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Tight contact structures on Seifert manifolds over T^2 with one singular fibre

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Abstract In this article we classify up to isotopy tight contact structures on Seifert manifolds over the torus with one singular fibre.

AMS Classification 57R17; 57M50

Keywords Contact structure, tight, Seifert 3–manifold, convex surface

1 Introduction

A *contact structure* on a 3–manifold M is a tangent 2–plane field ξ which is the kernel of a differentiable 1–form α such that $\alpha \wedge d\alpha$ is a nowhere vanishing 3–form. Contact structures on 3–manifolds split into two families. A contact structure ξ is *overtwisted* if there exists an embedded disc $D \subset M$ such that $TD|_{\partial D} \equiv \xi|_{\partial D}$. A contact structure is *tight* if it is not overtwisted. The disc D is called, with an abuse of terminology, an overtwisted disc.

Overtwisted contact structures are much more common and flexible objects than the tight ones, in fact any 3–manifold admits an overtwisted contact structure and on a closed 3–manifold two overtwisted contact structures are isotopic if and only if they are homotopic as plane fields (Eliashberg [7]). On the contrary, the classification of tight contact structures is still at its beginning. For a survey of contact structures, see [1, 8, 10, 16].

In the last decade there has been a dramatic growth of the three–dimensional methods in contact topology starting from the definition of convex surfaces in Giroux's paper [15]. Convex surfaces are the main tool to perform cut-and-paste operations on contact manifolds. Applying this technique, Kanda [28] and, independently, Giroux, classified the tight contact structures on the three–torus T^3. Later, Honda [22] and Giroux [18] classified the tight contact structures on lens spaces, the solid torus $D^2 \times S^1$ and the thickened torus $T^2 \times I$. In [22], Honda introduced the notion of *bypass*, a tool which allows one to handle contact topological problems in a combinatorial way (see [22], Section 3.4). In this paper we will assume that the reader is familiar with the material in [15] and [22].

The solid torus and the thickened torus can be thought of as basic building blocks for a number of other three dimensional manifolds. In fact, shortly after, Honda [23] gave a complete classification of tight contact structures on T^2–bundles over S^1 and S^1–bundles over surfaces. At the same time Giroux [19] obtained almost complete results on the same manifolds.

Other classification results are partial or sporadic. The most important of them are the non existence of tight contact structures on the Poincaré homology sphere with opposite orientation $-\Sigma(2,3,5)$ in [11] and the coarse classification which characterises the three–manifolds which carry infinitely many tight contact structures, [2, 3, 4, 26]. A complete classification is also known for the Seifert manifolds over S^2 with three singular fibres $\pm\Sigma(2,3,11)$, [13]. Moreover, there are partial results on fibred hyperbolic three–manifolds [27], which are the only non Seifert manifolds in the list so far. During the preparation of this article tight contact structures have been classified on small Seifert manifolds with integer Euler class $e_0 \neq -2, -1$, [14, 32].

Our aim is to give a complete isotopy classification of tight contact structures on Seifert manifolds over the torus T^2 with one singular fibre. Fix $e_0 \in \mathbb{Z}$ and $r \in (0,1) \cap \mathbb{Q}$, and let $T(e_0)$ be the circle bundle over T^2 with Euler class e_0. We denote by $M(e_0, r)$ the Seifert manifold obtained by $(-\frac{1}{r})$–surgery along a fibre of $T(e_0)$. The tight contact structures on $M(e_0, r)$ and $T(e_0)$ split into two families, according to their behaviour with respect to the finite coverings induced by a finite covering of T^2. We will call *generic* those tight contact structures which remain tight after pulling back to such coverings, and *exceptional* those ones which become overtwisted. The set of isotopy classes of generic tight contact structures on $M(e_0, r)$ splits into infinitely many sub-families parametrised by the isotopy classes of the generic tight contact structures on $T(e_0)$. Each subfamily contains finitely many isotopy classes of tight contact structures which are obtained by Legendrian surgery on the generic tight contact structure on $T(e_0)$ labelling the sub-family.

The isotopy classes of exceptional tight contact structures on $M(e_0, r)$ form a finite family, whose cardinality depends on e_0 and r. If $e_0 \leq 0$ there are no exceptional tight contact structures on $M(e_0, r)$. If $e_0 \geq 2$, all exceptional tight contact structures on $M(e_0, r)$ are obtained by Legendrian surgery on the exceptional tight contact structures on $T(e_0)$ which, however, are not fillable by [30]. If $e_0 = 1$, the exceptional tight contact structures on $M(e_0, r)$ have no tight analogue on $T(e_0)$. They are obtained by Legendrian surgery on overtwisted contact structures and there seems to be no natural way to express them as Legendrian surgery on a tight contact structure. When $e_0 = 1, 2$ the exceptional tight contact structures show an unexpected interplay between the corresponding contact structure on $T(e_0)$ and the surgery data. See Theorem 6.10.

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2 Statement of results

Let M be an oriented 3–manifold. The set of isotopy classes of tight contact structures on M will be denoted by $\text{Tight}(M)$. If $\partial M \neq \emptyset$, and $\mathcal{F}$ is a singular foliation on ∂M, $\text{Tight}(M, \mathcal{F})$ will denote the set of tight contact structures on M which induce the characteristic foliation $\mathcal{F}$ on ∂M, modulo isotopies fixed on the boundary. If $\mathcal{F}$ and $\mathcal{G}$ are two singular foliations on ∂M adapted to the same dividing set $\Gamma_{\partial M}$, then $\text{Tight}(M, \mathcal{F})$ and $\text{Tight}(M, \mathcal{G})$ are canonically identified, therefore we will write $\text{Tight}(M, \Gamma_{\partial M})$ in place of $\text{Tight}(M, \mathcal{F})$ for any $\mathcal{F}$ adapted to $\Gamma_{\partial M}$.

Recall that we denote by $T(e_0)$, for $e_0 \in \mathbb{Z}$, the S^1–bundle over T^2 with Euler class e_0, and by $M(e_0, r)$, for $r \in \mathbb{Q} \cap (0, 1)$, the Seifert manifold over T^2 obtained by $(-\frac{1}{r})$–surgery along a fibre R of $T(e_0)$. Here the surgery coefficient is calculated with respect to the standard framing on R. More explicitly, consider a tubular neighbourhood $\nu R \subset T(e_0)$ of R, and identify $-\partial(T(e_0) \setminus \nu R)$ to $\mathbb{R}^2/\mathbb{Z}^2$ so that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the direction of the meridian of

νR and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the direction of the fibres. Then $M(e_0, r)$ is the manifold obtained by gluing a solid torus $D^2 \times S^1$ to $T(e_0) \setminus \nu R$ by the map

$$A(r)\colon\ \partial D^2 \times S^1 \to -\partial(T(e_0) \setminus \nu R)$$

represented by the matrix

$$A(r) = \begin{pmatrix} \alpha & \alpha' \\ -\beta & -\beta' \end{pmatrix} \in SL(2, \mathbb{Z})$$

where $r = \frac{\beta}{\alpha}$ and $0 \leq \alpha' < \alpha$. The image of $\{0\} \times S^1 \subset D^2 \times S^1$ in $M(e_0, r)$ is called the *singular fibre*. The images of the fibres of $T(e_0)$ are called *regular fibres*. See [12, 21, 31] for more about Seifert manifolds.

Let M be a Seifert manifold, possibly without singular fibres, with non simply connected base. Let $R \subset M$ be a curve isotopic to a regular fibre. In the following such curve will be called a *vertical curve*. Following Kanda [28], we define the *canonical framing* of R as the framing induced by any incompressible torus $T \subset M$ containing R. Unless stated otherwise, the twisting number of Legendrian vertical curves will be calculated with respect to the canonical framing.

Definition 2.1 Let M be a Seifert fibred manifold over an oriented non simply connected surface. Given a regular fibre $R \subset M$ and a contact structure ξ on M, we define the *maximal twisting number* of ξ as

$$t(\xi) = \max_{L \in \mathcal{S}} \min\{tb(L), 0\}$$

where $\mathcal{S}$ is the set of all Legendrian curves $L \subset M$ isotopic to R.

It is clear that the number $t(\xi)$ does not depend on the choice of R, and is an isotopy invariant of ξ, therefore it defines a function

$$t\colon\ \mathrm{Tight}(M) \to \mathbb{Z}_{\leq 0}.$$

Seifert fibred manifolds over a surface of genus $g > 0$ have a distinguished family of coverings: namely, the coverings induced by a covering of the base.

Definition 2.2 A tight contact structure on a Seifert fibred manifold M is of *generic type* if it remains tight after pull-back with respect to any covering of M induced by a finite covering of the base. A tight contact structure on M is *exceptional* if it becomes overtwisted after pull-back with respect to any covering of M induced by a finite covering of the base.

We denote the set of the isotopy classes of the generic tight contact structures on M by $\mathrm{Tight}_0(M)$.

In the following theorem Γ_s will be a dividing set on T^2 with $\#\Gamma_s = 2$ and slope s. Every rational number $-\frac{p}{q} < -1$ has a unique finite continued fraction expansion

$$-\frac{p}{q} = d_0 - \cfrac{1}{d_1 - \cfrac{1}{\ddots - \frac{1}{d_n}}}$$

with $d_i \leq -2$ for $i > 0$. We denote this expansion by $-\frac{p}{q} = [d_0, \dots, d_n]$.

Theorem 2.3 *All tight contact structures on $M(e_0, r)$ are either of generic type or exceptional. There exists a map*

$$bg\colon \mathrm{Tight}_0(M(e_0, r)) \longrightarrow \mathrm{Tight}_0(T(e_0))$$

such that, given $\xi_0 \in \mathrm{Tight}_0(T(e_0))$,

- $bg^{-1}(\xi_0) = \emptyset$ *if* $t(\xi_0) \leq -\frac{1}{r}$,
- $bg^{-1}(\xi_0)$ *is in natural bijection with* $\mathrm{Tight}(D^2 \times S^1,\, A(r)^{-1}\Gamma_{\frac{1}{t(\xi_0)}})$ *and has cardinality* $|(d_0 - t(\xi_0))(d_1 + 1) \dots (d_n + 1)|$, *where* $[d_0, \dots, d_n]$ *is the continued fraction expansion of* $-\frac{1}{r}$, *if* $t(\xi_0) > -\frac{1}{r}$.

The exceptional tight contact structures exist only when $e_0 > 0$ and all have maximal twisting number $t = 0$. Their number is always finite and is

- $2|(d_0 + 1) \dots (d_k + 1)|$ *if* $e_0 > 2$,
- $|(d_0 - 1)(d_1 + 1) \dots (d_k + 1))|$ *if* $e_0 = 2$,
- $|d_1(d_2 + 1) \dots (d_k + 1)|$ *if* $e_0 = 1$.

