , and  $a \cup b$  bounds a disk  $D$  in  $F$  such that  $D \cap \gamma = \emptyset$ , and  $a$  is  $\gamma$ -essential if it is not  $\gamma$ -inessential.

**Definition 2.12** We say that  $\Delta$  is a  $\gamma$ -boundary compressing disk for  $F$  if  $\Delta$  is a disk disjoint from  $\gamma$ ,  $\Delta \cap F = \partial\Delta \cap F = \alpha$  is a  $\gamma$ -essential arc in  $F$ , and  $\Delta \cap \partial N = \partial\Delta \cap \partial N = \text{cl}(\partial\Delta - \alpha)$ . The surface  $F$  is  $\gamma$ -boundary compressible if it admits a  $\gamma$ -boundary compressing disk. The surface  $F$  is  $\gamma$ -boundary incompressible if it is not  $\gamma$ -boundary compressible. We note that if  $\Delta$  is a  $\gamma$ -boundary compressing disk for  $F$ , then we can perform a  $\gamma$ -boundary compression on  $F$  along  $\Delta$ .

**Definition 2.13** We say that  $F_1$  and  $F_2$  are  $\gamma$ -isotopic if there is an ambient isotopy  $\phi_t$  ( $0 \leq t \leq 1$ ) of  $N$  such that  $\phi_0 = \text{id}_N$ ,  $\phi_1(F_1) = F_2$ , and  $\phi_t(\gamma) = \gamma$  for each  $t$ .

The next definition gives an orbifold version of compression body (cf (2) of Remark 2.1).

**Definition 2.14** Suppose that  $N$  is a cobordism rel  $\partial$  between two surfaces  $G_+$ ,  $G_-$ . We say that  $(N, \gamma)$  is an orbifold compression body (or  $O$ -compression body) (with  $\partial_+ N = G_+$ , and  $\partial_- N = G_-$ ) if the following conditions are satisfied.

- (1)  $G_+$  is not empty, and is connected (possibly,  $G_-$  is empty).
- (2) No component of  $G_-$  is a  $\gamma$ -sphere.
- (3)  $\partial\gamma \subset \text{Int}(G_+ \cup G_-)$ .
- (4) There exists a union of mutually disjoint  $\gamma$ -compressing disks, say  $\mathcal{D}$ , for  $G_+$  such that, for each component  $E$  of the manifold obtained from  $N$  by cutting along  $\mathcal{D}$ , either  $E$  is a  $\gamma$ -ball with  $E \cap G_- = \emptyset$ , or  $(E, \gamma \cap E)$  is homeomorphic to  $(G \times [0, 1], \mathcal{P} \times [0, 1])$ , where  $G$  is a component of  $G_-$  with  $E \cap G_- = G \times \{0\} = G$  and  $\mathcal{P}$  is a union of mutually disjoint (possibly empty) points in  $G$  (see Figure 2).

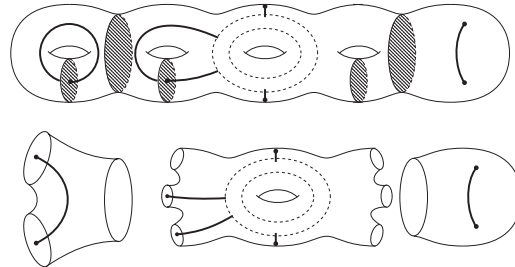


Figure 2

Note that the condition 1 of Definition 2.14 implies that  $N$  is connected. We say that an  $O$ -compression body  $(N, \gamma)$  is *trivial* if either  $N$  is a  $\gamma$ -ball with  $\partial_+ N = \partial N$ , or  $(N, \gamma)$  is homeomorphic to  $(G_- \times [0, 1], \mathcal{P}' \times [0, 1])$  with  $G_- (\subset \partial N)$  corresponding to  $G_- \times \{1\}$ , and  $\mathcal{P}'$  a union of mutually disjoint points in  $G_-$ . An  $O$ -compression body  $(N, \gamma)$  is called an  $O$ -*handlebody* if  $\partial_- N = \emptyset$ . A  $\gamma$ -compressing disk of  $\partial_+ N$  is called a  $(\gamma)$ -*meridian disk* of the  $O$ -compression body  $(N, \gamma)$ .

By  $\mathbb{Z}_2$ -equivariant loop theorem [12, Lemma 3], and  $\mathbb{Z}_2$ -equivariant cut and paste argument as in [10, Proof of 10.3], we can prove the following (the proof is omitted).

**Proposition 2.15** *Let  $N$  be a compact, orientable 3-manifold, and  $\gamma$  a 1-manifold properly embedded in  $N$ . Suppose that  $N$  admits a 2-fold branched cover  $p: \tilde{N} \rightarrow N$  with branch set  $\gamma$ . Let  $F$  be a (possibly closed) surface properly embedded in  $N$ , which is in general position with respect to  $\gamma$ . Then  $F$  is  $\gamma$ -incompressible ( $\gamma$ -boundary incompressible respectively) if and only if  $p^{-1}(F)$  is incompressible (boundary incompressible respectively) in  $\tilde{N}$ .*

By (2) of Remark 2.1, Definition 2.14,  $\mathbb{Z}_2$ -equivariant cut and paste argument as in [10, Proof of 10.3], and  $\mathbb{Z}_2$ -Smith conjecture [21], we immediately have the following.

**Proposition 2.16** *Let  $N, \gamma$  be as in Proposition 2.15. Then  $(N, \gamma)$  is an  $O$ -compression body with  $\partial_{\pm} N = G_{\pm}$ , if and only if  $\tilde{N}$  is a compression body with  $\partial_{\pm} \tilde{N} = p^{-1}(G_{\pm})$ .*

Since the compression bodies are irreducible (see (1) of Remark 2.1), Proposition 2.16 together with  $\mathbb{Z}_2$ -Smith conjecture [21] implies the following.

**Corollary 2.17** *Let  $(N, \gamma)$  be an  $O$ -compression body. Suppose that  $N$  admits a 2-fold branched cover with branch set  $\gamma$ . Then  $N$  is  $\gamma$ -irreducible.*

By (4) of Remark 2.1, and  $\mathbb{Z}_2$ -equivariant cut and paste argument as in [10, Proof of 10.3], we have the following.

**Corollary 2.18** *Let  $(N, \gamma)$  be an  $O$ -compression body such that  $N$  admits a 2-fold branched cover with branch set  $\gamma$ . Let  $F$  be a connected  $\gamma$ -incompressible surface properly embedded in  $N$ , which is not a  $\gamma$ -meridian disk. Suppose that  $\partial F \subset \partial_+ N$ . Then there exists a  $\gamma$ -boundary compressing disk  $\Delta$  for  $F$  such that  $\Delta \cap \partial N \subset \partial_+ N$ .*

Let  $M$  be a compact, orientable 3-manifold, and  $\delta$  a 1-manifold properly embedded in  $M$ . Let  $C$  be a 3-dimensional manifold embedded in  $M$ . We say that  $C$  is a  $\delta$ -compression body if  $(C, \delta \cap C)$  is an  $O$ -compression body. Suppose that  $M$  is a cobordism rel  $\partial$  between two surfaces  $G_1, G_2$  (possibly  $G_1 = \emptyset$  or  $G_2 = \emptyset$ ) such that  $\partial\delta \subset \text{Int}(G_1 \cup G_2)$ .

**Definition 2.19** We say that  $C_1 \cup_P C_2$  is a *Heegaard splitting* of  $(M, \delta, G_1, G_2)$  (or simply  $(M, \delta)$ ) if it satisfies the following conditions.

- (1)  $C_i$  ( $i = 1, 2$ ) is a  $\delta$ -compression body such that  $\partial_- C_i = G_i$ ,
- (2)  $C_1 \cup C_2 = M$ , and
- (3)  $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$ .

The surface  $P$  is called a *Heegaard surface* of  $(M, \delta, G_1, G_2)$  (or  $(M, \delta)$ ).

**Definition 2.20**

- (1) A Heegaard splitting  $C_1 \cup_P C_2$  of  $(M, \delta)$  is  $\delta$ -reducible if there exist  $\delta$ -meridian disks  $D_1, D_2$  of the  $\delta$ -compression bodies  $C_1, C_2$  respectively such that  $\partial D_1 = \partial D_2$ .
- (2) A Heegaard splitting  $C_1 \cup_P C_2$  of  $(M, \delta)$  is *weakly  $\delta$ -reducible* if there exist  $\delta$ -meridian disks  $D_1, D_2$  of the  $\delta$ -compression bodies  $C_1, C_2$  respectively such that  $\partial D_1 \cap \partial D_2 = \emptyset$ . If  $C_1 \cup_P C_2$  is not weakly  $\delta$ -reducible, then it is called *strongly  $\delta$ -irreducible*.
- (3) A Heegaard splitting  $C_1 \cup_P C_2$  of  $(M, \delta)$  is *trivial* if either  $C_1$  or  $C_2$  is a trivial  $\delta$ -compression body.



## 2.C Genus $g$ , $n$ -bridge positions

We first recall the definition of a genus  $g$ ,  $n$ -bridge position of H.Doll [4]. Let  $\Gamma = \gamma_1 \cup \cdots \cup \gamma_n$  be a union of mutually disjoint arcs  $\gamma_i$  properly embedded in a 3-manifold  $N$ .

**Definition 2.21** We say that  $\Gamma$  is *trivial* if there exist mutually disjoint disks  $D_1, \dots, D_n$  in  $N$  such that (1)  $D_i \cap \Gamma = \partial D_i \cap \gamma_i = \gamma_i$ , and (2)  $D_i \cap \partial N = \text{cl}(\partial D_i - \gamma_i)$ .

Let  $K$  be a link in a closed 3-manifold  $M$ . Let  $X \cup_Q Y$  be a genus  $g$  Heegaard splitting of  $M$ . Then, with following [4], we say that  $K$  is in a *genus  $g$ ,  $n$ -bridge position* (with respect to the Heegaard splitting  $X \cup_Q Y$ ) if  $K \cap X$  ( $K \cap Y$  respectively) is a union of  $n$  arcs which is trivial in  $X$  ( $Y$  respectively).

A proof of the next lemma is elementary, and we omit it.

**Lemma 2.22** *Let  $\Gamma$  be a union of mutually disjoint arcs properly embedded in a handlebody  $H$ . Then  $\Gamma$  is trivial if and only if  $(H, \Gamma)$  is a  $O$ -handlebody.*

This lemma allows us to generalize the definition of genus  $g$ ,  $n$ -bridge positions as in the following form. Let  $K$ ,  $M$ , and  $X \cup_Q Y$  be as above.

**Definition 2.23** We say that  $K$  is in a *genus  $g$ ,  $n$ -bridge position* (with respect to the Heegaard splitting  $X \cup_Q Y$ ) if  $X \cup_Q Y$  gives a Heegaard splitting of  $(M, K)$  such that  $\text{genus}(Q) = g$ , and  $K \cap Q$  consists of  $2n$  points.

**Remark 2.24** This definition allows genus  $g$ , 0-bridge position of  $K$ .

In this paper, we abbreviate genus 0,  $n$ -bridge position to  *$n$ -bridge position*.

**Definition 2.25** A knot  $K$  in the 3-sphere  $S^3$  is called a  *$n$ -bridge knot*, if it admits a  $n$ -bridge position.

## 3 Weakly $\gamma$ -reducible Heegaard splittings

In [8], W Haken proved that the Heegaard splittings of a reducible 3-manifold are reducible. As a sequel of this, Casson-Gordon [2] proved that each non-trivial Heegaard splitting of a  $\partial$ -reducible 3-manifold is weakly reducible. In this section, we prove orbifold versions of these results. In fact, we prove the following.

**Proposition 3.1** *Let  $N$  be a compact orientable 3-manifold, and  $\gamma$  a 1-manifold properly embedded in  $N$  such that  $N$  admits a 2-fold branched cover with branch set  $\gamma$ . Suppose that  $N$  is a cobordism rel  $\partial$  between two surfaces  $F_1, F_2$  (possibly  $F_1 = \emptyset$  or  $F_2 = \emptyset$ ) such that  $\partial\gamma \subset \text{Int}(F_1 \cup F_2)$ , and no component of  $F_1 \cup F_2$  is a  $\gamma$ -disk. If  $N$  is  $\gamma$ -reducible, then every Heegaard splitting of  $(N, \gamma, F_1, F_2)$  is weakly  $\gamma$ -reducible.*

**Proposition 3.2** *Let  $N, \gamma, F_1, F_2$  be as in Proposition 3.1. If  $F_1 \cup F_2$  is  $\gamma$ -compressible in  $N$ , then every non-trivial Heegaard splitting of  $(N, \gamma, F_1, F_2)$  is weakly  $\gamma$ -reducible.*

**Remark 3.3** In the conclusion of Proposition 3.1, we can have just “weakly  $\gamma$ -reducible”, not “ $\gamma$ -reducible”. For example, let  $K$  be a connected sum of two trefoil knots, and  $C_1 \cup C_2$  the Heegaard splitting of  $(S^3, K)$  as in Figure 3. We note that  $(S^3, K)$  is  $K$ -reducible (in fact, a 2-sphere giving prime decomposition of  $K$  is  $K$ -incompressible). Since the Heegaard splitting gives a minimal genus Heegaard splitting of  $E(K)$ , we can show that  $C_1 \cup C_2$  is not  $\gamma$ -reducible. But  $C_1 \cup C_2$  admits a pair of weakly  $K$ -reducing disks  $D_1, D_2$  as in Figure 3.

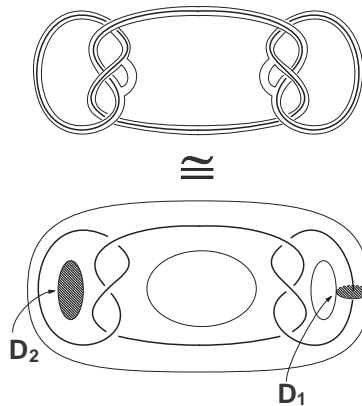


Figure 3

Then, by using Proposition 3.2, we prove an orbifold version of a lemma of Rubinstein–Scharlemann [17, Lemma 4.5].

**Proposition 3.4** *Let  $M$  be a closed orientable 3-manifold, and  $K$  a link in  $M$  such that  $M$  admits a 2-fold branched cover with branch set  $K$ . Let  $A \cup_P B, X \cup_Q Y$  be Heegaard splittings of  $(M, K)$ . Suppose that  $A \subset \text{Int}X$ , and there*

exists a  $K$ -meridian disk  $D$  of  $X$  such that  $D \cap A = \emptyset$ . Then we have one of the following.