The last expression has to be interpreted as 2 when $-\frac{1}{r} = d_0 \in \mathbb{Z}$.

The map bg is constructed by removing a tubular neighbourhood of the singular fibre V such that $-\partial(M(e_0, r) \setminus V)$ is convex with slope $\frac{1}{t(\xi)}$ and gluing $D^2 \times S^1$ with the unique tight contact structure with boundary slope $\frac{1}{t(\xi)}$ to $-\partial(M(e_0, r) \setminus V)$ via the identity map. The identification of $bg^{-1}(\xi_0)$ with $\mathrm{Tight}(D^2 \times S^1,\, A(r)^{-1}\Gamma_{\frac{1}{t(\xi_0)}})$ is given by the restriction $(M(e_0, r),\, \xi) \mapsto (V,\, \xi|_V)$. The fact that the map bg and the restriction $(M(e_0, r),\, \xi) \mapsto (V,\, \xi|_V)$ are well defined up to isotopy is part of the statement. Theorem 2.3 exhibits each generic tight contact structure on $M(e_0, r)$ as a contact surgery in the sense of [6] on a generic tight contact structure on $T(e_0)$. Moreover, the condition $t(\xi_0) > -\frac{1}{r}$ implies that it is a negative contact surgery, which means that the surgery coefficient, calculated with respect to the contact framing, is negative. The expression for the cardinality of $bg^{-1}(\xi_0)$ is a consequence of the following lemma, which is simply the classification of tight contact structures on solid tori [22], Theorem 2.3 applied to $\xi|_V$ after a change of coordinates. For benefit of the reader we sketch here how to deduce this lemma from Honda's Theorem.

Lemma 2.4 $\mathrm{Tight}(D^2 \times S^1,\, A(r)^{-1}\Gamma_{\frac{1}{t(\xi_0)}})$ *is a nonempty finite set with cardinality*

$$|\mathrm{Tight}(D^2 \times S^1,\, A(r)^{-1}\Gamma_{\frac{1}{t(\xi_0)}})| = |(d_0 - t(\xi_0))(d_1+1)\dots(d_n+1)|,$$

where $[d_0, \dots, d_n]$ *is the continued fraction expansion of* $-\frac{1}{r}$.

Proof Let $r' = \frac{1}{\frac{1}{r}+t(\xi)+1}$ so, by a direct check, $A(r)^{-1}\Gamma_{\frac{1}{t(\xi_0)}}$ and $A(r')^{-1}\Gamma_{-1}$ have the same slope s'. By [6], proof of Proposition 3, if $-\frac{1}{r'}$ has the continued fraction expansion $-\frac{1}{r'} = [d'_0, \dots, d'_n]$, then s' has the continued fraction expansion $s' = [r'_n, \dots, r'_0 + 1]$. By [22], Theorem 2.3,

$$|\mathrm{Tight}(D^2 \times S^1,\, A(r)^{-1}\Gamma_{\frac{1}{t(\xi_0)}})| = |(d'_0+1)(d'_1+1)\dots(d'_n+1)|$$

provided that $s' < -1$. As $d'_0 = d_0 - (t(\xi)+1)$ and $d'_i = d_i$ for $i > 0$, we have $s' < d_n + 1 \leq -1$ and $|d'_0(d'_1+1)\dots(d'_n+1)| = |(d_0 - t(\xi_0))(d_1+1)\dots(d_n+1)|$. □

3 Tight contact structures on $T(e_0)$

The tight contact structures on $T(e_0)$ have been classified in [23] and in [18]. The material in this section is taken primarily from [23], adapting statements and notation to our purposes. In order to fix terminology and notations, we start with a digression about characteristic foliations on tori in tight contact manifold before focusing on the classification of tight contact structures on $T(e_0)$.

3.1 Characteristic foliation on tori

If T is a convex torus in a tight contact manifold (M, ξ), by Giroux's Tightness Criterion [22] Lemma 4.2, its dividing set Γ_T contains no dividing curve bounding a disc in T, therefore it consists of an even number of closed, parallel, homotopically non trivial curves.

Definition 3.1 If γ is a dividing curve of T, we call the quantity $s(T) = [\gamma] \in \mathbb{P}(H_1(T, \mathbb{Q}))$ the slope of the convex torus T.

The choice of an identification $T \cong \mathbb{R}^2/\mathbb{Z}^2$ gives an identification $\mathbb{P}(H_1(T, \mathbb{Q})) \cong \mathbb{Q} \cup \{\infty\}$, hence we will more often see the slope as a rational number.

Definition 3.2 We call the *division number* of T the number $\mathrm{div}(T) = \frac{1}{2}\#\Gamma_T$. If $\mathrm{div}(T) = 1$ we say that T is *minimal*.

Given a dividing set Γ_T on a torus T in a tight contact manifold, there is a canonical family of characteristic foliations adapted to Γ_T. Fix a slope $r \neq s(T)$ and consider on T the singular foliation consisting of a 1-parameter family of closed curves with slope r, called *Legendrian rulings*, and a closed curve of singularities with slope $s(T)$ called *Legendrian divide* in each component of $T \setminus \Gamma_T$. See Figure 3.1 for an illustration. A torus with a characteristic foliation of this type is called a convex torus in *standard form*, or a *standard torus*.

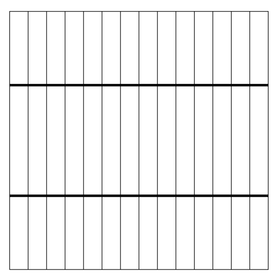

Figure 3.1: Characteristic foliation on a convex torus in standard form with vertical Legendrian ruling and two horizontal Legendrian divides

As an immediate consequence of Giroux's Flexibility theorem, any convex torus T with slope $s(T)$ in a tight contact manifold can be put in standard form with ruling slope r by a C^0-small perturbation, provided that $r \neq s(T)$.

Sometimes we will need to consider non convex tori of a particular kind.

Definition 3.3 A *pre-Lagrangian torus* is a torus embedded in a contact manifold, whose characteristic foliation after a change of coordinates is isotopic to a linear foliation with closed leaves.

Suppose we have chosen coordinates on a neighbourhood of a pre-Lagrangian torus T so that $T = \{y = 0\}$, and the characteristic foliation of T has slope 0. Then the contact form in a neighbourhood of T is given by $dz - y\,dx$. Pre-Lagrangian tori can be perturbed into convex tori, as explained in the following lemma.

Lemma 3.4 *Let T be a pre-Lagrangian torus whose characteristic foliation has closed leaves with slope s. Then, for any natural number $n > 0$, T can be put in standard form with $2n$ dividing curves with slope s by a C^∞-small perturbation.*

Proof Let T be the given pre-Lagrangian torus. Put coordinates $(x, y, z) \in \mathbb{R}/\mathbb{Z} \times I \times \mathbb{R}/\mathbb{Z}$ in a tubular neighbourhood N of T such that $T = \{y = 0\}$ and the contact form is $\alpha = dz - y\,dx$, then consider the embedding $i : T^2 \to N$ given by $i : (u, v) \mapsto (u, \epsilon \sin(2\pi n v), v)$. After identifying T^2 with the image of i, the characteristic foliation is given by the form $i^*\alpha = dv - \epsilon \sin(2\pi n v)du$.

Fix the area form $\omega = du \wedge dv$ on T^2, then the characteristic foliation is directed by a vector field X such that $\iota_X(\omega) = i^*\alpha$. Since $L_X\omega = di^*\alpha = 2\pi n\epsilon \cos(2\pi n v)du \wedge dv$, the set $\Gamma = \{L_X\omega = 0\}$ consists of $2n$ parallel simple closed curves with slope 0. The vector field X expands ω where $L_X\omega$ is a positive multiple of ω, and $-X$ expands ω where $L_X\omega$ is a negative multiple of ω, therefore, by [15] Proposition II.2.1, Γ is dividing set for the characteristic foliation of T. □

3.2 Tight contact structures with $t < 0$

Theorem 3.5 ([23], Lemma 2.7) *If $e_0 < 0$, then on $T(e_0)$ there are $|e_0 - 1|$ distinct tight contact structures with $t < 0$.*

By a direct check of the definition of such tight contact structures, see [23], Case 9, it follows that only 2 of the $|e_0 - 1|$ are universally tight, but all remain tight if lifted to a covering of $T(e_0)$ induced by a finite covering of the base T^2.

Theorem 3.6 *The tight contact structures with $t < 0$ on $T(e_0)$, when $e_0 < 0$, are Stein fillable.*

Proof In [20] Gompf constructed $|e_0 - 1|$ Stein fillings of $T(e_0)$ when $e_0 \leq 0$: see [20], Figure 36 (c) for a surgery presentation of the Stein filling of $T(0) = T^3$. When $e_0 \leq -1$, the Stein fillings of $T(e_0)$ are obtained by Legendrian surgery on a stabilisation of the knot in [20], Figure 36(c). All the Stein fillings obtained in such way are diffeomorphic to the disc bundle over T^2 with Euler class e_0, but their complex structures have different first Chern classes determined by the rotation number of the Legendrian knot, as shown in [20], Proposition 2.3. The tight contact structures induced on the boundary are pairwise non isotopic by [29], Corollary 4.2.

To prove Theorem 3.6, we need to show that the $|e_0-1|$ tight contact structures on $T(e_0)$ induced by the different Stein structures described above have $t<0$. Let W be the disk bundle over T^2 with Euler class e_0, and $D \subset W$ a fibre with Legendrian boundary $\partial D = K$. The *slice Thurston–Bennequin invariant* $tb_D(K) \in \mathbb{Z}$ is defined in [29], Definition 3.1 as the obstruction to extending the positively oriented normal of the contact structure restricted to K to a nowhere vanishing section of the normal bundle of D. It can be defined equivalently as the twisting number of K computed with respect to the framing induced on K by the restriction of a nowhere vanishing section of the normal bundle of D. The framing on the normal bundle of D induced by the disc bundle structure over W restricts to the framing on the normal bundle of $\partial D = K \subset T(e_0)$ induced by the circle bundle structure on $T(e_0)$.

The bundle framing of K coincides with the canonical framing, therefore the Thurston–Bennequin number $\mathrm{tb}_D(K)$ and the twisting number $\mathrm{tb}(K)$ defined by the canonical framing coincide. By the slice Thurston–Bennequin inequality [29], Theorem 3.4 $tb_D(K) \leq -1$ for any Legendrian knot K in $(T(e_0), \xi)$ smoothly isotopic to a fibre of $T(e_0)$, therefore $t(\xi)<0$. On the other hand there are exactly $|e_0-1|$ tight contact structures on $T(e_0)$ with $t<0$, so any tight contact structure on $T(e_0)$ with $t<0$ must be Stein fillable for cardinality reasons. □

For $n \in \mathbb{N}^+$, let ζ_n be the tight contact structures on T^3 defined as

$$\zeta_n = \ker(\sin(2\pi nz)dx + \cos(2\pi nz)dy).$$

Theorem 3.7 (Giroux, [17]) *For any $n \in \mathbb{N}^+$, the contact structure ζ_n is universally tight and weakly symplectically fillable. Moreover (T^3, ζ_n) is contactomorphic to (T^3, ζ_m) if and only if $n=m$.*

Theorem 3.8 ([28], Theorem 0.1) *Any tight contact structure ξ on T^3 is contactomorphic to ζ_n for some n.*

Corollary 3.9 *All tight contact structures on T^3 are universally tight and weakly symplectically fillable.*

Take a primitive vector $(c_1, c_2, c_3) \in \mathbb{Z}^3$ with $c_3 \neq 0$ and complete it as the third row of a matrix $\Phi \in SL(3, \mathbb{Z})$. The isotopy class of $\Phi_*^{-1}\zeta_n$ does not depend on the choice of the first and second rows of Φ because the stabiliser of ζ_n in $SL(3, \mathbb{Z})$ acts transitively on them: see [28] Theorem 0.2.

Definition 3.10 Let $(c_1, c_2, c_3) \in \mathbb{Z}^3$ be a primitive vector and let n be a positive natural number. We set $\xi_{(n,c_1,c_2,c_3)} = \Phi^* \zeta_n$.

By [28], Theorem 7.6, $t(\xi_{(n,c_1,c_2,c_3)}) = -|nc_3|$.

Theorem 3.11 ([23], Lemma 2.6) *The tight contact structures $\xi_{(n,c_1,c_2,c_3)}$ and $\xi_{(n',c_1',c_2',c_3')}$ are isotopic if and only if $n = n'$ and $(c_1, c_2, c_3) = \pm(c_1', c_2', c_3')$. Moreover, any tight contact structure ξ on T^3 with $t(\xi) < 0$ is isotopic to $\xi_{(n,c_1,c_2,c_3)}$ for some (n, c_1, c_2, c_3) with $c_3 \neq 0$.*

Theorem 3.12 ([23], Section 2.5 Case 5) *If $e_0 > 0$ there is no tight contact structure ξ on $T(e_0)$ with $t(\xi) < 0$.*

3.3 Tight contact structures with $t = 0$

Theorem 3.13 ([23], Section 2.2 and Lemma 2.5) *The universally tight contact structures on $T(e_0)$ with maximal twisting number $t = 0$ are in bijection with the set $\mathbb{N}^+ \times \mathbb{P}(H_1(T^2; \mathbb{Q}))$.*

The bijection in the theorem is given in the following way. $T(e_0)$ is also a T^2–bundle over S^1. Consider a convex T^2–fibre with infinite slope (i.e. whose dividing curves are isotopic in $T(e_0)$ to S^1–fibres) and cut $T(e_0)$ along it obtaining a $T^2 \times I$ with infinite boundary slopes. Make the boundary of $T^2 \times I$ standard with horizontal ruling, and take a convex horizontal annulus $A \subset T^2 \times I$. Gluing the boundary components of A together, we obtain a torus T with a multicurve Γ_T. Let $n = \mathrm{div}(T) \in \mathbb{N}^+$, and $s \in \mathbb{P}(H_1(T^2; \mathbb{Q}))$ the class of a connected component of Γ_T, then (n, s) is the element in $\mathbb{N}^+ \times \mathbb{P}(H_1(T^2; \mathbb{Q}))$ associated to the tight contact structure on $T(e_0)$.

Theorem 3.14 ([5], Proposition 16) *The universally tight contact structures on $T(e_0)$ with maximal twisting number $t = 0$ are all weakly symplectically fillable.*

Remark When $e_0 = 0$, i. e. when $T(e_0) = T^3$, the maximal twisting number t reflects no geometric property of the tight contact structure, but depends only on the choice of a bundle structure $T^3 \to T^2$.

For T^3 we have
$$\mathrm{Tight}(T^3) \cong \mathbb{N}^+ \times \mathbb{P}(H_2(T^3; \mathbb{Q})).$$

A tight (T^3, ξ) corresponds to $(n, [T])$ such that ξ is contactomorphic to ζ_n and $[T]$ is the unique homology class represented by a pre-Lagrangian torus in (T^3, ξ). The set $\mathbb{N}^+ \times \mathbb{P}(H_1(T^2;\mathbb{Q}))$ of the isotopy classes of the tight contact structures on T^3 with maximal twisting number $t = 0$ embeds into $\mathbb{N}^+ \times \mathbb{P}(H_2(T^3;\mathbb{Q}))$ as $\mathbb{N}^+ \times H$, where $H \subset \mathbb{P}(H_2(T^3;\mathbb{Q}))$ is the hyperplane of the homology classes represented by the fibred tori.

Theorem 3.15 ([23], Proposition 2.3) *There exist virtually overtwisted contact structures with $t = 0$ on $T(e_0)$ only when $e_0 > 1$. There is one if $e_0 = 2$ and two if $e_0 > 2$.*

The virtually overtwisted contact structures with maximal twisting number $t = 0$ become overtwisted when pulled back to any covering of $T(e_0)$ induced by a covering of the base T^2 and, by [30], are not weakly symplectically fillable.

4 Construction of the tight contact structures on $M(e_0, r)$

4.1 Thickening the singular fibre

Lemma 4.1 *If ξ is a tight contact structure on $M(e_0, r)$ with maximal twisting number $t(\xi)$, then there exists a neighbourhood V of the singular fibre F such that $-\partial(M(e_0, r) \setminus V)$ is convex with slope $\frac{1}{t(\xi)}$. Moreover:*

(1) *If $e_0 < 0$, then $t(\xi) \geq -1$.*

(2) *If $e_0 = 0$, then $t(\xi) > -\frac{1}{r}$.*

(3) *If $e_0 > 0$, then $t(\xi) = 0$.*

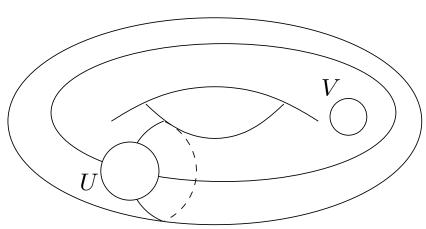

Figure 4.1: How to cut $M \setminus (V \cup U)$

Proof In the following, we will call $M = M(e_0, r)$. After an isotopy, we can find a Legendrian regular fibre R with twisting number $t(\xi)$. The singular fibre F can be made Legendrian with a very low twisting number n. We choose a standard neighbourhood V of F such that $-\partial(M \setminus V)$ has slope

$$s_V = \frac{-n\beta - \beta'}{n\alpha + \alpha'} = -\frac{\beta}{\alpha} + \frac{1}{\alpha(n\alpha + \alpha')} < -\frac{\beta}{\alpha}$$

where $\frac{\beta}{\alpha} = r$ and α', β' are defined by $0 \leq \alpha' < \alpha$ and $\alpha'\beta - \alpha\beta' = 1$.

If $t(\xi) = 0$, choose a convex annulus A so that one boundary component is a Legendrian ruling curve of $\partial(M \setminus V)$ and the other one is the Legendrian fibre R with twisting number $t(\xi)$. By the imbalance principle [22] Proposition 3.17, we can perturb A so that it contains a bypass attached to ∂V. By using this bypass we can thicken V as far as there are singular points on ∂A, therefore we eventually get a solid torus V with infinite boundary slope.

When $t(\xi) < 0$, we choose a standard neighbourhood U of R such that $-\partial(M \setminus U)$ has boundary slope $s_U = -e_0 + \frac{1}{t(\xi)}$. In the convex annuli in figure 4.1, whose boundary components are Legendrian ruling curves of $\partial(M \setminus U)$, all the dividing curves go from one boundary component to the other one, otherwise there would be a bypass attached vertically to U which would increase the twisting number of R by the twisting number lemma. When we cut $M \setminus (U \cup V)$ open along these two annuli, we obtain a thickened torus with corners as shown in figure 4.2.

From slope $e_0 - \frac{1}{t(\xi)}$ on $\partial(M \setminus U)$ by [22], Lemma 3.11 we get, after rounding the edges, slope $e_0 + \frac{1}{t(\xi)}$, so the thickened torus we have obtained has boundary slopes $s_0 = s_V < -r$ and $s_1 = e_0 + \frac{1}{t(\xi)}$. If $e_0 + \frac{1}{t(\xi)} > -r$, we have $s_1 > s_0$ and there is an intermediate torus with infinite slope by [22], Proposition 4.16. This torus would contradict the assumption about the maximality of the twisting number $t(\xi)$ of R, therefore $e_0 + \frac{1}{t(\xi)} \leq -r$. This implies that if $e_0 > 0$ than $t(\xi) = 0$. If $e_0 + \frac{1}{t(\xi)} = -r$, there is an overtwisted disc in a tubular neighbourhood of the singular fibre with boundary on a Legendrian divide with slope $-r$. We now divide into cases according to the sign of e_0.

(1) If $e_0 < 0$, then $e_0 + \frac{1}{t(\xi)} < -1 < -r$, therefore there is always an intermediate torus with slope -1 which forces the maximal twisting number $t(\xi)$ to be greater than or equal to -1.

(2) If $e_0 = 0$, then $\frac{1}{t(\xi)} < -r$.

In cases 1 and 2 we can find an intermediate convex torus with slope $\frac{1}{t(\xi)}$ because $\frac{1}{t(\xi)} \in [e_0 + \frac{1}{t(\xi)}, -r)$, and this torus bounds a neighbourhood V of the singular fibre F such that $-\partial(M \setminus V)$ has slope $\frac{1}{t(\xi)}$. □

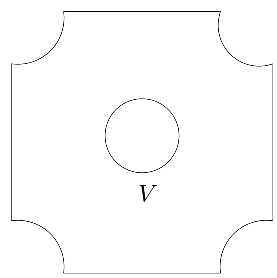

Figure 4.2: The thickened torus with corners

Definition 4.2 Let $(M(e_0, r), \xi)$ be a tight contact manifold with maximal twisting number $t(\xi)$, and let V be a tubular neighbourhood of the singular fibre F as in Lemma 4.1 such that $-\partial(M \setminus V)$ has slope $\frac{1}{t(\xi)}$. Then the contact manifold $(M(e_0, r) \setminus V, \xi|_{M(e_0,r)\setminus V})$ will be called a *background* of $(M(e_0, r), \xi)$.