- (1)  $M$  is homeomorphic to the 3-sphere, and either  $K = \emptyset$  or  $K$  is a trivial knot.
- (2)  $X \cup_Q Y$  is weakly  $K$ -reducible.

### 3.A Heegaard splittings of $(\hat{N}, \hat{F}_1, \hat{F}_2)$

For the proofs of Propositions 3.1, and 3.2, we show that we can derive Heegaard splittings of  $\text{cl}(N - N(\gamma))$  from Heegaard splittings of  $(N, \gamma)$ .

**Lemma 3.5** *Let  $(C, \beta)$  be a  $O$ -compression body such that  $C$  admits a 2-fold branched cover  $q: \tilde{C} \rightarrow C$  with branch set  $\beta$ . Let  $\hat{C} = \text{cl}(C - N(\beta))$ ,  $S = \text{cl}(\partial_+ C - N(\beta))$ . Then  $\hat{C}$  is a compression body with  $\partial_+ \hat{C} = S$ .*

**Proof** Let  $\mathcal{D}$  be the union of mutually disjoint  $\beta$ -compressing disks for  $\partial_+ C$  as in Definition 2.14. Let  $\mathcal{D}_0$  ( $\mathcal{D}_1$  respectively) be the union of the components of  $\mathcal{D}$  which are disjoint from  $\beta$  (which intersect  $\beta$  respectively). Let  $E$  be a component of the manifold obtained from  $C$  by cutting along  $\mathcal{D}_0$ , and  $\hat{E} = \text{cl}(E - N(\beta))$ . Let  $\mathcal{D}_{1,E}$  be the union of the components of  $\mathcal{D}_1$  that are contained in  $E$ . Let  $E'$  be the manifold obtained from  $E$  by cutting along  $\mathcal{D}_{1,E}$ , and  $\hat{E}' = \text{cl}(E' - N(\beta))$ . Then we have the following cases.

**Case 1**  $E \cap \beta = \emptyset$ .

In this case,  $\mathcal{D}_{1,E} = \emptyset$ , and we have  $E = \hat{E} = E' = \hat{E}'$ . By the definition of  $\beta$ -compression body (Definition 2.14), we see that  $\hat{E}(= E)$  is a trivial compression body such that  $\hat{E} \cap \mathcal{D}_0 \subset \partial_+ \hat{E}$ .

**Case 2**  $E \cap \beta \neq \emptyset$ , and  $E \cap \partial_- C = \emptyset$ .

By the definition of  $\beta$ -compression body, we see that each component of  $E'$  is a  $\beta$ -ball intersecting  $\beta$  with  $E' \cap \partial_- C = \emptyset$ . Hence each component of  $\hat{E}'$  is a solid torus, say  $T$ , such that  $T \cap N(\beta)$  is an annulus which is a neighborhood of a longitude of  $T$ . This implies that each component of  $\hat{E}'$  is a trivial compression body such that the union of the  $\partial_+$  boundaries is  $\text{cl}(\partial \hat{E}' - N(\beta))$ . Since  $\hat{E}$  is recovered from  $\hat{E}'$  by identifying pairs of annuli corresponding to  $\text{cl}(\mathcal{D}_{1,E} - N(\beta))$ , we see that the triviality can be pulled back to show that  $\hat{E}$  is a trivial

compression body with  $\partial_+ \hat{E} = \text{cl}(\partial \hat{E} - N(\beta)) = \text{cl}(\partial_+ E - N(\beta))$ , where  $\hat{E} \cap \mathcal{D}_0 \subset \partial_+ \hat{E}$ . In fact, we see that either  $E$  is a  $\beta$ -ball or  $E$  is a solid torus with  $\beta \cap E$  a core circle.

**Case 3**  $E \cap \beta \neq \emptyset$ , and  $E \cap \partial_- C \neq \emptyset$ .

By the definition of  $\beta$ -compression body, for each component  $E^*$  of  $E'$ , we have either  $E^*$  is a  $\beta$ -ball intersecting  $\beta$  with  $E^* \cap \partial_- C = \emptyset$ , or  $(E^*, E^* \cap \beta)$  is a trivial  $\beta$ -compression body such that the  $\partial_-$  boundary is a component of  $\partial_- C$ . In either case,  $\hat{E}^* = \text{cl}(E^* - N(\beta))$  is a trivial compression body such that  $\partial_+ \hat{E}^* = \text{cl}(\partial_+ E^* - N(\beta))$ . Hence  $\hat{E}'$  is a union of trivial compression bodies such that the union of the  $\partial_+$  boundaries is  $\text{cl}(\partial_+ E' - N(\beta))$ . Since  $\hat{E}$  is recovered from  $\hat{E}'$  by identifying pairs of annuli corresponding to  $\text{cl}(\mathcal{D}_{1,E} - N(\beta))$ , we see that the triviality can be pulled back to show that  $\hat{E}$  is a trivial compression body with  $\partial_+ \hat{E} = \text{cl}(\partial_+ E - N(\beta))$ , where  $\hat{E} \cap \mathcal{D}_0 \subset \partial_+ \hat{E}$ .

By the conclusions of Cases 1, 2 and 3, we see that  $\hat{C}$  is recovered from a union of trivial compression bodies by identifying the pairs of disks in  $\partial_+$  boundaries, which are corresponding to  $\mathcal{D}_0$ , and this implies that  $\hat{C}$  is a compression body (see (2) of Remark 2.1). Moreover, since the  $\partial_+$  boundary of each trivial compression body  $\hat{E}$  is  $\text{cl}(\partial_+ E - N(\beta))$ , we see that  $\partial_+ \hat{C} = \text{cl}(\partial_+ C - N(\beta))$ .  $\square$

Let  $C_1 \cup_P C_2$  be a Heegaard splitting of  $(N, \gamma, F_1, F_2)$ . Then let  $\hat{N} = \text{cl}(N - N(\gamma))$ ,  $\hat{P} = \text{cl}(P - N(\gamma))$ ,  $\hat{C}_i = \text{cl}(C_i - N(\gamma))$ , and  $\hat{F}_i = \text{cl}(\partial \hat{C}_i - N(\hat{P}, \partial \hat{C}_i))$  ( $i = 1, 2$ ). By Lemma 3.5, we see that  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  is a Heegaard splitting of  $(\hat{N}, \hat{F}_1, \hat{F}_2)$ . By the definitions of strongly irreducible Heegaard splittings, and strongly  $\gamma$ -irreducible Heegaard splittings, we immediately have the following.

**Lemma 3.6** *If  $C_1 \cup_P C_2$  is strongly  $\gamma$ -irreducible, then  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  is strongly irreducible.*

### 3.B Proof of Proposition 3.1

Let  $N, \gamma$  be as in Proposition 3.1, and  $C_1 \cup_P C_2$  a Heegaard splitting of  $(N, \gamma)$ . Let  $\hat{N} = \text{cl}(N - N(\gamma))$ , and  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  a Heegaard splitting of  $(\hat{N}, \hat{F}_1, \hat{F}_2)$  obtained from  $C_1 \cup_P C_2$  as in Section 3.A. Since  $(N, \gamma)$  is  $\gamma$ -reducible, there exists a  $\gamma$ -incompressible  $\gamma$ -sphere  $S$  in  $N$ . Then we have the following two cases.

**Case 1**  $S \cap \gamma = \emptyset$ .

In this case, we may regard that  $S$  is a 2-sphere in  $\hat{N}$ . It is clear that  $S$  is an incompressible 2-sphere in  $\hat{N}$ . Hence, by [2, Lemma 1.1], we see that there exists an incompressible 2-sphere  $S'$  in  $\hat{N}$  such that  $S'$  intersects  $\hat{P}$  in a circle. Since  $\hat{N} \subset N$ , we may regard  $S'$  is a 2-sphere in  $N$ . It is clear that  $S' \cap P$  is a  $\gamma$ -essential simple closed curve in  $P$ , hence,  $S' \cap C_i$  ( $i = 1, 2$ ) is a  $\gamma$ -meridian disk of  $C_i$ . This shows that  $C_1 \cup_P C_2$  is  $\gamma$ -reducible.

**Case 2**  $S \cap \gamma \neq \emptyset$  (ie,  $S \cap \gamma$  consists of two points).

We may suppose that  $(S \cap \gamma) \cap P = \emptyset$ . Let  $\hat{S} = \text{cl}(S - N(\gamma))$ . Then  $\hat{S}$  is an annulus properly embedded in  $\hat{N}$  such that  $\partial\hat{S} \subset \text{Fr}_N N(\gamma)$ , and  $\partial\hat{S} \cap \hat{P} = \emptyset$ .

**Claim 1**  $\hat{S}$  is incompressible in  $\hat{N}$ .

**Proof** If there is a compressing disk  $D$  for  $\hat{S}$ , then by compressing  $S$  along  $D$ , we obtain two 2-spheres, each of which intersects  $\gamma$  in one point. This contradicts the existence of a 2-fold branched cover of  $N$  with branch set  $\gamma$ .  $\square$

**Claim 2**  $\hat{S}$  is not  $\partial$ -parallel in  $\hat{N}$ .

**Proof** Suppose that  $\hat{S}$  is parallel to an annulus  $\mathcal{A}$  in  $\partial\hat{N}$ . Let  $s = \text{cl}(\partial N - (F_1 \cup F_2))$ . Note that  $s$  is a (possibly empty) union of annulus. Let  $F'_i = \text{cl}(F_i - N(\gamma))$ . Then  $\partial\hat{N} = s \cup F'_1 \cup F'_2 \cup \text{Fr}_N N(\gamma)$ . Since  $S$  is  $\gamma$ -incompressible, we see that  $(F'_1 \cup F'_2) \cap \mathcal{A} \neq \emptyset$ . Since no component of  $F_1 \cup F_2$  is a  $\gamma$ -disk, each component of  $(F'_1 \cup F'_2) \cap \mathcal{A}$  is an annulus. Let  $A^*$  be a component of  $\text{Fr}_N N(\gamma)$  such that  $A^*$  contains a component of  $\partial\hat{S}$ . Let  $F^*$  be the component of  $(F'_1 \cup F'_2) \cap \mathcal{A}$  such that  $F^* \cap A^* \neq \emptyset$ . Note that  $F^* \cap A^*$  is a component of  $\partial A^*$  and is also a component of  $\partial F^*$ . Let  $A'$  be the component of  $\text{cl}(\partial\hat{N} - (F'_1 \cup F'_2))$  such that  $A' \cap F^*$  is the component of  $\partial F^*$  other than  $F^* \cap A^*$ . Then  $A'$  is an annulus which is either a component of  $\text{Fr}_N N(\gamma)$ , or a component of  $s$ . If  $A'$  is a component of  $\text{Fr}_N N(\gamma)$ , then the component of  $F_1 \cup F_2$  corresponding to  $F^*$  is a  $\gamma$ -sphere, hence, a component of  $\partial_- C_1$  or  $\partial_- C_2$  is a  $\gamma$ -sphere, a contradiction. If  $A'$  is a component of  $s$ , then the component of  $F_1 \cup F_2$  corresponding to  $F^*$  is a  $\gamma$ -disk, contradicting the assumption of Proposition 3.1.  $\square$

By Claims 1 and 2,  $\hat{S}$  is  $\gamma$ -essential in  $\hat{N}$ . Suppose, for a contradiction, that  $C_1 \cup_P C_2$  is strongly  $\gamma$ -irreducible. By Lemma 3.6,  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  is strongly irreducible. Then, by [19, Lemma 6] or [16, Lemma 2.3],  $\hat{S}$  is ambient isotopic rel  $\partial$  to a surface  $\hat{S}'$  such that  $\hat{S}' \cap \hat{P}$  consists of essential simple closed curves

in  $\hat{S}'$ . We regard  $\hat{S} = \hat{S}'$ . This means that each component of  $S \cap P$  is a simple closed curve which separates the points  $S \cap \gamma$ . We suppose that  $|S \cap P|$  is minimal among the  $\gamma$ -incompressible  $\gamma$ -spheres with this property. Let  $n = |S \cap P|$ . Suppose that  $n = 1$ , ie,  $S \cap P$  consists of a simple closed curve, say  $\ell_1$ . Then  $\ell_1$  separates  $S$  into two  $\gamma$ -disks, which are  $\gamma$ -meridian disks in  $C_1$  and  $C_2$  respectively. This shows that  $C_1 \cup_P C_2$  is  $\gamma$ -reducible, a contradiction. Suppose that  $n \geq 2$ . Let  $D_1$  be the closure of a component of  $S - P$  such that  $D_1 \cap \gamma \neq \emptyset$ . Note that  $D_1$  is a  $\gamma$ -disk. Without loss of generality, we may suppose that  $D_1 \subset C_1$ . By the minimality of  $|S \cap P|$ , we see that  $D_1$  is a  $\gamma$ -meridian disk of  $C_1$ . Let  $A_2$  be the closure of the component of  $S - P$  such that  $A_2 \cap D_1 \neq \emptyset$ .

**Claim 3**  $A_2$  is  $\gamma$ -incompressible in  $C_2$ .

**Proof** Suppose that there is a  $\gamma$ -compressing disk  $D$  for  $A_2$  in  $C_2$ . If  $D \cap \gamma = \emptyset$ , then we have a contradiction as in the proof of Claim 1. Suppose that  $D \cap \gamma \neq \emptyset$ . Let  $D_2$  be the disk obtained from  $A_2$  by  $\gamma$ -compressing  $A_2$  along  $D$  such that  $\partial D_2 = \partial D_1$ . Since  $\partial D_1$  is  $\gamma$ -essential in  $P$ , this shows that  $D_2$  is a  $\gamma$ -meridian disk of  $C_2$ . Hence  $C_1 \cup_P C_2$  is  $\gamma$ -reducible, a contradiction.  $\square$

Note that  $\partial A_2 \subset \partial_+ C_2$ . There is a  $\gamma$ -boundary compressing disk  $\Delta$  for  $A_2$  in  $C_2$  such that  $\Delta \cap \partial C_2 \subset \partial_+ C_2$  (Corollary 2.18). By the minimality of  $|S \cap P|$ , we see that  $A_2$  is not  $\gamma$ -parallel to a surface in  $\partial_+ C_2$ . Hence, by  $\gamma$ -boundary compressing  $A_2$  along  $\Delta$ , and applying a tiny isotopy, we obtain a  $\gamma$ -meridian disk  $D_2$  in  $C_2$  such that  $D_1 \cap D_2 = \emptyset$ . Hence  $C_1 \cup_P C_2$  is weakly  $\gamma$ -reducible, a contradiction.