Definition 4.3 If ξ_0 is a contact structure on $M \setminus V$ and η is a contact structure on V which match along the boundary, we will denote the glued contact structure on M by $\xi_0(\eta)$.

Generally, on a manifold with nonempty boundary we consider tight contact structures up to isotopies fixed on the boundary. On the contrary, in the classification of the backgrounds we will allow isotopies to move the boundary because of the following lemma.

Lemma 4.4 *Suppose that ξ_1 and ξ_2 are tight contact structures on M and $V \subset M$ is a solid torus with convex boundary with respect to both ξ_1 and ξ_2. If $\xi_1|_{M\setminus V}$ is isotopic to $\xi_2|_{M\setminus V}$ by an isotopy not necessarily fixed at the boundary, and $\xi_1|_V$ is isotopic to $\xi_2|_V$, then the contact structures ξ_1 and ξ_2 are isotopic.*

Proof Let ϕ_s be the isotopy of $M \setminus V$ such that ϕ_0 is the identity and $(\phi_1)_*(\xi_1|_{M\setminus V}) = \xi_2|_{M\setminus V}$. We can extend ϕ_s to $\widetilde{\phi}_s$ on all of M so that $\widetilde{\phi}_0$ is the identity on M and consider $(\widetilde{\phi}_1)_*(\xi_1)$. By construction, $(\widetilde{\phi}_1)_*(\xi_1|_{M\setminus V}) = \xi_2|_{M\setminus V}$, and by the classification of tight contact structures in [22], $(\widetilde{\phi}_1)_*(\xi_1|_V)$

is isotopic relative to the boundary to $\xi_2|_V$ because they have the same boundary slope and the same relative Euler class. Let ψ_s be an isotopy between them, and $\widetilde{\psi}_s$ its extension to M by putting it constantly equal to the identity outside V, then $\widetilde{\phi}_s \circ \widetilde{\psi}_s$ is an isotopy between ξ_1 and ξ_2. □

4.2 Tight contact structures with $t < 0$

In this section we present all tight contact structures ξ on $M(e_0, r)$ with $t(\xi) < 0$ as negative contact surgery on fillable contact structures on $T(e_0)$. This result is obtained by showing that the background of $(M(e_0, r), \xi)$ is contactomorphic to the complement of a standard neighbourhood of a vertical Legendrian curve in $T(e_0)$. For conciseness of notation, in the following we will often write M instead of $M(e_0, r)$.

Proposition 4.5 *The background $(M \setminus V, \xi|_{M\setminus V})$ of (M, ξ) with maximal twisting number $t(\xi) < 0$ and integer Euler number $e_0 = 0$ is contactomorphic to the complement of a standard neighbourhood of a vertical Legendrian curve with twisting number $t(\xi)$ in $(T^3, \xi_{(n,c_1,c_2,c_3)})$ for some $(n, c_1, c_2, c_3) \in \mathbb{N}^+ \times \mathbb{P}(H_2(T^3, \mathbb{Q}))$. Moreover, (n, c_1, c_2, c_3) is uniquely determined by the dividing sets of two non-isotopic, incompressible standard tori intersecting along a common vertical Legendrian ruling curve with twisting number $t(\xi)$.*

Proof We choose a vertical Legendrian curve R with twisting number $t(\xi)$ in $M \setminus V$, and two standard tori T_1 and T_2 intersecting along R as in the statement. Let n_i be the division numbers and let $\frac{p_i}{q_i}$ be the slope of T_i. These numbers satisfy the relations $-n_i q_i = t(\xi)$ for $i = 1, 2$ because $tb(R) = -\frac{1}{2}|R \cap \Gamma_{T_i}|$.

Take a small standard neighbourhood U of R such that $T_i \cap \partial U$ is Legendrian. After cutting $(M \setminus V \cup U)$ along the two annuli $T_i \setminus U$ and rounding the edges as shown in Figures 4.1 and 4.2, by [22], Lemma 3.11 we obtain a thickened torus $T^2 \times I$ with minimal boundary and boundary slopes $\frac{1}{t(\xi)}$. This thickened torus is nonrotative, otherwise an intermediate standard torus with slope $-r$ would produce an overtwisted disc. By [22], Lemma 5.7, up to an isotopy which fixes one boundary component, there is a unique nonrotative tight contact structure on $T^2 \times I$ with minimal boundary and boundary slopes $\frac{1}{t(\xi)}$, therefore there is at most one tight contact structure on $M \setminus V$ which induces on T_i a dividing set with division number n_i and slope $\frac{p_i}{q_i}$ for $i = 1, 2$.

Let $n = (n_1, n_2)$ be the greatest common divisor and set $c_1 = -\frac{n_1 p_1}{n}$, $c_2 = -\frac{n_2 p_2}{n}$ and $c_3 = -\frac{t(\xi)}{n}$. As their greatest common divisor is $(c_1, c_2, c_3) = 1$, we can complete $\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$ to a matrix

$$\Phi = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in SL(3, \mathbb{Z})$$

Fix coordinates (x, y, z) on T^3, and consider the contact structure $\xi_{(n,c_1,c_2,c_3)} = \Phi_*^{-1}\zeta_n$. We claim that $(M \setminus V, \xi|_{M \setminus V})$ is contactomorphic to the complement of a standard neighbourhood of a vertical Legendrian curve in $(T^3, \xi_{(n,c_1,c_2,c_3)})$. In order to prove the claim, it is enough to show that the linear torus $T_1 \subset (T^3, \xi_{(n,c_1,c_2,c_3)})$ generated by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has division number n_1 and slope $\frac{p_1}{q_1}$, and the linear torus $T_2 \subset (T^3, \xi_{(n,c_1,c_2,c_3)})$ generated by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has division number n_2 and slope $\frac{p_2}{q_2}$. Equivalently, we can work with the tori $A(T_i) \subset (T^3, \zeta_n)$ generated by $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}$ and $\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}$ for $i = 1, 2$. Since $c_i \neq 0$ for $i = 1, 2, 3$, there is a linear combination X of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ transverse to both $A(T_1)$ and $A(T_2)$. X is a contact vector field of (T^3, ζ_n) for each n, and the set $\Sigma = \{p \in T^3 \mid X(p) \in \zeta_n(p)\}$ consists of $2n$ parallel copies of a horizontal torus of the form $\{z \in \mathbb{Z}\}$.

The embeddings $\iota_i \colon T^2 \to T^3$ induced by the embeddings $\widetilde{\iota}_i \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\widetilde{\iota}_i(u, v) = \begin{pmatrix} ua_3 + va_i \\ ub_3 + vb_i \\ uc_3 + vc_i \end{pmatrix}$$

are parametrisations of $A(T_i)$, for $i = 1, 2$. The dividing set $\Gamma_{A(T_i)} = \Sigma \cap A(T_i)$ is the image of $2n$ parallel copies of the set

$$\{vc_i + uc_3 \in \mathbb{Z}\} = \bigcup_{j=0}^{n_i/n} \{-vp_i + uq_i \in \frac{jn}{n_i}\mathbb{Z}\},$$

which in turn consists of $\frac{n_i}{n}$ parallel copies of a curves with slope $\frac{p_i}{q_i}$, therefore the dividing set $\Gamma_{A(T_i)}$ is the same dividing set induced by $\xi|_{M \setminus V}$ on T_i. $\square$

Theorem 4.6 *Any tight contact structure ξ on $M(0,r)$ with $t(\xi) \in (-\frac{1}{r}, 0)$ is a negative contact surgery on a vertical Legendrian curve with twisting number $t(\xi)$ in $(T^3, \xi_{(n,c_1,c_2,c_3)})$ for some (n, c_1, c_2, c_3). Conversely, any contact structure $\xi_{(n,c_1,c_2,c_3)}(\eta)$ on $M(0,r)$ obtained by negative contact surgery on $(T^3, \xi_{(n,c_1,c_2,c_3)})$ is tight.*

Proof The first half of the theorem comes from the previous proposition and from $-\frac{1}{r} < t(\xi)$. All contact structures $\xi_{(n,c_1,c_2,c_3)}(\eta)$ obtained by negative contact surgery on $(T^3, \xi_{(n,c_1,c_2,c_3)})$ are tight because all tight contact structures on T^3 are weakly symplectically fillable by Corollary 3.9. □

Theorem 4.7 *Let $\xi_{(n,c_1,c_2,c_3)}(\eta)$ be a tight contact structure on $M(0,r)$ obtained by negative contact surgery on a vertical Legendrian curve in the tight contact manifold $(T^3, \xi_{(n,c_1,c_2,c_3)})$. Let $\pi^*\xi_{(n,c_1,c_2,c_3)}(\eta)$ be the contact structure on $M(0,r,\ldots,r)$ obtained as pull-back of $\xi_{(n,c_1,c_2,c_3)}(\eta)$ with respect to the finite covering*

$$\pi\colon\ M(0,r,\ldots,r) \to M(0,r)$$

induced by a finite covering of T^2. Then $\pi^*\xi_{(n,c_1,c_2,c_3)}(\eta)$ is tight.

Proof Let $\widetilde{\xi}_{(n,c_1,c_2,c_3)}$ be the pull-back of $\xi_{(n,c_1,c_2,c_3)}$ with respect to the the finite covering of T^3 induced by the finite covering of T^2. By construction, $\pi^*\xi_{(n,c_1,c_2,c_3)}(\eta)$ is the contact structure $\widetilde{\xi}_{(n,c_1,c_2,c_3)}(\eta,\ldots,\eta)$, obtained by negative contact surgery along a finite number of fibres of T^3.

The contact structure $\widetilde{\xi}_{(n,c_1,c_2,c_3)}$ is tight because all tight contact structures on T^3 are universally tight, so it is also weakly symplectically fillable by Corollary 3.9. The contact manifold $(M(0,r,\ldots,r), \widetilde{\xi}_{(n,c_1,c_2,c_3)}(\eta,\ldots,\eta))$ is obtained by negative contact surgery on a weakly symplectically fillable contact manifold, therefore it is tight. □

Proposition 4.8 *Let ξ be a tight contact structure with maximal twisting number $t(\xi) = -1$ on the Seifert manifold $M = M(e_0, r)$ with integer Euler number $e_0 < 0$. Then any background $(M \setminus V, \xi|_{M\setminus V})$ is contactomorphic to the complement of a standard neighbourhood of a vertical Legendrian curve with twisting number -1 in $(T(e_0), \xi_0)$, where ξ_0 is a tight contact structure with $t(\xi_0) = -1$.*

Proof Let $T \subset M \setminus V$ be a standard vertical torus so that the manifold $M \setminus (T \cup V)$ is diffeomorphic to $\Sigma_0 \times S^1$, where Σ_0 is a pair of pants. We can assume that T has vertical Legendrian ruling and its dividing set intersects the Legendrian ruling curves in two points. If this were not the case, an annulus A between a Legendrian ruling curve of T and a Legendrian ruling curve of $\partial(M \setminus V)$ would give a bypass along T by the Imbalance Principle [22] Proposition 3.17, therefore we could decrease the number of intersection points between the dividing set and the Legendrian ruling curves of T.

Let T_+ and T_- be the boundary tori of $\partial(M \setminus (V \cup T))$ corresponding to T. $\xi|_{M \setminus (V \cup T)}$ is a tight contact structure with boundary slopes 1 on $\partial(M \setminus V)$, n on T_+, and $-n + e_0$ on T_-. Since the sum of the slopes is $1 + e_0 \leq 0$ and there are no vertical Legendrian curves with twisting number 0, by [23], Lemma 5.1 case 4(b), there are $1 - e_0$ tight contact structures on $\Sigma_0 \times S^1$ with those boundary slopes. Such contact structures are constructed by removing a standard neighbourhood of a vertical Legendrian curve with twisting number -1 from a minimally twisting $T^2 \times I$ with boundary slopes $n - e_0$ and n. Note here the effect of the orientation reversing identification $T_- \cong T^2 \times \{0\}$ on the slope. We can also assume that the standard neighbourhood of the vertical Legendrian curve is removed from an invariant collar of the boundary.

To have M back from $M \setminus T$, we glue T_+ to $-T_-$ by the map $A(e_0) = \begin{pmatrix} 1 & 0 \\ -e_0 & 1 \end{pmatrix}$, therefore, by comparing with the construction in [23], section 2.5, case 9, $(M \setminus V, \xi_{M \setminus V})$ is the complement of a vertical Legendrian curve with twisting number -1 in a circle bundle over the torus with Euler class e_0 with a tight contact structure with maximal twisting number $t = -1$.

Given any slope $\overline{s} \in [s(-T_-), s(T_+)]$, we can find a convex torus $T' \subset M \setminus (T \cup V)$ with slope $\overline{s}$ such that T_- and T' bound a thickened torus $T^2 \times [0, \frac{1}{2}] \subset M \setminus (T \cup V)$. Choose $\overline{s} = n - 1$, then remove $T^2 \times [0, \frac{1}{2}]$ from $M \setminus T$ and glue it back with $A(e_0)$ to the front, so that $M \setminus T'$ has boundary slopes $n - e_0 - 1$ and $n - 1$. Here one component of $\partial(M \setminus T')$ is oriented with the outward normal and the other one with the inward normal. In a similar way we can replace n with $n + 1$, so we have proved that the tight contact structure on M does not depend on n. □

Conversely, given any tight contact structure ξ_n on $T(e_0)$ with $t(\xi_n) = -1$, for $n \in \mathbb{Z}/(1 - e_0)\mathbb{Z}$ any negative contact surgery $(M(e_0, r), \xi_n(\eta))$ is tight.

Theorem 4.9 *Let $e_0 < 0$. Any tight contact structure with $t = -1$ on $M(e_0, r)$ is negative contact surgery on a tight contact structure with $t = -1$ on $T(e_0)$. Conversely, given any tight contact structure ξ_n on $T(e_0)$ with*

$t(\xi_n) = -1$, for $n \in \mathbb{Z}/(1-e_0)\mathbb{Z}$ any negative contact surgery $(M(e_0, r), \xi_n(\eta))$ is tight.

Proof Any tight contact structure with $t = -1$ on $M(e_0, r)$ is negative contact surgery on a tight contact structure ξ_n with $t = -1$ on $T(e_0)$ because $-\frac{1}{r} < -1$. Conversely, any negative contact surgery on $(T(e_0), \xi_n)$ is tight by [6], Proposition 3 because $(T(e_0), \xi_n)$ is Stein fillable. $\square$

Theorem 4.10 *Let $\pi^*\xi_n(\eta)$ be the contact structure on $M(ke_0, r, \dots, r)$ obtained as pull-back of $\xi_n(\eta)$ with respect to a degree k finite covering*

$$\pi\colon\ M(ke_0, r, \dots, r) \to M(e_0, r)$$

induced by a covering of T^2. Then $\pi^*\xi_n(\eta)$ is tight.

Proof By construction, $\pi^*\xi_n(\eta) = \widetilde{\xi}_n(\eta, \dots, \eta)$, where $\widetilde{\xi}_n$ is the pull-back of ξ_n to $T(ke_0)$. By [23], Section 2.5, Case 9, $\widetilde{\xi}_n$ is a tight contact structure with maximal twisting number $t(\widetilde{\xi}_n) = -1$, hence it is Stein fillable by Theorem 3.6. The contact manifold $(M(ke_0, r, \dots, r), \widetilde{\xi}_n(\eta, \dots, \eta))$ is tight because it is obtained by negative contact surgery on the Stein fillable contact manifold $(T(ke_0), \widetilde{\xi}_n)$. $\square$

4.3 Tight contact structures with $t = 0$

In this subsection we construct all tight contact structures ξ on $M(e_0, r)$ with maximal twisting number $t(\xi) = 0$. By Lemma 4.1, there is a tubular neighbourhood V of the singular fibre such that $-\partial(M(e_0, r) \setminus V)$ is a convex torus with infinite slope. $M(e_0, r) \setminus V$ is diffeomorphic to $\Sigma \times S^1$, where Σ is a punctured torus. We will abusively identify Σ with the image of a section $\Sigma \to \Sigma \times S^1$ and assume it is convex with Legendrian boundary and $\#\Gamma$–minimising in its isotopy class.

The dividing set Γ_Σ of Σ consists of one arc with endpoints on $\partial\Sigma$ and some simple homotopically nontrivial closed curves.

Definition 4.11 We define an *abstract dividing set* on an oriented surface Σ as a multicurve Γ_Σ together with a map $\pi_0(\Sigma \setminus \Gamma_\Sigma) \to \{+, -\}$ such that any connected component of Γ_Σ belongs to the boundary of both a positive and a negative region. We say that an abstract dividing set is *tight* if its underlying multicurve does not have closed, homotopically trivial connected components. We say that it is *overtwisted* if it is not tight.

In the following, we will almost always use the same symbol for both an abstract dividing set and for its underlying multicurve. However, we will always specify what we are referring to, whenever it is relevant.

Definition 4.12 Given an abstract dividing set Γ_Σ on Σ, we denote by ξ_{Γ_Σ} the S^1–invariant contact structure on $\Sigma \times S^1$ which induces the dividing set Γ_Σ on a convex $\#\Gamma$–minimising section.

By Giroux's tightness criterion, [23] Lemma 4.2, ξ_{Γ_Σ} is tight (and in fact universally tight) if and only if Γ_Σ is a tight abstract dividing set. By [23], Section 4.3, $(M(e_0, r) \setminus V, \xi|_{M(e_0,r)\setminus V})$ is contactomorphic to an S^1–invariant tight contact manifold $(\Sigma \times S^1, \xi_{\Gamma_\Sigma})$. We call $\eta = \xi|_V$ and $\xi = \xi_{\Gamma_\Sigma}(\eta)$.

We recall that we have chosen the basis on $-\partial(M(e_0, r) \setminus V)$ so that $\partial\Sigma$ has slope e_0 and the fibres have infinite slope and the basis on $-\partial(\Sigma \times S^1)$ so that $\partial\Sigma$ has slope 0 and the fibres have infinite slope.

Proposition 4.13 *Let ξ be a tight contact structure on $M(e_0, r)$ with maximal twisting number $t(\xi) = 0$ and fix a diffeomorphism $M \setminus V \cong \Sigma \times S^1$ so that Σ is $\#\Gamma$–minimising. If $e_0 \leq 0$, then Γ_Σ has no boundary parallel dividing curves. If $e_0 > 0$ and Γ_Σ has a boundary parallel dividing curve, then $\#\Gamma = 1$.*

Proof If Γ contains a boundary parallel dividing arc, then there is a singular bypass on Σ by [22], Proposition 3.18. By [22], Lemma 3.15, attaching this bypass to $-\partial(M \setminus V)$ we thicken V to V' so that $-\partial(M \setminus V')$ has slope e_0. If $\#\Gamma \geq 2$, and $p \in \Sigma$ belongs to some other dividing curve, then $\{p\} \times S^1$ is a Legendrian fibre with twisting number 0 because $\xi|_{M\setminus V}$ is S^1–invariant by [23], section 4.3. Applying the Imbalance principle, [22], Proposition 3.17, we use this curve to find a vertical bypass attached to $\partial(M \setminus V')$. The attachment of this bypass gives a further thickening of V' to V'' so that $-\partial(M \setminus V'')$ has infinite boundary slope again. By [22], Proposition 4.16, there is a standard torus with slope $-r$ in $V'' \setminus V$. This torus produces an overtwisted disc.

If $\#\Gamma = 1$, we pick a simple closed curve $C \subset \Sigma \setminus V'$ which does not disconnect Σ and is disjoint from the dividing curve. By the Legendrian Realization Principle, [22], Theorem 3.7, we can arrange the characteristic foliation on Σ so that C is a closed leaf. Because of the S^1–invariance of $\xi|_{M\setminus V}$, $C \times S^1$ is a pre-Lagrangian torus with slope 0. By Lemma 3.4 we can perturb this torus in order to obtain a convex torus T in standard form with slope 0 and two dividing curves. The torus T can be assumed to be disjoint from V' because C is disjoint from the boundary parallel dividing arc producing the bypass.