This completes the proof of Proposition 3.1.

### 3.C Proof of Proposition 3.2

Let  $N, \gamma$  be as in Proposition 3.2 and  $C_1 \cup_P C_2$  a Heegaard splitting of  $(N, \gamma)$ . Let  $\hat{N} = \text{cl}(N - N(\gamma))$ , and  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  the Heegaard splitting of  $(\hat{N}, \hat{F}_1, \hat{F}_2)$  obtained from  $C_1 \cup_P C_2$  as in Section 3.A. Let  $D$  be a  $\gamma$ -compressing disk for  $F_1 \cup F_2$ .

**Case 1**  $D \cap \gamma = \emptyset$ .

In this case, we may regard that  $D$  is a disk in  $\hat{N}$ . It is clear that  $D$  is a compressing disk of  $\hat{F}_1 \cup \hat{F}_2$ . Hence, by [2, Lemma 1.1], we see that  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  is weakly reducible. This implies that  $C_1 \cup_P C_2$  is weakly  $\gamma$ -reducible.

**Case 2**  $D \cap \gamma \neq \emptyset$  (ie,  $D \cap \gamma$  consists of a point).

Let  $\hat{D} = \text{cl}(D - N(\gamma))$ .

**Claim**  $\hat{D}$  is an essential annulus in  $\hat{N}$ .

**Proof** By using the argument as in Claim 1 of Case 2 of Section 3.B, we can show that  $\hat{D}$  is incompressible in  $\hat{N}$ . Suppose that  $\hat{D}$  is parallel to an annulus  $\mathcal{A}$  in  $\partial\hat{N}$ . Let  $s, F'_i$  ( $i = 1, 2$ ) be as in Claim 2 of Case 2 of Section 3.B. Let  $A^*$  be the component of  $\text{Fr}_N N(\gamma)$  such that  $\partial D \subset A^*$ , and  $F^*$  the component of  $F'_1 \cup F'_2$  such that  $F^* \supset \partial D$ . By using the argument of the proof of Claim 2 of Case 2 of Section 3.B, we see that  $\mathcal{A}$  is disjoint from  $s \cup (\text{Fr}_N N(\gamma) - A^*)$ , hence  $\text{cl}(\mathcal{A} - A^*) \subset F^*$ . Hence  $F^* \cap \mathcal{A}$  is an annulus, and this shows that  $\partial D$  bounds a  $\gamma$ -disk in  $F_1 \cup F_2$ , a contradiction.  $\square$

Suppose, for a contradiction, that  $C_1 \cup_P C_2$  is strongly  $\gamma$ -irreducible. By Lemma 3.6,  $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$  is strongly irreducible. Then, by [19, Lemma 6] or [16, Lemma 2.3],  $\hat{D}$  is ambient isotopic rel  $\partial$  to a surface  $\hat{D}'$  such that  $\hat{D}' \cap \hat{P}$  consists of essential simple closed curves in  $\hat{D}'$ . We regard  $\hat{D} = \hat{D}'$ . This means that each component of  $D \cap P$  is a simple closed curve bounding a disk in  $D$ , which contains the point  $D \cap \gamma$ . We suppose that  $|D \cap P|$  is minimal among the  $\gamma$ -compressing disks for  $F_1 \cup F_2$  with this property. Let  $n = |D \cap P|$ .

Suppose that  $n = 1$ , ie,  $D \cap P$  consists of a simple closed curve, say  $\ell_1$ . Then the closures of the components of  $D - \ell_1$  consists of a disk, say  $D_1$ , and an annulus, say  $A_2$ . Without loss of generality, we may suppose that  $D_1 \subset C_1$ , and  $A_2 \subset C_2$ . Note that a component of  $\partial A_2$  is contained in  $\partial_- C_2$ , and the other in  $\partial_+ C_2$ . Since  $C_2$  is not trivial, there exists a  $\gamma$ -meridian disk  $D_2$  in  $C_2$ . It is elementary to show, by applying cut and paste arguments on  $D_2$  and  $A_2$ , that there is a  $\gamma$ -meridian disk  $D'_2$  in  $C_2$  such that  $D'_2 \cap A_2 = \emptyset$ . Hence  $D_1 \cap D'_2 = \emptyset$ , and this shows that  $C_1 \cup_P C_2$  is weakly  $\gamma$ -reducible, a contradiction.

Suppose that  $n \geq 2$ . Let  $D_1$  be the closure of the component of  $D - P$  such that  $D_1 \cap \gamma \neq \emptyset$ . Note that  $D_1$  is a  $\gamma$ -disk. Without loss of generality, we may suppose that  $D_1 \subset C_1$ . By the minimality of  $|D \cap P|$ , we see that  $D_1$  is a  $\gamma$ -meridian disk of  $C_1$ . Let  $A_2$  be the closure of the component of  $D - P$  such that  $A_2 \cap D_1 \neq \emptyset$ . Then, by using the arguments as in the proof of Claim 3 of Case 2 of Section 3.B, we can show that  $A_2$  is  $\gamma$ -incompressible in  $C_2$ . Note that  $\partial A_2 \subset \partial_+ C_2$ . There is a  $\gamma$ -boundary compressing disk  $\Delta$  for  $A_2$  in  $C_2$  such that  $\Delta \cap \partial C_2 \subset \partial_+ C_2$  (Corollary 2.18). By the minimality of  $|S \cap P|$ ,

we see that  $A_2$  is not  $\gamma$ -parallel to a surface in  $\partial_+ C_2$ . Hence, by  $\gamma$ -boundary compressing  $A_2$  along  $\Delta$ , and applying a tiny isotopy, we obtain a  $\gamma$ -meridian disk  $D_2$  of  $C_2$  such that  $D_1 \cap D_2 = \emptyset$ . Hence  $C_1 \cup_P C_2$  is weakly  $\gamma$ -reducible, a contradiction.

This completes the proof of Proposition 3.2.

### 3.D Proof of Proposition 3.4

Let  $\mathcal{D}$  be a union of mutually disjoint, non  $K$ -parallel,  $K$ -meridian disks for  $X$  such that  $\mathcal{D} \cap A = \emptyset$ . We suppose that  $\mathcal{D}$  is maximal among the unions of  $K$ -meridian disks with the above properties. Let  $Z' = N(\partial X, X) \cup N(\mathcal{D}, X)$ . Then we have the following two cases.

**Case 1** A component of  $\partial Z' - \partial X$  bounds a  $K$ -ball, say  $B_K$ , such that  $B_K \supset A$ .

In this case, since  $\partial B_K \subset B$ , and  $B$  is  $K$ -irreducible,  $\partial B_K$  bounds a  $K$ -ball  $B'_K$  in  $B$  (Corollary 2.17). Hence  $M = B_K \cup B'_K$  is a 3-sphere. In particular, if  $K \neq \emptyset$ , then  $K \cap B_K$  ( $K \cap B'_K$  respectively) is a trivial arc properly embedded in  $B_K$  ( $B'_K$  respectively). Hence  $K$  is a trivial knot. This shows that we have conclusion 1.

**Case 2** No component of  $\partial Z' - \partial X$  bounds a  $K$ -ball which contains  $A$ .

Since  $X$  is  $K$ -irreducible, each of the  $K$ -sphere components of  $\partial Z' - \partial X$  (if exists) bounds  $K$ -balls in  $X$ . By the construction of  $Z'$ , it is easy to see that the  $K$ -balls are mutually disjoint. Let  $Z = Z' \cup$  (the  $K$ -balls). By (3) of Remark 2.1 and Proposition 2.16, we see that  $Z$  is a  $K$ -compression body with  $\partial_+ Z = \partial X$ , and by the maximality of  $\mathcal{D}$ , we see that  $\partial_- Z$  consists of one component, say  $F$ , such that  $F$  bounds a  $K$ -handlebody which contains  $A$ . Let  $N = Y \cup Z$ . Note that  $Y \cup_Q Z$  is a Heegaard splitting of  $(N, K \cap N)$ . Since  $\partial N = F$  is a closed surface contained in  $B$ , it is  $K$ -compressible in  $B$  (Proposition 2.15). By the maximality of  $\mathcal{D}$ , we see that the compressing disk lies in  $N$ . Hence, by Proposition 3.2, we see that  $Y \cup_Q Z$  is weakly  $K$ -reducible. This obviously implies that  $X \cup_Q Y$  is weakly  $K$ -reducible, and we have conclusion 2.

This completes the proof of Proposition 3.4.



## 4 The Casson–Gordon theorem

A Casson and C McA Gordon proved that if a Heegaard splitting of a closed 3–manifold  $M$  is weakly reducible, then either the splitting is reducible, or  $M$  contains an incompressible surface [2, Theorem 3.1]. In this section, we generalize this result for compact  $M$ . The author thinks that this generalization is well known (eg, [20]). However, the formulation given here will be useful for the proof of Theorem 1.1 (Section 7.C).

Let  $M$  be a compact, orientable 3–manifold, and  $C_1 \cup_P C_2$  a Heegaard splitting of  $M$  such that  $P$  is a closed surface, ie,  $\partial_- C_1 \cup \partial_- C_2 = \partial M$ . Let  $\Delta = \Delta_1 \cup \Delta_2$  be a weakly reducing collection of disks for  $P$ , ie,  $\Delta_i$  ( $i = 1, 2$ ) is a union of mutually disjoint, non-empty meridian disks of  $C_i$  such that  $\Delta_1 \cap \Delta_2 = \emptyset$ . Then  $P(\Delta)$  denotes the surface obtained from  $P$  by compressing  $P$  along  $\Delta$ . Let  $\hat{P}(\Delta) = P(\Delta) -$  (the components of  $P(\Delta)$  which are contained in  $C_1$  or  $C_2$ ).

**Lemma 4.1** *If there is a 2–sphere component in  $\hat{P}(\Delta)$ , then  $C_1 \cup_P C_2$  is reducible.*

**Proof** Suppose that there is a 2–sphere component  $S$  of  $\hat{P}(\Delta)$ . We note that  $S \cap C_i$  ( $i = 1, 2$ ) is a union of non-empty meridian disks of  $C_i$ . Let  $\hat{S} = \text{cl}(S - (C_1 \cup C_2))$ . Note that  $\hat{S}$  is a planar surface in  $P$ . Let  $\mathcal{A}_1 \cup \mathcal{A}_2$  be a union of mutually disjoint arcs properly embedded in  $\hat{S}$  such that  $\partial \mathcal{A}_i \subset \partial(S \cap C_i)$ , and that  $\text{cl}(\hat{S} - N(\mathcal{A}_1 \cup \mathcal{A}_2, \hat{S}))$  is an annulus, say  $A'$ . Let  $S'$  be a 2–sphere obtained from  $S$  by pushing  $\mathcal{A}_1$  into  $C_1$ , and  $\mathcal{A}_2$  into  $C_2$  such that  $S' \cap P = A'$ . It is clear that  $S' \cap C_i$  ( $i = 1, 2$ ) consists of a disk, say  $D_i$ , obtained from  $S \cap C_i$  by banding along  $\mathcal{A}_i$ .

**Claim**  $D_i$  is a meridian disk of the compression body  $C_i$  ( $i = 1, 2$ ).

**Proof** Suppose, for a contradiction, that either  $D_1$  or  $D_2$ , say  $D_1$ , is not a meridian disk, ie, there is a disk  $D$  in  $P$  such that  $\partial D = \partial D_1$ . Note that we have either  $N(\mathcal{A}_1, \hat{S}) \subset D$ , or  $N(\mathcal{A}_1, \hat{S}) \subset \text{cl}(P - D)$ . If  $N(\mathcal{A}_1, \hat{S}) \subset D$ , then  $\partial(S \cap C_1)$  is recovered from  $\partial D$  by banding along arcs properly embedded in  $D$ . This shows that  $\partial(S \cap C_1) \subset D$ , and this implies that each component of  $S \cap C_1$  is not a meridian disk, a contradiction. On the other hand, if  $N(\mathcal{A}_1, \hat{S}) \subset \text{cl}(P - D)$ , then  $\text{cl}(\hat{S} - N(\mathcal{A}_1, \hat{S})) \subset D$ . This shows that  $\partial(S \cap C_2) \subset D$ , and this implies that each component of  $S \cap C_2$  is not a meridian disk, a contradiction. □

Since  $\partial D_1$  and  $\partial D_2$  are parallel in  $P$ , we see by Claim that  $C_1 \cup_P C_2$  is reducible.  $\square$

Now we define a complexity  $c(F)$  of a closed surface  $F$  as follows.

$$c(F) = \sum (\chi(F_i) - 1),$$

where the sum is taken for all components of  $F$ . Then we suppose that  $c(\hat{P}(\Delta))$  is maximal among all weakly reducing collections of disks for  $P$ . By Lemma 4.1, we see that if the complexity of a component of  $\hat{P}(\Delta)$  is positive, then  $C_1 \cup_P C_2$  is reducible. Suppose that the complexities of the components of  $\hat{P}(\Delta)$  are strictly negative, ie, each component of  $\hat{P}(\Delta)$  is not a 2-sphere. Then, by the argument of the proof of [2, Theorem 3.1], we see that  $\hat{P}(\Delta)$  is incompressible in  $M$ . Hence we have the next proposition.

**Proposition 4.2** *Let  $M$  be a compact, orientable 3-manifold, and  $C_1 \cup_P C_2$  a Heegaard splitting of  $M$  with  $\partial_- C_1 \cup \partial_- C_2 = \partial M$ . Suppose that  $C_1 \cup_P C_2$  is weakly reducible. Then either*

- (1)  $C_1 \cup_P C_2$  is reducible, or
- (2) there exists a weakly reducing collection of disks  $\Delta$  for  $P$  such that each component of  $\hat{P}(\Delta)$  is an incompressible surface in  $M$ , which is not a 2-sphere.

Note that, in [2],  $M$  is assumed to be closed. However, it is easy to see that the arguments there work for Heegaard splittings  $C_1 \cup C_2$  such that  $\partial_- C_1 \cup \partial_- C_2 = \partial M$ .