If we cut $M \setminus V'$ open along T, we obtain $\Sigma_0 \times S^1$, where Σ_0 is a pair of pants, and all the three boundary tori have slope 0 calculated with respect to the product structure on $\Sigma_0 \times S^1$. Let $T_\pm$ be the two boundary tori corresponding to T, and take a convex vertical annulus A with Legendrian boundary between T_+ and T_-. If the dividing curves on A do not go from T_+ to T_-, then there is a vertical bypass along T. The attachment of this bypass produces a torus T' with infinite slope. Using a vertical Legendrian divide of T' we can thicken V' to V'' so that $-\partial(M \setminus V'')$ has infinite slope again, thus obtaining a standard torus with slope $-r$ in $V'' \setminus V$. Again, this torus produces an overtwisted disc. If the dividing curves on A go from one boundary component to the other, then, after cutting along A and rounding the edges, by [22], Lemma 3.11 we obtain a torus with slope -1 parallel to $-\partial(\Sigma \times S^1)$ which has slope $e_0 - 1$ calculated with respect to the basis of $-\partial(M \setminus V)$. If $e_0 \leq 0$, by [22], Proposition 4.16, there is a convex torus with slope $-r$ parallel to $-\partial(M \setminus V)$ which gives an overtwisted disc. □

Proposition 4.14 *Let Σ be a punctured torus and Γ_Σ a tight abstract dividing set on Σ without boundary parallel dividing arcs. Then $(\Sigma \times S^1, \xi_{\Gamma_\Sigma})$ can be contact embedded into a tight contact manifold $(T(e_0), \xi_{\Gamma_\Sigma})$ as the complement of a standard neighbourhood of a vertical Legendrian curve with twisting number 0.*

Proof Take a curve $C \subset \Sigma$ so that C intersect each dividing arc in one single point. If we make C Legendrian using the Legendrian realisation principle [22] Corollary 3.8, the torus $T = C \times S^1$ is in standard form with infinite slope because ξ_{Γ_Σ} is S^1–invariant. The contact structure ξ_{Γ_Σ} restricted to $\Sigma \times S^1 \setminus T$ is still S^1–invariant and $\Gamma_{\Sigma \setminus C} = \Gamma_\Sigma \setminus C$ is a $\#\Gamma$–minimising section of $\Sigma \times S^1 \setminus T$. Let S be the surface diffeomorphic to an annulus obtained by gluing a disc D to the boundary component of $\Sigma \setminus C$ corresponding to $\partial\Sigma$, and let Γ_S be the natural extension of $\Gamma_{\Sigma \setminus C}$ to an abstract dividing set on S. The S^1–invariant tight contact manifold $(S \times S^1, \xi_{\Gamma_S})$ is contactomorphic to an I–invariant tight contact structure on $T^2 \times I$ by [22], Theorem 2.3(4) because Γ_S consists of parallel arcs joining the two different boundary components of S. The S^1–invariant contact manifold $(D \times S^1, \xi_{\Gamma_D})$ is a tight solid torus with infinite boundary slope and $\#\Gamma_{\partial D \times S^1} = 2$. By [22] Theorem 2.3 there is a unique tight contact structure with such boundary conditions on the solid torus, therefore it is contactomorphic to a standard neighbourhood of a Legendrian curve with twisting number 0. Gluing $T^2 \times \{0\}$ to $T^2 \times \{1\}$ with the matrix $\begin{pmatrix} 1 & 0 \\ -e_0 & 1 \end{pmatrix}$ we get a tight contact structure on $T(e_0)$ which we call ξ_{Γ_Σ} again, then $(\Sigma \times S^1, \xi_{\Gamma_\Sigma})$ contact embeds in $(T(e_0), \xi_{\Gamma_\Sigma})$ as the complement of a vertical Legendrian curve with twisting number 0. □

Theorem 4.15 *let Γ_Σ be a tight abstract dividing set on a punctured torus Σ without boundary parallel dividing arcs, and let $\nu L \subset (T(e_0), \xi_{\Gamma_\Sigma})$ be a standard neighbourhood of a vertical Legendrian curve L with twisting number 0. Then, for any tight contact structure η on $D^2 \times S^1$ whose characteristic foliation on $\partial D^2 \times \Sigma$ is mapped by*

$$A(r)\colon\ \partial(D^2 \times S^1) \to -\partial(T(e_0) \setminus \nu L)$$

to the characteristic foliation of $-\partial(T(e_0) \setminus \nu L)$, the contact structure $\xi_{\Gamma_\Sigma}(\eta)$ on $M(e_0, r)$ is tight.

Proof By Proposition 4.14, the contact manifold $(M, \xi_{\Gamma_\Sigma}(\eta))$ is obtained by negative contact surgery on $(T(e_0), \xi_{\Gamma_\Sigma})$, which is a weakly symplectically fillable contact manifold by Theorem 3.14 because it is universally tight by S^1–invariance. □

Proposition 4.16 *Let Γ_Σ^+ and Γ_Σ^- be the two tight abstract dividing sets on the punctured torus Σ with underlying multicurve Γ_Σ with no boundary parallel dividing arcs. Then, for any tight contact structure η on $D^2 \times S^1$ as in Theorem 4.15, $(M(e_0, r), \xi_{\Gamma_\Sigma^+}(\eta))$ is isotopic to $(M(e_0, r), \xi_{\Gamma_\Sigma^-}(\eta))$.*

Most of the proof of Proposition 4.16 relies on the following lemma.

Lemma 4.17 *Let Γ_{T^2} be a tight abstract dividing set on T^2, and γ_1, $\gamma_2 \subset T^2$ dividing curves bounding a negative (positive) region $C \subset T^2$. Given points $p_i \in \gamma_i$, for $i = 1, 2$, the curves $\{p_i\} \times S^1$ in $(T^3, \xi_{\Gamma_{T^2}})$ are Legendrian and have twisting number $tb(\{p_i\} \times S^1) = 0$. If L_1 and L_2 are positive (negative) stabilisations of $\{p_1\} \times S^1$ and $\{p_2\} \times S^1$ respectively, then they are contact isotopic.*

Proof The curves $\{p_i\} \times S^1$ are Legendrian because $(T^3, \xi_{\Gamma_{T^2}})$ is S^1–invariant.

Let $(T^2 \times [0, \frac{1}{2}], \xi)$ be a positive (negative) basic slice with standard boundary and boundary slopes $s_0 = 0$ and $s_{\frac{1}{2}} = \infty$, contact embedded in $(T^3, \xi_{\Gamma_{T^2}})$ so that $\{p_1\} \times S^1$ is a Legendrian divide of $T^2 \times \{\frac{1}{2}\}$ and $T^2 \times \{0\} \subset C \times S^1$. Make the Legendrian ruling of $T^2 \times \{0\}$ vertical, and consider a convex vertical annulus A between $\{p_1\} \times S^1 \subset T^2 \times \{\frac{1}{2}\}$, and a vertical Legendrian ruling curve of $T^2 \times \{0\}$. The dividing set of A consists of a single dividing arc with both endpoints on $T^2 \times \{0\}$, and the simply connected region of $A \setminus \Gamma_A$ is positive (negative). Then, by [9], Lemma 2.20, a vertical Legendrian ruling curve of $T^2 \times \{0\}$ is a positive (negative) stabilisation of $\{p_1\} \times S^1$. From the well-definedness up to isotopy of the stabilisation, it follows that L_1 is contact isotopic to a vertical Legendrian ruling curve of $T^2 \times \{0\}$. We can repeat the same argument with a basic slice ¡$(T^2 \times [0, \frac{1}{2}], \xi)$ with the same sign and the same boundary slopes so that $\{p_2\} \times S^1$ is a Legendrian divide of $T^2 \times \{\frac{1}{2}\}$, and conclude that L_2 is contact isotopic to a vertical Legendrian ruling of $T^2 \times \{0\}$. Then L_1 and L_2 are Legendrian isotopic. □

Proof of Proposition 4.16 Let $V \subset M(e_0, r)$ be a tubular neighbourhood of the singular fibre such that $(M(e_0, r) \setminus V, \xi_{\Gamma_\Sigma^\pm}(\eta)|_{M(e_0,r)\setminus V}) = (\Sigma \times S^1, \xi_{\Gamma_\Sigma^\pm})$, and $(V, \xi_{\Gamma_\Sigma^\pm}(\eta)|_V)$ is contactomorphic to $(D^2 \times S^1, \eta)$. We can find a solid torus $V' \subset V$ such that $-\partial(M(e_0, r) \setminus V')$ has slope -1 because $-\frac{1}{r} < -1$. By proposition 4.14, $(\Sigma \times S^1, \xi_{\Gamma_\Sigma^\pm})$ is the complement of a vertical Legendrian curve $\{p_1\} \times S^1$ with twisting number 0 with $p_1 \in \gamma_1$, where γ_1 is the completion in T^2 of the dividing arc in Σ. Analogously, $(M(e_0, r) \setminus V', \xi_{\Gamma_\Sigma^\pm}(\eta)|_{M(e_0,r)\setminus V'}) = (M(e_0, r) \setminus V', \xi_\pm)$ is the complement of a vertical Legendrian curve L_1 with twisting number -1 which is a stabilisation of $\{p_1\} \times S^1$ by [9], Lemma 2.20. The sign of the stabilisation is determined by the sign of the basic slice $V \setminus V'$. In order to fix the notation, let us suppose it is positive. We claim that $(M(e_0, r) \setminus V', \xi_+)$ is contact isotopic to $(M(e_0, r) \setminus V', \xi_-)$. Since our argument will be semi-local, we can assume without loss of generality that $T(e_0)$ is a trivial S^1–bundle.

Let γ_2 be another dividing curve on T^2 such that γ_1 and γ_2 bound a positive region. Choose a point $p_2 \in \gamma_2$ and consider the vertical Legendrian curve $\{p_2\} \times S^1$ and its positive stabilisation L_2. Then the complement of a standard neighbourhood of L_2 in $(T(e_0), \xi_{\Gamma_\Sigma^+})$ is contactomorphic to $(M(e_0, r) \setminus V', \xi_-)$. By Lemma 4.17, L_1 and L_2 are Legendrian isotopic, therefore, by [9], Theorem 2.12, there is a contact isotopy $\varphi_t\colon (T(e_0), \xi_{\Gamma_\Sigma^+}) \to (T(e_0), \xi_{\Gamma_\Sigma^+})$ such that $\varphi_0 = id$ and $\varphi_1(L_1) = L_2$. This implies that $\varphi_1(M(e_0, r) \setminus V')$ is the complement of a standard neighbourhood of L_2, so $(\varphi_t|_{M(e_0,r)\setminus V'})^* \xi_{\Gamma_\Sigma^+}$ is a 1–parameter family of contact structures on $M(e_0, r) \setminus V'$ all with the same boundary condition joining ξ_+ to ξ_-. By Gray's Theorem, this implies that $(M(e_0, r) \setminus V', \xi_+)$ and $(M(e_0, r) \setminus V', \xi_-)$ are contact isotopic. □

Theorem 4.18 *Let $\pi^* \xi_{\Gamma_\Sigma}(\eta)$ be the contact structure on $M(ke_0, r, \dots, r)$ obtained by pull-back of $\xi_{\Gamma_\Sigma}(\eta)$ with respect to a degree k finite covering*

$$\pi\colon M(ke_0, r \dots, r) \to M(e_0, r)$$

induced by a covering of T^2. Then $\pi^*\xi_{\Gamma_\Sigma}(\eta)$ is tight when $\#\Gamma_\Sigma > 1$ and is overtwisted when $\#\Gamma_\Sigma = 1$.

Proof By construction, $\pi^*\xi_{\Gamma_\Sigma}(\eta) = \xi_{\Gamma_{\widetilde{\Sigma}}}(\eta, \dots, \eta)$, where $\Gamma_{\widetilde{\Sigma}}$ is the pull-back of Γ_Σ with respect to the finite cover of T^2 (we remind that the inclusion $\iota\colon \Sigma \to T^2$ can be lifted to an inclusion $\widetilde{\iota}\colon \widetilde{\Sigma} \to T^2$). If $\#\Gamma_\Sigma > 1$, then Γ_Σ contains no boundary parallel arcs. In this case, $\Gamma_{\widetilde{\Sigma}}$ does not contain boundary parallel dividing curves either, then the contact manifold $(M(ke_0, r, \dots, r), \xi_{\Gamma_{\widetilde{\Sigma}}}(\eta, \dots, \eta))$ is obtained by negative contact surgery on the weakly symplectically fillable contact manifold $(T(ke_0), \xi_{\Gamma_{\widetilde{\Sigma}}})$, therefore $\xi_{\Gamma_{\widetilde{\Sigma}}}(\eta, \dots, \eta))$ is tight. If $\#\Gamma_\Sigma = 1$, then $\Gamma_{\widetilde{\Sigma}}$ consists of k boundary parallel arcs, one for each boundary component of $\widetilde{\Sigma}$. When $\#\Gamma > 1$ a boundary parallel dividing arc produces an overtwisted disc as in the proof of Proposition 4.13, therefore $\xi_{\Gamma_{\widetilde{\Sigma}}}(\eta, \dots, \eta))$ is overtwisted. □

5 Classification of the generic tight contact structures

5.1 Tight contact structures with $t < 0$

Theorem 5.1 *Let $e_0 < 0$. The tight contact manifolds $(M(e_0, r), \xi_n(\eta))$ and $(M(e_0, r), \xi_m(\eta'))$ obtained by negative contact surgery on $(T(e_0), \xi_n)$ and $(T(e_0), \xi_m)$ respectively are isotopic if and only if $m = n$ and η is isotopic to η' relative to the boundary.*

Proof By Theorem 3.5 there are $|e_0 - 1|$ choices for the background and by Lemma 2.4, there are $|(d_0 + 1) \dots (d_k + 1)|$ choices for η. On the other hand, Theorem 5.4 in [20] shows how to produce Stein fillings (W, J) for $M(e_0, r)$. As a smooth manifold, W is the same for all the Stein fillings, and choosing all possible rotation numbers in the Legendrian realisation of the surgery link presenting W, by [20], Proposition 2.3, we obtain $|(e_0 - 1)(d_0 + 1) \dots (d_0 + k)|$ Stein structures on W with different $c_1(J)$. By [29], Corollary 4.2, these Stein structures induce $|(e_0 - 1)(d_0 + 1) \dots (d_0 + k)|$ mutually non isotopic tight tight contact structures on $T(e_0)$. □

Now we turn our attention to the tight contact structures on $M(0, r)$ with $t < 0$. The result proved here is a generalisation of Theorem 4.7 in [25] for the part concerning the distinction of the tight contact structures obtained by negative contact surgery on T^3. We start with a preliminary digression about negative contact surgery on nonrotative tight contact structures on $T^2 \times I$. We fix a S^1–bundle structure $T^2 \times I \to S^1 \times I$ and, consequently, a Seifert fibration $M' \to S^1 \times I$ with one singular fibre F on any manifold M' obtained by surgery along a fibre of $T^2 \times I$.

Proposition 5.2 *Let ξ be a nonrotative tight contact structure on $T^2 \times I$ and r a rational number such that $-\frac{1}{r} < t(\xi)$. Let $L \subset (T^2 \times I, \xi)$ be a vertical Legendrian curve with twisting number $tb(L) = t(\xi)$ and $(M', \xi(\eta))$ a contact manifold obtained from $(T^2 \times I, \xi)$ by contact surgery along L with surgery coefficient $-\frac{1}{r}$ with respect to the canonical framing. Then $\xi(\eta)$ is tight and any two properly embedded convex vertical annuli A_0, A_1 with common Legendrian boundary are contact isotopic, possibly after perturbing the characteristic foliation of A_1. In particular, Γ_{A_0} is isotopic to Γ_{A_1} and $\xi(\eta)|_{M' \setminus A_i}$ is isotopic to η for $i = 0, 1$.*

Proof The contact structure $\xi(\eta)$ is tight by [6], Proposition 3 because $(T^2 \times I, \xi)$ can be contact embedded into a weakly symplectically fillable contact manifold.

First we prove that, if A_0 and A_1 are disjoint from the surgery support, then Γ_{A_0} is isotopic to Γ_{A_1}. In this case, we can think of A_0 and A_1 also as convex annuli in $(T^2 \times I, \xi)$. Passing to a finite covering in the horizontal direction, we can assume that ξ is nonrotative with integer boundary slopes, therefore, by [22], Lemma 5.7, Γ_{A_0} is isotopic to Γ_{A_1}.

Now we turn to the proof of the general case. We can assume without loss of generality that one of the two annuli, say A_0, is disjoint from the surgery support. If this is not the case, we introduce a third annulus A_2 disjoint from the surgery support and isotope A_0 to A_2 first, and then isotope A_2 to A_1. By Isotopy discretisation [25], Lemma 3.10, there is a sequence of convex vertical annuli $A_0, \dots, A_{\frac{i}{n}}, \dots, A_{\frac{n}{n}} = A_1$ all with the same Legendrian boundary such that $A_{\frac{i+1}{n}}$ is obtained from $A_{\frac{i}{n}}$ by attaching a single bypass. By the following Lemma 5.3, if $A_{\frac{i}{n}}$ is disjoint from the surgery support, we can find another contact surgery presentation of $(M', \xi(\eta))$ with surgery support disjoint from both $A_{\frac{i}{n}}$ and $A_{\frac{i+1}{n}}$. Once $A_{\frac{i}{n}}$ and $A_{\frac{i+1}{n}}$ are disjoint from the surgery support, we can conclude that $\Gamma_{A_{\frac{i}{n}}}$ is isotopic to $\Gamma_{A_{\frac{i+1}{n}}}$, so the bypass between $A_{\frac{i}{n}}$ and $A_{\frac{i+1}{n}}$ is trivial. By the triviality of trivial bypass attachments, [25], Lemma 2.10, $\xi(\eta)$ restricted to the layer between $A_{\frac{i}{n}}$ and $A_{\frac{i+1}{n}}$ is invariant, therefore $A_{\frac{i}{n}}$ and $A_{\frac{i+1}{n}}$ are contact isotopic, possibly after perturbing the characteristic foliation of $A_{\frac{i+1}{n}}$. □

Lemma 5.3 *Let A_0 and A_1 be convex annuli in M' as in the statement of Proposition 5.2 such that they intersect only at the boundary. Also assume that A_0 is disjoint from the surgery support V. Then we can find another contact surgery presentation of $(M', \xi(\eta))$ such that the surgery support is disjoint from both A_0 and A_1.*

Proof Let N be the component of $M' \setminus (A_0 \cup A_1)$ homeomorphic to $D^2 \times S^1$. By the monotonicity of the slope, [22], Proposition 4.16, $M' \setminus N$ has boundary slope $\frac{p}{q} \in [\frac{1}{t(\xi)}, -r)$ because $M' \setminus N \subset M' \setminus A_0$. Let γ be a vertical Legendrian curve contained in $\partial(M' \setminus N)$, then $q \leq \frac{1}{2}\#(\gamma \cap \Gamma_{\partial(M' \setminus N)})$. In particular, if we take $\gamma \subset A_0$, then $\frac{1}{2}\#(\gamma \cap \Gamma_{\partial(M' \setminus N)}) = -t(\xi)$, so $q \leq -t(\xi)$. This is possible only if $\frac{p}{q} = \frac{1}{t(\xi)}$, therefore there is a solid torus $V' \subset M' \setminus N$ with convex boundary with slope $-\frac{1}{t(\xi)}$ such that $(M' \setminus V', \xi(\eta)|_{M' \setminus V'})$ is contactomorphic to the complement of a standard neighbourhood of a vertical Legendrian curve L' with twisting number $tb(L') = t(\xi)$ in $(T^2 \times I, \xi)$. In fact, we can identify $M' \setminus V'$ and $T^2 \times I \setminus \nu L$ so that ∂V corresponds to $\partial(\nu L)$. Then, both $\xi(\eta)|_{M' \setminus V'}$ and $\xi|_{T^2 \times I \setminus \nu L}$ have the same boundary slopes, induce the same dividing set on A_0 and, after cutting along A_0 and rounding the edges, yield a nonrotative tight contact structure on $T^2 \times I$ with slope $\frac{1}{t(\xi)}$. By [22], Lemma 5.7, there is only one such tight contact structure on $T^2 \times I$ up to contactomorphism, therefore $\xi(\eta)|_{M' \setminus V'}$ and $\xi|_{T^2 \times I \setminus \nu L}$ are contactomorphic. □

Let $T_1,\ T_2 \subset (M(e_0, r), \xi_{(n,c_1,c_2,c_3)}(\eta))$ be convex tori in the direction (x, z) and (y, z) disjoint from the surgery support V such that their intersection is a common vertical Legendrian ruling curve R with twisting number t: see Figure 4.1. Let s_i be the slope of T_i and n_i its division number. The fact that $R = T_1 \cap T_2$ is a common Legendrian ruling curve implies that the intersection between R and Γ_{T_i} is minimal for both $i = 1, 2$. Let $U \subset V$ be a solid torus such that $-\partial(M \setminus U)$ is convex and has slope $\frac{1}{d_0+1}$, where $d_0 = [-\frac{1}{r}]$.

Proposition 5.4 *Let $T' \subset M = M(e_0, r)$ be a standard torus isotopic to T_1 with vertical Legendrian ruling. Then:*

(1) *T' has slope s_1 and division number $n' \geq n_1$.*

(2) *Any convex torus T'' intersecting T' in a vertical Legendrian ruling curve of T' is contact isotopic to T_2, possibly after perturbing its characteristic foliation.*

Proof To simplify the notation, in the proof we will fix $\xi = \xi_{(n,c_1,c_2,c_3)}(\eta)$. By Isotopy Discretisation [25] Lemma 3.10, there is a sequence of convex tori

$T_1 = T_{(1)}, \dots, T_{(n)} = T'$ such that $T_{(i+1)}$ is obtained from $T_{(i)}$ by attaching a bypass. In particular, $T_{(i+1)}$ and $T_{(i)}$ bound N_i diffeomorphic to $T^2 \times I$. We can assume inductively that $T_{(i)}$ satisfies:

(1) $s(T_{(i)}) = s_1$ and $\mathrm{div}(T_{(i)}) \geq n_1$.