The following is a slight extension of [1, Lemme 1.4]. Let  $M$ ,  $C_1 \cup_P C_2$ ,  $\Delta$  be as above. Suppose that we have conclusion 2 of Proposition 4.2. Let  $M_1, \dots, M_n$  be the closures of the components of  $M - \hat{P}(\Delta)$ . Let  $M_{j,i} = M_j \cap C_i$  ( $j = 1, \dots, n, i = 1, 2$ ).

**Lemma 4.3** *For each  $j$ , we have either one of the following.*

- (1)  $M_{j,2} \cap P \subset \text{Int}(M_{j,1} \cap P)$ , and  $M_{j,1}$  is connected.
- (2)  $M_{j,1} \cap P \subset \text{Int}(M_{j,2} \cap P)$ , and  $M_{j,2}$  is connected.

**Proof** Recall that  $\Delta_i$  is the union of the components of  $\Delta$  that are contained in  $C_i$  ( $i = 1, 2$ ). We see, from the definition of  $\hat{P}(\Delta)$ , that each  $M_j$  is obtained as in the following manner.

(\*) Take a component  $N$  of  $\text{cl}(C_i - N(\Delta_i, C_i))$  ( $i = 1$  or  $2$ , say  $1$ ) such that there exists a component  $D_2$  of  $\Delta_2$  such that  $\partial D_2 \subset N$ . Let  $N' = N \cup (\text{the components of } N(\Delta_2, C_2) \text{ intersecting } N)$ . Then  $M_j = N' \cup (\text{the union of components } N_2 \text{ of } \text{cl}(C_2 - N(\Delta_2, C_2)) \text{ such that } (N_2 \cap P) \subset (N \cap P))$ .

It is clear that this construction process gives conclusion 1. If  $N$  is a component of  $\text{cl}(C_2 - N(\Delta_2, C_2))$ , then we have conclusion 2. □

We note that each component of  $\text{Fr}_{C_i}(M_{j,i})$  is a meridian disk of  $C_i$ , which is parallel to a component of  $\Delta$ . Recall that  $\hat{P}(\Delta)$  is obtained from  $P(\Delta)$  by discarding the components each of which is contained in  $C_1$  or  $C_2$ . These imply that each component  $E$  of  $M_{j,i}$  inherits a compression body structure from  $C_i$  (see (3) of Remark 2.1), ie,  $\partial_+ E = (E \cap \partial_+ C_i) \cup \text{Fr}_{C_i}(E)$ . Then we can obtain a splitting, denoted by  $C_{j,1} \cup_{P_j} C_{j,2}$ , of  $M_j$  as follows ([1, Lemme 1.4]).

Suppose that  $M_j$  satisfies conclusion 1 (2 respectively) of Lemma 4.3. Recall that  $M_{j,1}$  ( $M_{j,2}$  respectively) inherits a compression body structure from  $C_1$  ( $C_2$  respectively). Then let  $C_{j,1} = \text{cl}(M_{j,1} - N(\partial_+ M_{j,1}, M_{j,1}))$  ( $C_{j,2} = \text{cl}(M_{j,2} - N(\partial_+ M_{j,2}, M_{j,2}))$  respectively), and  $C_{j,2} = N(\partial_+ M_{j,1}, M_{j,1}) \cup M_{j,2}$  ( $C_{j,1} = N(\partial_+ M_{j,2}, M_{j,2}) \cup M_{j,1}$  respectively).

**Lemma 4.4** *Suppose that each component of  $\hat{P}(\Delta)$  is not a 2-sphere. Then each  $C_{j,i}$  is a compression body such that, for each  $j$ , we have  $\partial_+ C_{j,2} = \partial_+ C_{j,1} = C_{j,1} \cap C_{j,2}$ , ie,  $C_{j,1} \cup_{P_j} C_{j,2}$  is a Heegaard splitting of  $M_j$ .*

**Proof** Since the argument is symmetric, we may suppose that  $M_j$  satisfies conclusion 1 of Lemma 4.3. Since  $M_{j,1}$  is a compression body, it is clear that  $C_{j,1}$  is a compression body. Let  $\mathcal{D}_1 = \text{Fr}_{C_2} M_{j,2}$ . There is a union of mutually disjoint meridian disks, say  $\mathcal{D}_2$ , of  $M_{j,2}$  such that  $\mathcal{D}_2 \cap \mathcal{D}_1 = \emptyset$ , and each component of the manifold obtained from  $M_{j,2}$  by cutting along  $\mathcal{D}_2$  is homeomorphic to either a 3-ball or  $G \times [0, 1]$ , where  $G$  is a component of  $\partial_- M_{j,2}$  with  $G \times \{0\}$  corresponding to  $G$ . Hence  $C_{j,2} (= N(\partial_+ M_{j,1}, M_{j,1}) \cup M_{j,2})$  is homeomorphic to a manifold obtained from  $N(\partial_+ M_{j,1}, M_{j,1}) (\cong \partial_+ M_{j,1} \times [0, 1])$  by attaching 2-handles along the simple closed curves corresponding to  $\partial(\mathcal{D}_1 \cup \mathcal{D}_2)$  in  $\partial_+ M_{j,1} \times \{1\}$ , and capping off some of the resulting 2-sphere boundary components. By the definition of compression body (Section 2.A), this implies that  $C_{j,2}$  is a compression body, unless there exists a 2-sphere component  $S$  of  $\partial C_{j,2}$ , which is disjoint from  $N(\partial_+ M_{j,1}, M_{j,1}) \cap C_{j,1} (= \partial_+ M_{j,1} \times \{0\})$ . However such  $S$  must be a component of  $\hat{P}(\Delta)$ , a contradiction. □

Let  $M$ ,  $C_1 \cup_P C_2$ ,  $\Delta$ ,  $M_j$ ,  $M_{j,i}$ , and  $C_{j,1} \cup_{P_j} C_{j,2}$  be as above.

**Lemma 4.5** *Suppose that each component of  $\hat{P}(\Delta)$  is not a 2–sphere. If  $\partial M$  is incompressible in  $M$ , then each compression body  $C_{j,i}$  is not trivial.*

**Proof** Suppose that some compression body is trivial. By changing subscripts if necessary, we may suppose that  $C_{1,1}$  is trivial. Then we claim that  $M_{1,2} \cap P \subset \text{Int}(M_{1,1} \cap P)$ , ie, we have conclusion 1 of Lemma 4.3. In fact, if we have conclusion 2 of Lemma 4.3, then  $C_{1,1} = N(\partial_+ M_{1,2}, M_{1,2}) \cup M_{1,1}$ . However this expression obviously implies  $C_{1,1}$  is not trivial, a contradiction. Hence  $C_{1,1} = \text{cl}(M_{1,1} - N(\partial_+ M_{1,1}, M_{1,1}))$ , and this implies that  $M_{1,1}$  is a trivial compression body such that  $\partial_- M_{1,1}$  is a component of  $\partial M$ . Let  $D$  be any component of  $\text{Fr}_{C_2} M_{1,2}$ . Then by extending  $D$  vertically to  $M_{1,1}$ , we obtain a disk  $\tilde{D}$  properly embedded in  $M$ . Since each component of  $\hat{P}(\Delta)$  is not a 2–sphere,  $\partial \tilde{D}$  is not contractible in  $\partial M$ . Hence  $\tilde{D}$  is a compressing disk of  $\partial M$ , a contradiction.  $\square$

**Lemma 4.6** *If some  $C_{j,1} \cup_{P_j} C_{j,2}$  is reducible, then  $C_1 \cup_P C_2$  is reducible.*

**Proof** We prove this by using an argument of C.Frohman [5, Lemma 1.1]. If  $M$  is reducible, then by [2, Lemma 1.1], we see that any Heegaard splitting of  $M$  is reducible. Hence we may suppose that  $M$  is irreducible. If a component of  $\hat{P}(\Delta)$  is a 2–sphere, then  $C_1 \cup_P C_2$  is reducible (Lemma 4.1). Hence we may suppose that each component of  $\hat{P}(\Delta)$  is not a 2–sphere (hence,  $C_{j,1} \cup_{P_j} C_{j,2}$  is a Heegaard splitting of  $M_j$ ). Since the argument is symmetric, we may suppose that the pair  $M_{j,1}$ ,  $M_{j,2}$  satisfies conclusion 1 of Lemma 4.3. By [2, Lemma 1.1], there exists an incompressible 2–sphere  $S$  in  $M_j$  such that  $S$  intersects  $P_j$  in a circle. Let  $D_1 = S \cap C_{j,1}$ . Note that  $D_1$  is a meridian disk of  $C_{j,1}$ . Since  $M$  is irreducible,  $S$  bounds a 3–ball  $B^3$  in  $M_j$ . Let  $C'_{j,1}$  be the closure of the component of  $C_{j,1} - D_1$  such that  $C'_{j,1} \subset B^3$ . Since  $\partial M \cap B^3 = \emptyset$ , we see that  $C'_{j,1}$  is a handlebody, ie,  $\partial_- C'_{j,1} = \emptyset$ . Let  $X$  be a spine of  $C'_{j,1}$ , and  $M_X = \text{cl}(M - N(X, C_1))$ . It is clear that  $S$  is an incompressible 2–sphere in  $M_X$ , and  $P$  is a Heegaard surface of  $M_X$ . Hence, by [2, Lemma 1.1], there exists an incompressible 2–sphere  $S_X$  in  $M_X$  such that  $S_X$  intersects  $P$  in a circle. It is obvious that the 2–sphere  $S_X$  gives a reducibility of  $C_1 \cup_P C_2$ .  $\square$

## 5 Reducing genus $g$ , $n$ –bridge positions

Let  $K$  be a knot in a closed, orientable 3–manifold  $M$ . Let  $V_1 \cup V_2$  be a Heegaard splitting of  $M$ , which gives a genus  $g$ ,  $n$ –bridge position of  $K$  with

$n \geq 1$ . Let  $a$  be a component of  $K \cap V_i$  ( $i = 1$  or  $2$ , say  $2$ ). Let  $V'_1 = V_1 \cup N(a, V_2)$ , and  $V'_2 = \text{cl}(V_2 - N(a, V_2))$ . By the definition of genus  $g$ ,  $n$ -bridge positions, it is easy to see that  $V'_1 \cup V'_2$  gives a genus  $(g+1)$ ,  $(n-1)$ -bridge position of  $K$ . We say that the Heegaard splitting  $V'_1 \cup V'_2$  is obtained from  $V_1 \cup V_2$  by a *tubing* (along  $a$ ). See Figure 4.

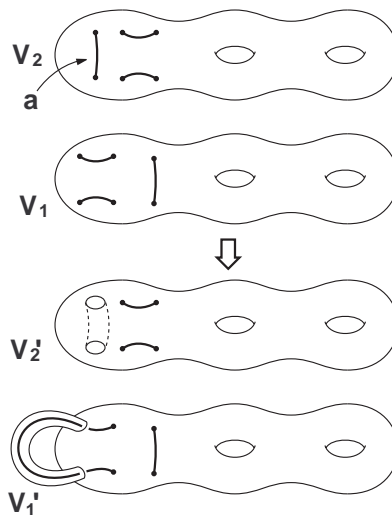


Figure 4

We say that a knot  $K$  in  $M$  is a *core knot* if there is a genus one Heegaard splitting  $V \cup W$  of  $M$  such that  $K$  is a core curve of the solid torus  $V$ , ie,  $K$  admits a genus one, 0-bridge position. Note that if  $M$  is a 3-sphere, then  $K$  is a core knot if and only if  $K$  is a trivial knot. We say that  $K$  is *small* if the exterior  $E(K)$  of  $K$  does not contain a closed essential surface. We say that a surface  $F$  properly embedded in  $E(K)$  is *meridional* if  $\partial F$  is a union of non-empty meridian loops. We note that [3, Theorem 2.0.3] implies that if  $M$  is a 3-sphere and  $K$  is small, then  $E(K)$  does not contain a meridional essential surface.

**Proposition 5.1** *Let  $K$  be a knot in a closed orientable 3-manifold  $M$  with the following properties.*

- (1)  $M$  is  $K$ -irreducible.
- (2) There exists a 2-fold branched covering space of  $M$  with branch set  $K$ .
- (3)  $K$  is not a core knot.
- (4)  $K$  is small and there does not exist a meridional essential surface in  $E(K)$ .

Let  $C_1 \cup_P C_2$  be a Heegaard splitting of  $M$ , which gives a genus  $g$ ,  $n$ -bridge position of  $K$ . Suppose that  $C_1 \cup_P C_2$  is weakly  $K$ -reducible. Then we have either one of the following.

- (1) There exists a weakly  $K$ -reducing pair of disks  $E_1, E_2$  in  $C_1, C_2$  respectively such that  $E_1 \cap K = \emptyset$ , and  $E_2 \cap K = \emptyset$ .
- (2) There exists a Heegaard splitting  $H_1 \cup H_2$  of  $M$ , which gives a genus  $(g - 1)$   $(n + 1)$ -bridge position of  $K$  such that  $C_1 \cup C_2$  is obtained from  $H_1 \cup H_2$  by a tubing.

**Remark 5.2** Note that, in Proposition 5.1, if  $g = 0$ , then we always have conclusion 1.

**Proof** Let  $D_1, D_2$  be a pair of  $K$ -essential disks in  $C_1, C_2$  respectively, which gives a weak  $K$ -reducibility of  $C_1 \cup C_2$ . If  $D_1 \cap K = \emptyset$  and  $D_2 \cap K = \emptyset$ , then we have conclusion 1. Hence in the rest of the proof, we may suppose that  $D_1 \cap K \neq \emptyset$ . We have the following two cases.

**Case 1**  $D_2 \cap K = \emptyset$ .

In this case, we first show the following.

**Claim 1** If  $D_1$  is separating in  $C_1$ , then we have conclusion 1.

**Proof** Let  $C'_1, C''_1$  be the closures of the components of  $C_1 - D_1$  such that  $\partial D_2 \subset \partial C''_1$ . Then  $C'_1$  is a  $K$ -handlebody which is not a  $K$ -ball. Hence there exists a  $K$ -essential disk  $D'_1$  in  $C'_1$  such that  $D'_1 \cap K = \emptyset$ , and  $D'_1 \cap D_1 = \emptyset$  (hence,  $D'_1$  is properly embedded in  $C_1$ ). See Figure 5. It is clear that  $D'_1$  is a  $K$ -meridian disk of  $C_1$ . Hence, by regarding,  $E_1 = D'_1, E_2 = D_2$ , we have conclusion 1.  $\square$

By Claim 1, we may suppose that  $D_1$  is non-separating in  $C_1$ . Let  $P'$  be the surface obtained from  $P$  by  $K$ -compressing along  $D_1$ , and  $\hat{P}' = P' \cap E(K)$ . We note that  $P'$  separates  $M$  into two components, say  $C'_1$  and  $C'_2$ , where  $C'_1$  is obtained from  $C_1$  by cutting along  $D_1$ . Let  $\hat{C}'_i = C'_i \cap E(K)$  ( $i = 1, 2$ ). Hence, by Section 3.B, we see that  $\hat{C}'_1$  is a compression body with  $\partial_+ \hat{C}'_1 = \hat{P}'$ . Let  $\mathcal{D}$  be a union of maximal mutually disjoint, non parallel compressing disks for  $\hat{P}'$  such that  $\mathcal{D} \subset \hat{C}'_2$ . Note that since  $D_2$  is a compressing disk for  $\hat{P}'$  such that  $D_2 \subset \hat{C}'_2$ , there actually exists such  $\mathcal{D}$ . Let  $\hat{C}''_2 = N(\hat{P}', \hat{C}'_2) \cup N(\mathcal{D}, \hat{C}'_2)$ . Note that  $\hat{C}''_2$  is homeomorphic to  $C_2 \cap E(K) = \text{cl}(C_2 - N(K))$ , hence,  $\hat{C}''_2$  is

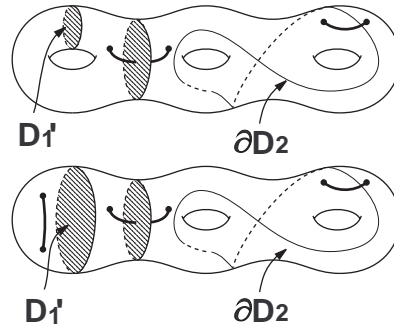


Figure 5

irreducible (Section 3.A). Hence the 2-sphere components  $\mathcal{S}$  (possibly  $\mathcal{S} = \emptyset$ ) of  $\partial\hat{C}_2'' - \hat{P}'$  bounds mutually disjoint 3-balls in  $\hat{C}_2'$ . Let  $C_2^* = \hat{C}_2'' \cup$  (the 3-balls). Then  $C_2^*$  is a compression body such that  $\partial_+ C_2^* = \hat{P}'$ . Let  $P^* = \partial_- C_2^*$ .

**Claim 2** If  $P^*$  is compressible in  $E(K)$ , then we have conclusion 1.

**Proof** Suppose that there exists a compressing disk  $E$  of  $P^*$  in  $E(K)$ . Let  $M^* = \hat{C}_1' \cup C_2^*$ . By the maximality of  $\mathcal{D}$ , we see that  $E$  is contained in  $M^*$ . Note that  $\hat{C}_1' \cup_{\hat{P}'} C_2^*$  is a Heegaard splitting of  $M^*$ , and  $\partial E \subset \partial_- C_2^*$ . Hence, by [2, Lemma 1.1], we see that  $\hat{C}_1' \cup_{\hat{P}'} C_2^*$  is weakly reducible, and this implies conclusion 1.  $\square$

By Claim 2, we may suppose that  $P^*$  is incompressible in  $E(K)$ . Note that each component of  $\partial P^*$  is a meridian loop of  $K$ . Since  $K$  is small and there does not exist a meridional essential surface in  $E(K)$ , we see that each component of  $P^*$  is a boundary parallel annulus properly embedded in  $E(K)$ . Recall that  $\mathcal{S}$  is the union of the 2-sphere components of  $\partial\hat{C}_2'' - \hat{P}'$ . Note that we can assign labels  $C_1$  and  $C_2$  to the components of  $E(K) - (P^* \cup \mathcal{S})$  alternately so that the  $C_2$  region are contained in  $\hat{C}_2'$ , and that  $\hat{P}'$  is recovered from  $P^* \cup \mathcal{S}$  by adding tubes along mutually disjoint arcs in  $C_1$ -regions. Recall that  $\hat{P}'$  is connected. Since each component of  $P^* \cup \mathcal{S}$  is separating in  $E(K)$ , this shows that exactly one component of  $E(K) - (P^* \cup \mathcal{S})$  is a  $C_1$ -region. Let  $\tilde{P}^*$  be a surface in  $M$  obtained from  $P^*$  by capping off the boundary components by mutually disjoint  $K$ -disks in  $N(K)$  (hence, via isotopy,  $P'$  is recovered from  $\tilde{P}^* \cup \mathcal{S}$  by adding the tubes used for recovering  $\hat{P}'$  from  $P^* \cup \mathcal{S}$ ). Then each component of  $\tilde{P}^* \cup \mathcal{S}$  is a  $K$ -sphere. Since  $M$  is  $K$ -irreducible, the components of  $\tilde{P}^* \cup \mathcal{S}$  bounds  $K$ -balls, say  $B_1, \dots, B_m$ , in  $M$ .

**Claim 3** The  $K$ -balls  $B_1, \dots, B_m$  are mutually disjoint.

**Proof** Suppose not. By exchanging the subscript if necessary, we may suppose that  $B_2 \subset B_1$ . Since there exists exactly one  $C_1$ -region, this implies that the  $K$ -balls  $B_2, \dots, B_m$  are included in  $B_1$  in a non-nested configuration. Hence  $P'$  is contained in the  $K$ -ball  $B_1$ . See Figure 6. Note that  $P$  is recovered from  $P'$  by adding a tube along the component of  $K - P'$ , which intersects  $D_1$ . Hence we see that  $P$  is contained in a regular neighborhood of  $K$ , say  $N_K$ .

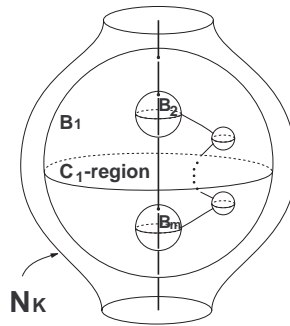


Figure 6

Note that  $\text{cl}(M - N_K)$  is contained in  $C_2$ . Since  $C_2$  is a  $K$ -compression body, there exists a  $K$ -compressing disk  $D_N$  for  $\partial N_K$  in  $C_2$ . Suppose that  $D_N \subset N_K$ . Since  $N_K$  is a regular neighborhood of  $K$ , we see that  $\text{cl}(N_K - N(D_N, N_K))$  is a  $K$ -ball. Since  $C_2$  is  $K$ -irreducible,  $\text{cl}(M - N_K) \cup N(D_N, N_K)$  is also a  $K$ -ball. These show that  $M$  is the 3-sphere, and  $K$  is a trivial knot, contradicting the condition 3 of the assumption of Proposition 5.1. Suppose that  $D_N \subset \text{cl}(M - N_K)$ . Since  $M$  is  $K$ -irreducible, we see that  $\text{cl}(M - N_K)$  is irreducible. This shows that we obtain a 3-ball by cutting  $\text{cl}(M - N_K)$  along  $D_N$ . This shows that  $\text{cl}(M - N_K)$  is a solid torus. Hence  $N_K \cup \text{cl}(M - N_K)$  is a genus one Heegaard splitting of  $M$ . Hence  $K$  is a core knot, contradicting the condition 3 of the assumption of Proposition 5.1.

This completes the proof of Claim 3.  $\square$

Recall that  $P'$  is the surface obtained from  $P$  by  $K$ -compressing along  $D_1$ , and  $C'_1, C'_2$  the closures of the components of  $M - P'$ , where  $C'_1$  is obtained from  $C_1$  by cutting along  $D_1$ . By Proposition 2.16 and (3) of Remark 2.1,  $C'_1$  is a  $K$ -handlebody. By Claim 3, we see that the  $C_1$ -region is  $\text{cl}(M - (\cup_{i=1}^m B_i)) \cap E(K)$ . Hence  $P'$  is recovered from  $\partial B_1 \cup \dots \cup \partial B_m$  by adding tubes along arcs properly embedded in  $\text{cl}(M - (\cup_{i=1}^m B_i))$ . Hence, we see that  $C'_2$  is obtained from the



$K$ -balls  $B_1, \dots, B_m$  by adding 1-handles disjoint from  $K$ . Hence  $C'_2$  is also a  $K$ -handlebody. These show that  $P'$  is a Heegaard surface for  $(M, K)$ . It is clear that  $C'_1 \cup C'_2$  gives a genus  $(g - 1)$ ,  $(n + 1)$  bridge position of  $K$ , and  $C_1 \cup C_2$  is obtained from  $C'_1 \cup C'_2$  by a tubing along a component of  $K \cap (\cup B_i)$ .

Hence, by regarding  $H_1 = C'_1$ ,  $H_2 = C'_2$ , we have conclusion 2 of Proposition 5.1.

**Case 2**  $D_2 \cap K \neq \emptyset$ .

In this case, we first show the following.

**Claim 1** If  $\partial D_i$  ( $i = 1$  or  $2$ ) is separating in  $P$ , then we have conclusion of Proposition 5.1.

**Proof** Since the argument is symmetric, we may suppose that  $\partial D_2$  is separating in  $P$ . This implies that  $D_2$  is separating in  $C_2$ . Let  $C'_2, C''_2$  be the closures of the components of  $C_2 - D_2$  such that  $\partial D_1 \subset C''_2$ . Then  $C'_2$  is a  $K$ -handlebody which is not a  $K$ -ball. Hence there exists a  $K$ -essential disk  $D'_2$  in  $C'_2$  such that  $D'_2 \cap K = \emptyset$ , and  $D'_2 \cap D_2 = \emptyset$  (hence,  $D'_2$  is properly embedded in  $C_2$ ). See Figure 5. It is clear that  $D'_2$  is a  $K$ -meridian disk of  $C_2$ . Hence by applying the arguments of Case 1 to the pair  $D_1, D'_2$ , we have conclusion of Proposition 5.1.  $\square$

Let  $P'$  be the surface obtained from  $P$  by  $K$ -compressing along  $D_1 \cup D_2$ , and  $\hat{P}' = P' \cap E(K)$ . Let  $C'_1, C'_2$  be the closures of the components of  $M - P'$  such that  $C'_1$  is obtained from  $C_1$  by cutting along  $D_1$  and attaching  $N(D_2, C_2)$ , and  $C'_2$  is obtained from  $C_2$  by cutting along  $D_2$  and attaching  $N(D_1, C_1)$ . Then let  $\hat{C}'_i = C'_i \cap E(K)$  ( $i = 1, 2$ ).

**Claim 2** If  $\hat{P}'$  is compressible in  $E(K)$ , then we have conclusion of Proposition 5.1.

**Proof** Suppose that there is a compressing disk  $D$  for  $\hat{P}'$  in  $E(K)$ . Since the argument is symmetric, we may suppose that  $D \subset \hat{C}'_2$ . We may regard that  $D$  is a compressing disk for  $P'$ . Since  $P$  is recovered from  $P'$  by adding two tubes along a component of  $K \cap C'_1$  and a component of  $K \cap C'_2$ , we may suppose that  $D \cap P = \partial D$ . Hence  $D$  is a  $K$ -meridian disk of  $C_2$  such that  $D \cap K = \emptyset$ . Hence, by applying the arguments of Case 1 to the pair  $D_1, D$ , we have the conclusion of Proposition 5.1.  $\square$

By Claims 1 and 2, we see that, for the proof of Proposition 5.1, it is enough to show that either (1)  $\partial D_i$  ( $i = 1$  or  $2$ ) is separating in  $P$ , or (2)  $\hat{P}'$  is compressible in  $E(K)$ . Suppose that  $\partial D_i$  ( $i = 1, 2$ ) is non-separating in  $P$ , and that  $\hat{P}'$  is incompressible in  $E(K)$ . Then, by the argument preceding Claim 3 of Case 1, we see that each component of  $P'$  is a  $K$ -sphere, and  $P$  is recovered from  $P'$  by adding tubes along two arcs  $a_1, a_2$  such that  $a_i$  is a component of  $K \cap C'_i$  ( $i = 1, 2$ ), and that  $a_1 \cap a_2 = \emptyset$ . Note that  $P$  is connected. Since  $\partial D_1, \partial D_2$  are non-separating in  $P$ , we see that  $P'$  consists of one  $K$ -sphere, or two  $K$ -spheres, and this shows that  $K \cap C'_i$  consists of one arc, or two arcs. But since  $K$  is a knot, we have  $a_1 \cap a_2 \neq \emptyset$  in either case, a contradiction. Hence we have the conclusion of Proposition 5.1 in Case 2.

This completes the proof of Proposition 5.1.  $\square$

## 6 Heegaard splittings of $(S^3, \text{two bridge knot})$

In this section, we prove the following.

**Proposition 6.1** *Let  $K$  be a non-trivial two bridge knot, and  $X \cup_Q Y$  a Heegaard splitting of  $S^3$ , which gives a genus  $g$ ,  $n$ -bridge position of  $K$ . Suppose that  $(g, n) \neq (0, 2)$ . Then  $X \cup_Q Y$  is weakly  $K$ -reducible.*

**Proposition 6.2** *Let  $K$  be a non-trivial two bridge knot. Then, for each  $g \geq 3$ , every genus  $g$  Heegaard splitting of the exterior  $E(K)$  of  $K$  is weakly reducible.*

### 6.A Comparing $X \cup_Q Y$ with a two bridge position

Let  $A \cup_P B$  be a genus 0 Heegaard splitting of  $S^3$ , which gives a 2-bridge position of  $K$ . Then, by [14, Corollary 6.22] (if  $n \geq 1$ ) or by [13, Corollary 3.2] (if  $n = 0$ ), we have the following.

**Proposition 6.3** *Let  $X \cup_Q Y$  be a Heegaard splitting of  $S^3$ , which gives a genus  $g$ ,  $n$ -bridge position of  $K$ . If  $X \cup_Q Y$  is strongly  $K$ -irreducible, then  $Q$  is  $K$ -isotopic to a position such that  $P \cap Q$  consists of non-empty collection of transverse simple closed curves which are  $K$ -essential in both  $P$  and  $Q$ .*

In this subsection, we prove the following proposition.

**Proposition 6.4** *Let  $X \cup_Q Y$  be a Heegaard splitting of  $S^3$ , which gives a genus  $g$ ,  $n$ -bridge position of  $K$  with  $(g, n) \neq (0, 2)$ . Suppose that  $P \cap Q$  consists of non-empty collection of transverse simple closed curves which are  $K$ -essential in both  $P$  and  $Q$ . Then  $X \cup_Q Y$  is weakly  $K$ -reducible.*

We note that Proposition 6.1 is a consequence of Propositions 6.3 and 6.4.

**Proof of Proposition 6.1 from Propositions 6.3 and 6.4** Let  $X \cup_Q Y$  be a Heegaard splitting of  $S^3$ , which gives a genus  $g$ ,  $n$ -bridge position of  $K$  with  $(g, n) \neq (0, 2)$ . Suppose, for a contradiction, that  $X \cup_Q Y$  is strongly  $K$ -irreducible. Then, by Propositions 6.3, we may suppose that  $P \cap Q$  consists of non-empty collection of transverse simple closed curves which are  $K$ -essential in both  $P$  and  $Q$ . By Propositions 6.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible, a contradiction.  $\square$

**Proof of Proposition 6.4** First of all, we would like to remark that the proof given below is just an orbifold version of the proof of [17, Corollary 6.4]. We suppose that  $|P \cap Q|$  is minimal among all surfaces  $P$  such that  $P$  gives a two bridge position of  $K$ , and that  $P \cap Q$  consists of non-empty collection of simple closed curves which are  $K$ -essential in both  $P$  and  $Q$ . Note that the closure of each component of  $P - Q$  is either an annulus which is disjoint from  $K$ , or a disk intersecting  $K$  in two points. We divide the proof into several cases.

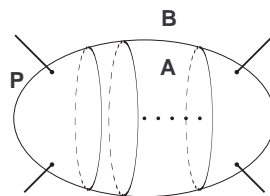


Figure 7

**Case 1** Each component of  $P \cap X$  is not  $K$ -boundary parallel in  $X$ , and each component of  $P \cap Y$  is not  $K$ -boundary parallel in  $Y$ .

Case 1 is divided into the following subcases.

**Case 1.1**  $P \cap X$  contains a component which is  $K$ -compressible in  $X$ , and  $P \cap Y$  contains a component which is  $K$ -compressible in  $Y$

In this case, by  $K$ -compressing the components in  $X$  and  $Y$ , we obtain  $K$ -meridian disks  $D_X, D_Y$  in  $X, Y$  respectively. By applying a slight  $K$ -isotopy if necessary, we may suppose that  $D_X \cap D_Y = \emptyset$ , and this shows that  $X \cup_Q Y$  is weakly  $K$ -reducible.

**Case 1.2** Either  $P \cap X$  or  $P \cap Y$ , say  $P \cap X$ , contains a component which is  $K$ -compressible in  $X$ , and each component of  $P \cap Y$  is  $K$ -incompressible in  $Y$ .

Let  $D_X$  be a  $K$ -meridian disk obtained by  $K$ -compressing the component of  $P \cap X$ . Note that  $P \cap Y$  is  $K$ -boundary compressible in  $Y$  (Corollary 2.18). Let  $D_Y$  be a disk in  $Y$  obtained by  $K$ -boundary compressing  $P \cap Y$ . By the minimality of  $|P \cap Q|$ , we see that  $D_Y$  is a  $K$ -meridian disk in  $Y$ . Since  $\partial D_X$  is a component of  $P \cap Q$ , we may suppose that  $D_X \cap D_Y = \emptyset$  by applying a slight  $K$ -isotopy if necessary. This shows that  $X \cup_Q Y$  is weakly  $K$ -reducible.

**Case 1.3** Each component of  $P \cap X$  is  $K$ -incompressible in  $X$ , and each component of  $P \cap Y$  is  $K$ -incompressible in  $Y$ .

Let  $D_X$  ( $D_Y$  respectively) be a  $K$ -meridian disk obtained by  $K$ -boundary compressing  $P \cap X$  ( $P \cap Y$  respectively). By applying slight isotopies, we may suppose that  $\partial D_X \cap P = \emptyset, \partial D_Y \cap P = \emptyset$  (hence,  $\partial D_X \subset A$  or  $B, \partial D_Y \subset A$  or  $B$ ). If one of  $\partial D_X$  or  $\partial D_Y$  is contained in  $A$ , and the other in  $B$ , then  $D_X \cap D_Y = \emptyset$ , and this shows that  $X \cup_Q Y$  is weakly  $K$ -reducible. Suppose that  $\partial D_X \cup \partial D_Y$  is contained in  $A$  or  $B$ , say  $A$ . Let  $D_B$  be a  $K$ -meridian disk in  $B$  (ie,  $D_B$  is a disk properly embedded in  $B$  such that  $D_B \cap K = \emptyset$ , and  $D_B$  separates the components of  $K \cap B$ ). Note that since each component of  $P \cap X, P \cap Y$  is  $K$ -incompressible,  $D_B \cap Q \neq \emptyset$ . We take  $D_B$  so that  $|D_B \cap Q|$  is minimal among all  $K$ -essential disks  $D'$  in  $B$  such that each component of  $D' \cap (P \cap X)$  ( $D' \cap (P \cap Y)$  respectively) is a  $K$ -essential arc properly embedded in  $P \cap X$  ( $P \cap Y$  respectively). Suppose that  $D_B \cap Q$  contains a simple closed curve component. Let  $D^*(\subset D_B)$  be an innermost disk. Since the argument is symmetric, we may suppose that  $D^* \subset X$ . By the minimality of  $|D_B \cap Q|$ , we see that  $D^*$  is a  $K$ -meridian disk in  $X$ . Since  $D^* \subset B, \partial D^* \cap \partial D_Y = \emptyset$ . Hence the pair  $D^*, D_Y$  gives a weak  $K$ -reducibility of  $X \cup_Q Y$ . Suppose that each component of  $D_B \cap Q$  is an arc. Let  $\Delta(\subset D_B)$  be an outermost disk. Since the argument is symmetric, we may suppose that  $\Delta \subset X$ . Recall that  $\Delta \cap (P \cap X)$  is a  $K$ -essential arc in  $P \cap X$ . By the minimality of  $|D_B \cap Q|$ , we see that at least one component, say  $D^{**}$ , of the surface obtained from  $P \cap X$  by  $K$ -boundary compressing along  $\Delta$  is a  $K$ -meridian disk in  $X$ . Since  $\Delta \subset B,$

$\partial D^{**} \cap \partial D_Y = \emptyset$ . Hence the pair  $D^{**}, D_Y$  gives a weak  $K$ -reducibility of  $X \cup_Q Y$ .

**Case 2** A component of  $P \cap X$  or  $P \cap Y$ , say  $P \cap Y$ , is  $K$ -boundary parallel in  $Y$ .

By the minimality of  $|P \cap Q|$ , we have either  $|P \cap Q| = 1$  (and  $P \cap Y$  ( $P \cap X$  respectively) is a disk intersecting  $K$  in two points) or,  $|P \cap Q| = 2$  (and  $P \cap Y$  is an annulus disjoint from  $K$ ).

**Case 2a**  $|P \cap Q| = 1$ .

Let  $P_X = P \cap X$  and  $P_Y = P \cap Y$ . Let  $E$  be the closure of the component of  $Q - P$  such that  $E$  and  $P_Y$  are  $K$ -parallel in  $Y$ . Since the argument is symmetric, we may suppose that  $E \subset A$ . We have the following subcases.

**Case 2a.1**  $P_X$  is  $K$ -boundary parallel in  $X$

Since  $(g, n) \neq (0, 2)$ ,  $P_X$  is parallel to  $E$  in  $A$ , and cannot be parallel to  $\text{cl}(Q - E)$ .

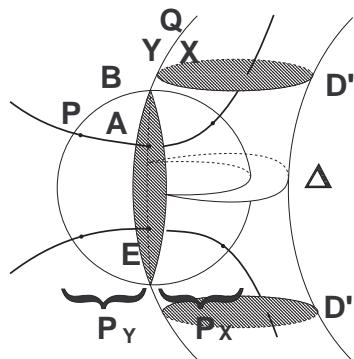


Figure 8

Let  $D_B$  be a  $K$ -meridian disk in  $B$ . Since  $K$  is not a trivial knot,  $\partial D_B$  and  $\partial E$  are not isotopic in  $P - K$ . Hence  $D_B \cap Q \neq \emptyset$ . We suppose that  $|D_B \cap Q|$  is minimal among all  $K$ -meridian disks  $D'$  in  $B$  such that each component of  $D' \cap P_X$  ( $D' \cap P_Y$  respectively) is a  $K$ -essential arc in  $P_X$  ( $P_Y$  respectively). Suppose that  $D_B \cap Q$  contains a simple closed curve. Let  $D^* (\subset D_B)$  be an innermost disk. Since the argument is symmetric, we may suppose that  $D^* \subset X$ . By the minimality of  $|D_B \cap Q|$ , we see that  $D^*$  is a

$K$ -meridian disk in  $X$ . Then by pushing  $D_Y$  into  $X$  along the parallelism through  $E$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D^* = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible. Suppose that each component of  $D_B \cap Q$  is an arc. Let  $\Delta (\subset D_B)$  be an outermost disk. Since the argument is symmetric, we may suppose that  $\Delta \subset X$ . See Figure 8.

**Claim** At least one component of the disks obtained from  $P_X$  by  $K$ -boundary compressing along  $\Delta$  is a  $K$ -meridian disk.

**Proof** Let  $D', D''$  be the disks obtained from  $P_X$  by  $K$ -boundary compressing along  $\Delta$ . Suppose that  $D'$  is  $K$ -boundary parallel, ie, there exists a  $K$ -disk  $D_Q$  in  $Q$  such that  $\partial D_Q = \partial D'$ . Note that since  $D' \cup D''$  is obtained from  $P_X$  by  $K$ -boundary compressing along  $\Delta$ , there is an annulus  $A_Q$  in  $Q$  such that  $\partial A_Q = \partial D' \cup \partial D''$ , and that  $A_Q \cap K = E \cap K$ : two points. Note also that  $D_Q \cap K$  consists of one point. Hence  $A_Q$  is not contained in  $D_Q$ , and this implies that  $A_Q \cap D_Q = \partial D'$ . Then  $A_Q \cup D_Q$  is a disk intersecting  $K$  in three points, whose boundary is  $\partial D''$ . Since  $(g, n) \neq (0, 2)$ ,  $\text{cl}(Q - (A_Q \cup D_Q))$  is not a  $K$ -disk. Hence  $D''$  is a  $K$ -meridian disk in  $X$ .  $\square$

Let  $D''$  be a  $K$ -meridian disk in  $X$  obtained as in Claim. By applying a slight isotopy, we may suppose that  $P \cap D'' = \emptyset$ . Then by pushing  $P_Y$  into  $X$  along the parallelism through  $E$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D'' = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible.

**Case 2a.2**  $P_X$  is not  $K$ -boundary parallel in  $X$ , and  $P_X$  is  $K$ -incompressible in  $X$ , ie,  $P_X$  is  $K$ -essential in  $X$ .

Since  $P_X$  is  $K$ -incompressible, there is a  $K$ -boundary compressing disk  $\Delta$  for  $P_X$  in  $X$ .

**Claim**  $\Delta \subset B$ .

**Proof** Suppose that  $\Delta \subset A$ . Note that  $K \cap E$  consists of two points in  $\text{Int}E$ , and  $\Delta \cap E$  is an arc properly embedded in  $E$ , which separates the points. Then, by  $K$ -boundary compressing  $P_X$  along  $\Delta$ , we obtain two  $K$ -disks. Since  $X$  is  $K$ -irreducible, these  $K$ -disks are  $K$ -boundary parallel in  $X$ . This shows that  $P_X$  is  $K$ -boundary parallel in  $X$ , contradicting the condition of Case 2a.2.  $\square$

Then, by using the argument of the proof of Claim of Case 2a.1, we see that at least one component, say  $D''$ , of the  $K$ -disks obtained from  $P_X$  by  $K$ -boundary compressing along  $\Delta$  is a  $K$ -meridian disk in  $X$ . By applying a slight isotopy, we may suppose that  $D'' \cap P = \emptyset$ . By Claim, we see that  $\partial D'' \subset B$ . Then by pushing  $P_Y$  into  $X$  along the parallelism through  $E$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D'' = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible.

**Case 2a.3**  $P_X$  is not  $K$ -boundary parallel in  $X$ , and  $P_X$  is  $K$ -compressible in  $X$ .

Let  $D$  be the  $K$ -compressing disk for  $P_X$ . Since there does not exist a 2-sphere ( $\subset S^3$ ) intersecting  $K$  in three points,  $D \cap K = \emptyset$ . Let  $D^*$  be the disk component of a surface obtained from  $P_X$  by  $K$ -compressing along  $D$ . Since  $(g, n) \neq (0, 2)$ , we see that  $D^*$  is a  $K$ -meridian disk of  $X$ . By applying a slight isotopy, we may suppose that  $D^* \cap P = \emptyset$ . Suppose that  $D^* \subset B$ . Then by pushing  $D_Y$  into  $X$  along the parallelism through  $E$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D^* = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible. Hence, in the rest of this subcase, we suppose that  $D^* \subset A$  (Figure 9). Let  $D_B$  be a  $K$ -meridian disk in  $B$ . Since  $K$  is not a trivial two component link,  $\partial D$  and  $\partial D_B$  are not isotopic in  $P - K$ . Hence  $D_B \cap Q \neq \emptyset$ . We suppose that  $|D_B \cap Q|$  is minimal among all  $K$ -meridian disks  $D'$  in  $B$  such that each component of  $D' \cap P_X$  ( $D' \cap P_Y$  respectively) is a  $K$ -essential arc in  $P_X$  ( $P_Y$  respectively).

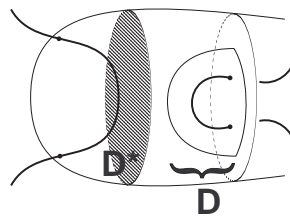


Figure 9

Suppose that  $D_B \cap Q$  contains a simple closed curve. Let  $D^{**}(\subset D_B)$  be an innermost disk. By the minimality of  $|D_B \cap Q|$ , we see that  $\partial D^{**}$  is  $K$ -essential in  $Q$ . Note that  $\partial D^{**} \subset B$ . If  $D^{**} \subset Y$ , then the pair  $D^{**}, D^*$  gives a weak  $K$ -reducibility of  $X \cup_Q Y$ . If  $D^{**} \subset X$ , then by pushing  $P_Y$  into  $X$  along the parallelism through  $E$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D^{**} = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible.

Suppose that each component of  $D_B \cap Q$  is an arc. Let  $\Delta(\subset D_B)$  be an outermost disk. If  $\Delta \subset X$ , then by using the argument as in the proof of Case 2a.1, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible. Suppose that  $\Delta \subset Y$ . Then, by using the argument as in the proof of Claim of Case 2a.1, we can show that at least one component, say  $D''$ , of the  $K$ -disks obtained from  $P_Y$  by  $K$ -boundary compressing along  $\Delta$  is a  $K$ -meridian disk in  $Y$ . By applying slight  $K$ -isotopy, we may suppose that  $D'' \subset B$ . Hence the pair  $D^*, D''$  gives a weak  $K$ -reducibility of  $X \cup_Q Y$ .

**Case 2b**  $|P \cap Q| = 2$ .

Let  $D_1, D_2$  be the components of  $P \cap X$ , and  $A_1 = P \cap Y$ . Recall that  $A_1$  is a  $K$ -boundary parallel annulus in  $Y$  such that  $A_1 \cap K = \emptyset$ , and that  $D_1, D_2$  are not  $K$ -boundary parallel. We also note that  $\partial D_1 \cup \partial D_2$  bounds an annulus  $A'$  in  $Q$  such that  $A_1$  and  $A'$  are  $K$ -parallel in  $Y$ . Without loss of generality, we may suppose that  $A'$  is contained in the 3-ball  $A$ .

**Case 2b.1**  $D_1 \cup D_2$  is  $K$ -incompressible in  $X$ .

Since  $D_1 \cup D_2$  is  $K$ -incompressible, there is a  $K$ -boundary compressing disk  $\Delta$  for  $D_1 \cup D_2$ . Without loss of generality, we may suppose that  $\Delta \cap D_1 \neq \emptyset$ ,  $\Delta \cap D_2 = \emptyset$ . Since  $D_1$  is not  $K$ -boundary parallel, at least one component, say  $D^{**}$ , of the  $K$ -disks obtained from  $D_1$  by  $K$ -boundary compressing along  $\Delta$  is a  $K$ -meridian disk in  $X$ . By applying a slight  $K$ -isotopy, we may suppose that  $D^{**} \cap P = \emptyset$ .

**Claim**  $D^{**} \subset B$ .

**Proof** Suppose, for a contradiction, that  $D^{**} \subset A$ . Then  $\partial D^{**}$  is contained in the annulus  $A'$  bounded by  $\partial D_1 \cup \partial D_2$ . We note that  $D^{**}$  intersects  $K$  in one point. Hence  $\partial D^{**}$  is not contractible in  $Q$ . This shows that  $\partial D^{**}$  is a core curve of  $A'$ . Let  $A''$  be the annulus in  $A'$  bounded by  $\partial D^{**} \cup \partial D_1$ . Then the 2-sphere  $D_1 \cup A'' \cup D^{**}$  intersects  $K$  in three points, a contradiction.  $\square$

By Claim we see that, by pushing  $A_1$  into  $X$  along the parallelism through  $A'$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ . By the above claim, we may suppose that  $P' \cap D^{**} = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible.

**Case 2b.2**  $D_1 \cup D_2$  is  $K$ -compressible.



Let  $D$  be the  $K$ -compressing disk for  $D_1 \cup D_2$ . Without loss of generality, we may suppose that  $D \cap D_1 \neq \emptyset$ ,  $D \cap D_2 = \emptyset$ . Let  $D^*$  be a  $K$ -meridian disk of  $X$  obtained from  $D_1$  by  $K$ -compressing along  $D$ . By applying slight isotopy, we may suppose that  $D^* \cap P = \emptyset$ . Suppose that  $D^* \subset B$ . By pushing  $A_1$  into  $X$  along the parallelism through  $A'$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D^* = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible.

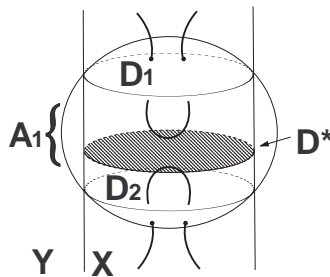


Figure 10

Suppose that  $D^* \subset A$  (Figure 10). Let  $D_B$  be a  $K$ -meridian disk in  $B$ . Since  $K$  is not a trivial two component link,  $\partial D$  and  $\partial D_B$  are not isotopic in  $P - K$ . Hence  $D_B \cap Q \neq \emptyset$ . We suppose that  $|D_B \cap Q|$  is minimal among all  $K$ -essential disks  $D'$  in  $B$  such that each component of  $D' \cap D_1$  ( $D' \cap D_2$ ,  $D' \cap A_1$  respectively) is a  $K$ -essential arc in  $D_1$  ( $D_2$ ,  $A_1$  respectively). Suppose that  $D_B \cap Q$  contains a simple closed curve component. Let  $D'$  be an innermost disk. By the minimality of  $|D_B \cap Q|$ , we see that  $\partial D'$  is  $K$ -essential in  $Q$ . Note that  $\partial D' \subset B$ . If  $D' \subset Y$ , then the pair  $D'$ ,  $D^*$  gives a weak  $K$ -reducibility of  $X \cup_Q Y$ . If  $D' \subset X$ , then by pushing  $A_1$  into  $X$  along the parallelism through  $A'$ , we can  $K$ -isotope  $P$  to  $P'$  such that  $P' \subset \text{Int}X$ , and  $P' \cap D' = \emptyset$ . Hence, by Proposition 3.4, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible. Suppose that each component of  $D_B \cap Q$  is an arc. Let  $\Delta (\subset D_B)$  be an outermost disk. If  $\Delta \subset X$ , then by using the argument as in the proof of Case 2b.1, we see that  $X \cup_Q Y$  is weakly  $K$ -reducible. Suppose that  $\Delta \subset Y$ . Let  $D^{**}$  be the disk obtained from  $A_1$  by  $K$ -boundary compressing along  $\Delta$ .

**Claim**  $D^{**}$  is a  $K$ -meridian disk of  $Y$ .

**Proof** Suppose that  $D^{**}$  is not a  $K$ -meridian disk of  $Y$ , ie,  $D^{**}$  is  $K$ -parallel to a disk, say  $D''$ , in  $\partial Y (= Q)$ . Since  $\Delta \subset B$ , we see that  $D'' \subset \text{cl}(Q - A')$ . Note that  $\text{cl}(Q - A')$  is recovered from  $D''$  by adding a band along an arc intersecting  $\Delta \cap Q$  in one point. This shows that  $\text{cl}(Q - A')$  is an annulus not

intersecting  $K$ . Hence  $Q$  is a torus, and  $X$  is a solid torus such that  $Q \cap K = \emptyset$ . However, since  $D^*$  is a meridian disk of  $X$ , this implies that  $X$  is  $K$ -reducible, contradicting Corollary 2.17.  $\square$

By Claim, we see that, by applying a slight isotopy, we may suppose that  $D^{**} \cap P = \emptyset$ , and  $D^{**} \subset B$ . Hence the pair  $D^*$ ,  $D^{**}$  gives a weak  $K$ -reducibility of  $X \cup_Q Y$ .

This completes the proof of Proposition 6.4  $\square$

## 6.B Proof of Proposition 6.2

Let  $K$  be a non-trivial two bridge knot, and  $C \cup_P V_2$  a genus  $g$  Heegaard splitting of  $E(K)$  with  $g \geq 3$ . Note that  $K$  satisfies the conditions of the assumption of Proposition 5.1. Let  $V_1$  be the handlebody in  $S^3$  such that  $\partial V_1 = P$ , and  $C \subset V_1$ . Then  $V_1 \cup V_2$  is a Heegaard splitting of  $S^3$  which gives a genus  $g$ , 0-bridge position of  $K$ . By Propositions 6.1 and 5.1, we have either one of the following.

(1.1) There exists a weakly  $K$ -reducing pair of disks  $D_1, D_2$  for  $V_1 \cup V_2$  such that  $D_1 \cap K = \emptyset$ , and  $D_2 \cap K = \emptyset$ .

(1.2) There exists a Heegaard splitting  $V_{1,1} \cup_{P_1} V_{1,2}$  of  $(S^3, K)$  which gives a genus  $(g - 1)$ , 1-bridge position of  $K$  such that  $V_1 \cup V_2$  is obtained from  $V_{1,1} \cup_{P_1} V_{1,2}$  by a tubing.

If (1.1) holds, then we immediately have the conclusion of Propositions 6.2. If (1.2) holds, then we further apply Propositions 6.1 and 5.1, and we have either one of the following.

(2.1) There exists a weakly  $K$ -reducing pair of disks  $D_1, D_2$  for  $V_{1,1} \cup_{P_1} V_{1,2}$  such that  $D_1 \cap K = \emptyset$ , and  $D_2 \cap K = \emptyset$ .

(2.2) There exists a Heegaard splitting  $V_{2,1} \cup_{P_2} V_{2,2}$  of  $(S^3, K)$  which gives a genus  $(g - 2)$ , 2-bridge position of  $K$  such that  $V_{1,1} \cup_{P_1} V_{1,2}$  is obtained from  $V_{2,1} \cup_{P_2} V_{2,2}$  by a tubing.

We claim that if (2.1) holds, then we have the conclusion of Propositions 6.2. In fact, since  $D_1 \cap K = \emptyset$ , and  $D_2 \cap K = \emptyset$ , and tubing operations are performed in a small neighborhood of  $K$ , the pair  $D_1, D_2$  survives in  $V_1 \cup V_2$  to give a

weak reducibility. If (2.2) holds, then we further apply Propositions 6.1 and 5.1, and we have either one of the following.

(3.1) There exists a weakly  $K$ -reducing pair of disks  $D_1, D_2$  for  $V_{2,1} \cup_{P_2} V_{2,2}$  such that  $D_1 \cap K = \emptyset$ , and  $D_2 \cap K = \emptyset$ .

(3.2) There exists a Heegaard splitting  $V_{3,1} \cup_{P_3} V_{3,2}$  of  $(S^3, K)$  which gives a genus  $(g - 3)$ , 3-bridge position of  $K$  such that  $V_{2,1} \cup_{P_2} V_{2,2}$  is obtained from  $V_{3,1} \cup_{P_3} V_{3,2}$  by a tubing.

Then we apply the same argument as above, and so on. Then either we have the conclusion of Propositions 6.2, or the procedures are repeated  $(g - 1)$  times to give the following.

( $g.1$ ) There exists a weakly  $K$ -reducing pair of disks  $D_1, D_2$  for  $V_{g-1,1} \cup_{P_{g-1}} V_{g-1,2}$  such that  $D_1 \cap K = \emptyset$ , and  $D_2 \cap K = \emptyset$ .

( $g.2$ ) There exists a Heegaard splitting  $V_{g,1} \cup_{P_g} V_{g,2}$  of  $(S^3, K)$  which gives a genus 0,  $g$ -bridge position of  $K$  such that  $V_{g-1,1} \cup_{P_{g-1}} V_{g-1,2}$  is obtained from  $V_{g,1} \cup_{P_g} V_{g,2}$  by a tubing.

If ( $g.1$ ) holds, then by using the arguments as above, we see that we have the conclusion of Propositions 6.2. Suppose that ( $g.2$ ) holds. Then we see that there exists a weakly reducing pair of disks  $D_1, D_2$  for  $V_{g,1} \cup_{P_g} V_{g,2}$  such that  $D_1 \cap K = \emptyset$ , and  $D_2 \cap K = \emptyset$  (see Remark 5.2), and this together with the arguments as for the case ( $g.1$ ), we see that we have the conclusion of Propositions 6.2.

This completes the proof of Propositions 6.2.

## 7 Proof of Theorem 1.1

Let  $K$  be a knot in a closed 3-manifold  $M$ .

**Definition 7.1** A *tunnel* for  $K$  is an embedded arc  $\sigma$  in  $S^3$  such that  $\sigma \cap K = \partial\sigma$ . We say that a tunnel  $\sigma$  for  $K$  is *unknotting* if  $S^3 - \text{Int } N(K \cup \sigma, S^3)$  is a genus two handlebody.

For a tunnel  $\sigma$  for  $K$ , let  $\hat{\sigma} = \sigma \cap E(K)$ . Then  $\hat{\sigma}$  is an arc properly embedded in  $E(K)$ , and we may regard that  $N(K \cup \sigma)$  is obtained from  $N(K)$  by attaching  $N(\hat{\sigma}, E(K))$ , where  $N(\hat{\sigma}, E(K)) \cap N(K)$  consists of two disks, ie,  $N(\hat{\sigma}, E(K))$  is a 1-handle attached to  $N(K)$ .

**Definition 7.2** Let  $\sigma_1, \sigma_2$  be tunnels for  $K$ . We say that  $\sigma_1$  is *isotopic* to  $\sigma_2$  if there is an ambient isotopy  $h_t$  ( $0 \leq t \leq 1$ ) of  $E(K)$  such that  $h_0 = \text{id}_{E(K)}$ , and  $h_1(\hat{\sigma}_1) = \hat{\sigma}_2$ .

**Remark 7.3** Let  $\sigma$  be an unknotting tunnel for  $K$ , and let  $V = N(K \cup \sigma, M)$ , and  $W = \text{cl}(M - V)$ . Note that  $V \cup W$  is a Heegaard splitting of  $(M, K)$ , which gives a genus two, 0-bridge position of  $K$ . Let  $\sigma_1, \sigma_2$  be unknotting tunnels for  $K$ , and  $V_1 \cup_{P_1} W_1, V_2 \cup_{P_2} W_2$  Heegaard splittings obtained from  $\sigma_1, \sigma_2$  respectively as above. Then it is known that  $\sigma_1$  is isotopic to  $\sigma_2$  if and only if  $P_1$  is  $K$ -isotopic to  $P_2$ .

Now, in the rest of this paper, let  $K$  be a non-trivial 2-bridge knot, and  $A \cup_P B$  a genus 0 Heegaard splitting of  $S^3$ , which gives a two bridge position of  $K$  (Figure 11).

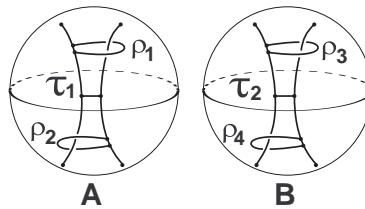


Figure 11

## 7.A Genus two Heegaard splittings of $E(K)$

Here we show the next lemma on unknotting tunnels of  $K$ , which is used in the proof of Theorem 1.1.

**Lemma 7.4** Let  $\sigma$  be an unknotting tunnel for  $K$ , and  $V \cup W$  a Heegaard splitting obtained from  $\sigma$  as in Remark 7.3. Then there exist meridian disks  $D_1, D_2$  of  $V, W$  respectively such that  $D_1$  intersects  $K$  transversely in one point,  $D_1 \cap N(\hat{\sigma}, E(K)) = \emptyset$ , and  $\partial D_1$  intersects  $\partial D_2$  transversely in one point.

**Proof** We note that  $\sigma$  is isotopic to either one of the six unknotting tunnels  $\tau_1, \tau_2, \rho_1, \rho_2, \rho_3,$  or  $\rho_4$  in Figure 11 (see [6] or [13]). Suppose that  $\sigma$  is isotopic to  $\tau_i, i = 1$  or  $2,$  say  $1.$  Then we may regard that  $V = A \cup N(K \cap B, B)$  (Figure 12). Here  $N(\hat{\sigma}, E(K)) = N(D_A, A),$  where  $D_A$  is a disk properly embedded in  $A,$  such that  $D_A$  separates the components of  $K \cap A,$  and  $N(D_A, A) \cap N(K \cap B, B) = \emptyset$  (hence,  $D_A$  is properly embedded in  $V).$  Then we can take a pair  $D_1, D_2$  satisfying the conclusion of Lemma 7.4 as in Figure 12.

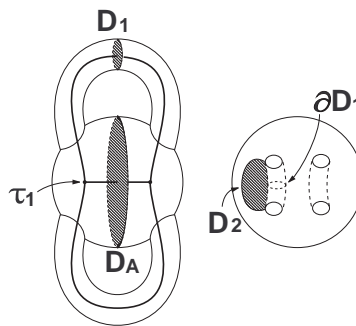


Figure 12

Suppose that  $\sigma$  is isotopic to  $\rho_i, i = 1, 2, 3,$  or  $4,$  say  $1.$  Then we may regard that  $V$  is obtained from the Heegaard splitting  $A \cup_P B$  of  $(S^3, K)$  as follows.

Let  $a$  be the component of  $K \cap A,$  which is disjoint from  $\sigma,$  and  $V' = \text{cl}(A - N(a, A)), W' = B \cup N(a, A).$  Let  $a' = a \cup (K \cap B).$  Note that  $a'$  is an arc properly embedded in  $W'.$  Then  $V = V' \cup N(a', W').$  See Figure 13. That is,  $V \cup W$  is obtained from  $A \cup_P B$  by successively tubing along  $a,$  and  $a'.$  We can take a pair  $D_1, D_2$  satisfying the conclusion of Lemma 7.4, as in Figure 13.  $\square$

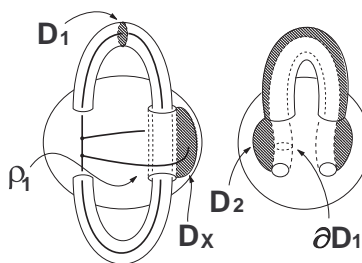


Figure 13

## 7.B Irreducible Heegaard splittings of (torus) $\times [0, 1]$

In [1], M Boileau, and J-P Otal gave a classification of Heegaard splittings of (torus)  $\times [0, 1]$ , and M.Scharlemann, and A.Thompson [18] proved that the same kind of results hold for  $F \times [0, 1]$ , where  $F$  is any closed orientable surface. The result of Boileau–Otal will be used for the proof of Theorem 1.1, and in this section we quickly state it.

Let  $T$  be a torus. Let  $Q_1$  be the surface  $T \times \{1/2\}$  in  $T \times [0, 1]$ . It is clear that  $Q_1$  separates  $T \times [0, 1]$  into two trivial compression bodies. Hence  $Q_1$  is a Heegaard surface of  $T \times [0, 1]$ . We call this Heegaard splitting *type 1*. Let  $a$  be a vertical arc in  $T \times [0, 1]$ . Let  $V_1 = N((T \times \{0, 1\}) \cup a, T \times [0, 1])$ , and  $V_2 = \text{cl}(T \times [0, 1] - V_1)$ . It is easy to see that  $V_1$  is a compression body,  $V_2$  is a genus two handlebody, and  $V_1 \cap V_2 = \partial_+ V_1 = \partial_+ V_2 (= \partial V_2)$ . Hence  $V_1 \cup V_2$  is a Heegaard splitting of  $T \times I$ . We call this Heegaard splitting *type 2*. Then in [1, Théorème 1.5], or [18, Main theorem 2.11], the following is shown.

**Theorem 7.5** *Every irreducible Heegaard splitting of  $T \times [0, 1]$  is isotopic to either a Heegaard splitting of type 1 or type 2.*

## 7.C Proof of Theorem 1.1

Let  $C_1 \cup_P C_2$  be a genus  $g$  Heegaard splitting of the exterior of  $K$ ,  $E(K) = \text{cl}(S^3 - N(K))$ , with  $g \geq 3$  and  $\partial_- C_1 = \partial E(K)$ . Then, by Proposition 6.2, we see that  $C_1 \cup_P C_2$  is weakly reducible. By Proposition 4.2, either  $C_1 \cup_P C_2$  is reducible, or there is a weakly reducing collection of disks  $\Delta$  for  $P$  such that each component of  $\hat{P}(\Delta)$  is an incompressible surface in  $E(K)$ , which is not a 2-sphere. Suppose that the second conclusion holds and let  $M_j$  ( $j = 1, \dots, n$ ),  $M_{j,i}$  ( $i = 1, 2$ ), and  $C_{1,1} \cup_{P_1} C_{1,2}, \dots, C_{n,1} \cup_{P_n} C_{n,2}$  be as in Section 4. Note that each component of  $\partial_- C_{i,j}$  is either  $\partial E(K)$  or a closed incompressible surface in  $\text{Int}E(K)$ . Since every closed incompressible surface in  $\text{Int}E(K)$  is a  $\partial$ -parallel torus, we see that the submanifolds  $M_1, \dots, M_n$  lie in  $E(K)$  in a linear configuration, ie, by exchanging the subscripts if necessary, we may suppose that

- (1)  $\partial_- C_{1,1} = \partial E(K)$ ,
- (2) For each  $i$  ( $1 \leq i \leq n - 1$ ),  $M_i$  is homeomorphic to (torus)  $\times [0, 1]$ , and  $M_i \cap M_{i+1} = F_i$ : a  $\partial$ -parallel torus in  $E(K)$ .

**Claim 1** If  $n > 2$ , then  $C_1 \cup_P C_2$  is reducible.

**Proof** Let  $M'_1 = \text{cl}(C_1 - M_{n,1})$ , and  $M'_2 = \text{cl}(C_2 - M_{n,2})$ . Then from the pair  $M'_1, M'_2$  we can obtain, as in Section 4, a Heegaard splitting, say  $C'_1 \cup_{P'} C'_2$ , of the product region between  $F_{n-1}$  and  $\partial E(K)$ . Since  $n > 2$ , we see, by [20, Remark 2.7], that  $\text{genus}(P') > 2$ . Hence by Theorem 7.5,  $C'_1 \cup_{P'} C'_2$  is reducible. Hence, by Lemma 4.6,  $C_1 \cup_P C_2$  is reducible.  $\square$

By Claim 1, we may suppose, in the rest of the proof, that  $n = 2$ . Now we prove Theorem 1.1 by the induction on  $g$ .

Suppose that  $g = 3$ . By Lemma 4.6, we may suppose that both  $C_{1,1} \cup_{P_1} C_{1,2}$ , and  $C_{2,1} \cup_{P_1} C_{2,2}$  are irreducible. By Lemma 4.5 and Theorem 7.5, we see that  $C_{1,1}$  is a genus 2 compression body with  $\partial_- C_{1,1} = \partial E(K) \cup F_1$ , and  $C_{1,2}$  is a genus 2 handlebody.

**Claim 2**  $(M_{1,1} \cap P) \subset (M_{1,2} \cap P)$ .

**Proof** Suppose not. Then, by Lemma 4.3, we see that  $(M_{1,1} \cap P) \supset (M_{1,2} \cap P)$ . Recall that  $C_{1,1} = \text{cl}(M_{1,1} - N(\partial_+ M_{1,1}, M_{1,1}))$ . This implies that  $\partial_- M_{1,1} = \partial_- C_{1,1}$ . Note that  $C_{1,1} \cup_{P_1} C_{1,2}$  is a Heegaard splitting of type 2 in Section 7.B. These show that  $\partial_- M_{1,1} = \partial E(K) \cup F_1$ . However, this is impossible since  $\partial_- M_{1,1} \subset \partial E(K)$ .  $\square$

By Claim 2, we see that  $M_{1,2}$  is a genus two handlebody. Hence  $\Delta_2$  is either one of Figure 14, ie, either (1)  $\Delta_2$  consists of a non-separating disk in  $C_2$ , (2)  $\Delta_2$  consists of a separating disk in  $C_2$ , or (3)  $\Delta_2$  consists of two disks, one of which is a separating disk, and the other is a non-separating disk in  $C_2$ .

Suppose that  $\Delta_2$  is of type (1) in Figure 14. Since no component of  $\hat{P}(\Delta)$  is a 2-sphere, we see that  $\partial \Delta_1 \subset M_{1,2}$ . By Claim 2, we see that  $(M_{2,1} \cap P) \supset (M_{2,2} \cap P)$ . Since  $\partial(M_{2,1} \cup M_{2,2}) = \partial M_2 = F_1$ : a torus, we see that  $M_{2,1}$  is a genus two handlebody, and  $\Delta_1$  consists of a separating disk in  $C_1$  (Figure 15).

Let  $N_K = \text{cl}(S^3 - M_2)$ . Since  $F_1$  is a  $\partial$ -parallel torus in  $E(K)$ , we see that  $N_K$  is a regular neighborhood of  $K$ , hence  $M_2$  is an exterior of  $K$ . Note that  $M_{2,2}$  is a 1-handle attached to  $N_K$  such that  $\text{cl}(S^3 - (N_K \cup M_{2,2})) = M_{2,1}$ , a genus two handlebody. This shows that  $M_{2,2}$  is a regular neighborhood of an arc properly embedded in  $M_2$ , which comes from an unknotting tunnel of  $K$ . Hence, by Lemma 7.4, we see that there is a pair of disks  $D_1, D_2$  in  $N_K \cup M_{2,2}$ ,  $M_{2,1}$  respectively such that  $D_1$  intersects  $K$  transversely in one point,  $D_1 \cap M_{2,2} = \emptyset$ , and  $\partial D_1$  intersects  $\partial D_2$  transversely in one point. Here, by deforming  $D_2$  by an ambient isotopy of  $M_{2,1}$  if necessary, we may suppose that  $D_2 \cap \Delta_1 = \emptyset$

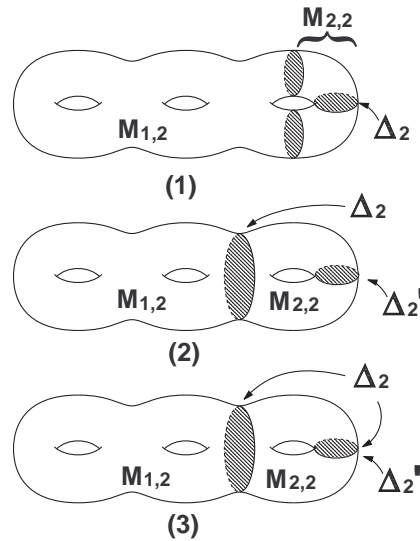


Figure 14

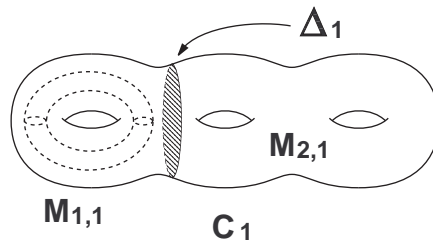


Figure 15

(hence,  $D_2$  is a meridian disk of  $C_1$ ). Since  $D_1$  and  $K$  intersect transversely in one point, we may suppose that  $D_1 \cap E(K)$  ( $= D_1 \cap M_1$ ) is a vertical annulus, say  $A_1$ , properly embedded in  $M_1$  ( $\cong T^2 \times [0, 1]$ ). Recall that  $C_{1,1} \cup_P C_{1,2}$  is a type 2 Heegaard splitting of  $M_1$ . This implies that there exists a vertical arc  $a$  in  $M_1$  such that  $M_{1,1} = N(\partial E(K) \cup a, M_1)$ . Since  $a$  is vertical, we may suppose, by isotopy, that  $a \subset A_1$ , ie,  $a$  is an essential arc properly embedded in  $A_1$ . Let  $\ell$  be the component of  $\partial A_1$  contained in  $\partial E(K)$ . Hence  $A_1 \cap C_2 = A_1 \cap M_{1,2} = \text{cl}(A_1 - N(\ell \cup a, M_1))$ , and this is a disk, say  $D_1'$ , properly embedded in  $C_2$ . Obviously  $\partial D_1'$  and  $\partial D_2$  intersect transversely in one point. Recall that  $D_2$  ( $D_1'$  respectively) is a disk properly embedded in  $C_1$  ( $C_2$  respectively). Hence  $C_1 \cup_P C_2$  is stabilized and this shows that  $C_1 \cup_P C_2$  is reducible if  $g = 3$  (see 2 of Remark 2.3).



Suppose that  $\Delta_2$  is of type (2) or (3) in Figure 14. Then we take  $\Delta'_2$  as in Figure 14, and let  $\Delta' = \Delta_1 \cup \Delta'_2$ . We note that  $\Delta'$  is a weakly reducing collection of disks for  $P$ , where  $\Delta'$  is of type (1) in Figure 14. Let  $F'_1$  be the torus obtained from  $\Delta'$ , which is corresponding to  $F_1$ . It is directly observed from Figure 14 that  $F'_1$  is isotopic to  $F_1$ . Hence we can apply the argument for type 1 weakly reducing collection of disks to  $\Delta'$ , and we can show that  $C_1 \cup_P C_2$  is reducible.

Suppose that  $g \geq 4$ . If  $\text{genus}(P_1) > 2$ , then by Theorem 7.5 and Lemma 4.6, we see that  $C_1 \cup_P C_2$  is reducible. Suppose that  $\text{genus}(P_1) = 2$ . Then, by [20, Remark 2.7], we see that  $\text{genus}(P_2) = g - 1$ . Hence, by the assumption of the induction, we see that  $C_{2,1} \cup_{P_2} C_{2,2}$  is reducible. Hence, by Lemma 4.6,  $C_1 \cup_P C_2$  is reducible.

This completes the proof of Theorem 1.1.

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