(2) There is a solid torus $U_i \subset M \setminus T_{(i)}$ isotopic in M to U such that $-\partial(M \setminus U_i)$ is convex with slope $\frac{1}{d_0+1}$.

(3) $\xi|_{U_i}$ is isotopic to $\xi|_U$ and $\xi|_{M \setminus U_i}$ is isotopic to $\xi|_{M\setminus U}$.

(4) There is a convex vertical annulus $A_i \subset M \setminus T_{(i)}$ with Legendrian boundary on $\partial(M \setminus T_{(i)})$ such that A closes to a convex torus $\overline{A}_i \subset M$ contact isotopic to T_2.

We observe that the inductive hypotheses are satisfied for T_1 taking as A_0 the annulus obtained by cutting T_2 open along $R = T_1 \cap T_2$, and $U_0 = U$. Assumptions 2 and 3 imply that $(M \setminus T_i, \xi|_{M \setminus T_i})$ is negative contact surgery along a vertical Legendrian curve in $T^2 \times I$ with a nonrotative tight contact structure, therefore all vertical annuli as in assumption 4 have the same dividing set by Proposition 5.2.

A priori there are three kinds of transitions from $T_{(i)}$ to $T_{(i+1)}$:

(1) $\mathrm{div}(T_{(i)}) = \mathrm{div}(T_{(i+1)}) = 1$ and $s(T_{(i+1)}) \neq s_1$

(2) $s(T_{(i+1)}) = s_1$ and $\mathrm{div}(T_{(i+1)}) = \mathrm{div}(T_{(i)}) + 1$

(3) $s(T_{(i+1)}) = s_1$ and $\mathrm{div}(T_{(i+1)}) = \mathrm{div}(T_{(i)}) - 1$

Case 1 We will prove that there are no transitions which change the slope. Suppose by contradiction that $\mathrm{div}(T_{(i)}) = \mathrm{div}(T_{(i+1)}) = 1$ and $s(T_{(i+1)}) = s_1' \neq s_1$. We can assume either that the bypass is attached to $T_{(i)}$ from the front and $s_1' < s_1$, or that the bypass is attached to $T_{(i)}$ from the back and $s_1 < s_1'$. We describe only the first possibility because the second one is symmetric.

Attaching bypasses coming from a convex vertical annulus with Legendrian boundary $S \subset M \setminus N_i$ as long as the Imbalance Principle can be applied, we eventually obtain tori $T_{(i)}'$ and $T_{(i+1)}'$ bounding $N_i' \supset N_i$. The tori $T_{(i)}'$ and $T_{(i+1)}'$ have either infinite slope or have slopes $s(T_{(i)}') = \frac{p}{q} > s(T_{(i+1)}') = \frac{p'}{q}$ and a convex vertical annulus with Legendrian boundary $S' \subset M \setminus N_i'$ between $T_{(i)}'$ and $T_{(i+1)}$ contains no more boundary parallel dividing arcs. In the first case we have a vertical Legendrian curve with twisting number 0 in $M \setminus (T_{(i)} \cup A_i)$. This is excluded by the classification of tight contact structures on solid tori because, by the inductive hypothesis, there is no such curve either in $(U_i, \xi|_{U_i})$ or in $(M \setminus (T_{(i)} \cup A_i \cup U_i),\ \xi|_{M \setminus (T_{(i)} \cup A_i \cup U_i)})$. In the second case, after cutting along S and rounding the edges, we get slope $\frac{p-p'-1}{q} \geq 0$ on $\partial(M \setminus (N_i' \cup S)) \subset M \setminus (T_{(i)} \cup A_i)$. This is also impossible because $M \setminus (T_{(i)} \cup A_i)$ has meridional slope $-r < 0$, and the existence of a torus with non negative slope contained in $M \setminus (T_{(i)} \cup A_i)$ would imply the existence of a vertical Legendrian curve with twisting number 0.

Case 2 Now we consider transitions which increase the division number. The main point here is to show that the surgery support can be assumed to be disjoint from the transition. Take convex vertical annuli $A_i' \subset N_i$ and $A_i'' \subset M \setminus N_i$ with common Legendrian boundary and call $B_i = A_i' \cup A_i'' \subset M \setminus T_{(i)}$. By Proposition 5.2, A_i is contact isotopic to B_i and $\xi|_{M \setminus (T_{(i)} \cup A_i)}$ is isotopic to $\xi|_{M \setminus (T_{(i)} \cup B_i)}$. If we set $A_{i+1} = A_i'' \cup A_i' \subset M \setminus T_{(i+1)}$, then $\overline{A}_i$ is contact isotopic to $\overline{A}_{i+1}$. The solid torus obtained by rounding the edges of $M \setminus (T_{(i)} \cup B_i)$ has boundary slope $-\frac{1}{k_i}$ for some positive integer $k_i < \frac{1}{r}$ and the solid torus obtained by rounding the edges of $M \setminus (N_i \cup A_i'')$ has boundary slope $-\frac{1}{k_i'} \in [-\frac{1}{k_i}, -r)$ because $M \setminus (T_{(i)} \cup B_i)$ has meridional slope $-r$. In $M \setminus (N_i \cup A_i'')$ there is a solid torus U_{i+1} such that $-\partial(M \setminus U_{i+1})$ is a convex torus with slope $\frac{1}{d_o+1}$ because $\frac{1}{d_o+1} \in [-\frac{1}{k_i'}, -r)$. Applying Proposition 6.5 to $M \setminus (T_{(i)} \cup B_i)$, we conclude that $\xi|_{U_i}$ is isotopic to $\xi|_{U_{i+1}}$ because slope $\frac{1}{d_o+1}$ is a border between continued fraction blocks in $M \setminus (T_{(i)} \cup B_i)$, and no shuffling can occur between the signs of basic slices belonging to different continued fraction blocks. For the same reason $\xi|_{M \setminus (T_{(i)} \cup A_i \cup U_i)}$ is isotopic to $\xi|_{M \setminus (T_{(i)} \cup B_i \cup U_{i+1})}$, then we can conclude that $\xi|_{M \setminus U_i}$ is isotopic to $\xi|_{M \setminus U_{i+1}}$.

Case 3 Transitions which decrease the division number can be handled in the same way as transitions which increase it, we need only to show that no transition can decrease $\operatorname{div}(T_{(i)})$ below n_1. Suppose that, on the contrary, $\operatorname{div}(T_{(i)}) = n_1$ and $\operatorname{div}(T_{(i+1)}) = n_1 - 1$ and take convex vertical annuli $A_i' \subset N_i$ and $A_i'' \subset M \setminus N_i$ with common Legendrian boundary. The dividing set of $B_i = A_i' \cup A_i'' \subset M \setminus T_{(i)}$ has at least one boundary parallel dividing arc, but the same total number of dividing arcs as A_0. This is a contradiction because, by Proposition 5.2 and the inductive hypothesis, the dividing set $\Gamma_{\overline{B}_i}$ on the torus $\overline{B}_i$ obtained by gluing the boundary components of B_i is isotopic to $\Gamma_{\overline{A}_0} = \Gamma_{T_2}$ and Γ_{A_0} contains no boundary parallel dividing arcs.

It remains to prove that any convex torus T'' isotopic to T_2 and intersecting T' in a Legendrian ruling curve of T' is contact isotopic to T_2. The Legendrian curves $T' \cap \overline{A}_n$ and $T' \cap T''$ are Legendrian isotopic because they are both Legendrian ruling curves of T'. Let $\varphi_t\colon M \to M$ be a contact isotopy extending the Legendrian isotopy between $T' \cap T''$ and $T' \cap \overline{A}_n$, so that $T' \cap \overline{A}_n = T' \cap \varphi_1(T'')$. By Proposition 5.2, $A_n = \overline{A}_n \setminus T'$ is contact isotopic to $\varphi_1(T'') \setminus T'$, therefore the proof is finished because $\overline{A}_n$ is contact isotopic to T_2. □

Theorem 5.5 *Let $\xi_{(n,c_1,c_2,c_3)}$ and $\xi_{(n',c_1',c_2',c_3')}$ be tight contact structures over T^3 with $c_3 \neq 0$ and $c_3' \neq 0$. Then the tight contact structures $\xi_{(n,c_1,c_2,c_3)}(\eta)$ and $\xi_{(n',c_1',c_2',c_3')}(\eta')$ over $M(0,r)$ constructed by negative contact surgery are isotopic if and only if $n = n'$, $(c_1, c_2, c_3) = \pm(c_1', c_2', c_3')$ and η is isotopic to η'.*

Proof Suppose $\xi_{(n,c_1,c_2,c_3)}(\eta)$ is isotopic to $\xi_{(n',c_1',c_2',c_3')}(\eta')$ and call it ξ. Because of the presentation of ξ as $\xi_{(n,c_1,c_2,c_3)}(\eta)$, in $M = M(0,r)$ we find a solid torus $V \subset M$ so that $-\partial(M \setminus V)$ is convex with slope $\frac{1}{t} = -\frac{1}{|nc_3|}$, $\xi|_V = \eta$ and $M \setminus V$ can be contact embedded in $(T^3, \xi_{(n,c_1,c_2,c_3)})$ as the complement of a vertical Legendrian curve. In $M \setminus V$ we choose tori T_1 and T_2 with slope $s(T_i) = \frac{c_i}{c_3}$ and division number $\mathrm{div}(T_i) = n(c_i, c_3)$ respectively intersecting along a common vertical Legendrian ruling curve. See the proof of Proposition 4.5 for details. Similarly, because of the the presentation of ξ as $\xi_{(n',c_1',c_2',c_3')}(\eta')$, in $M = M(0,r)$ we find tori T_1' and T_2' with slope $s(T_i') = \frac{c_i'}{c_3'}$ and division number $\mathrm{div}(T_i') = n'(c_i', c_3')$ respectively and a solid torus $V' \subset M \setminus (T_1' \cup T_2')$ such that $-\partial(M \setminus V')$ is convex with slope $\frac{1}{t'} = -\frac{1}{|n'c_3'|}$ and $\xi|_V' = \eta'$.

Proposition 5.4, implies that Γ_{T_i} is isotopic to $\Gamma_{T_i'}$ for $i = 1, 2$, therefore $n = n'$ and $(c_1, c_2, c_3) = \pm(c_1', c_2', c_3')$. Applying Proposition 5.4 to the first cut, and Proposition 5.2 to the second one, we prove that $\eta \cong \xi|_{M \setminus (T_1 \cup T_2)}$ is isotopic to $\eta' \cong \xi|_{M \setminus (T_1' \cup T_2')}$, thus concluding the proof. □

5.2 Tight contact structures with $t = 0$

The aim of this section is the classification of the tight contact structures on $M(e_0, r)$ constructed by negative contact surgery on a vertical Legendrian curve in a tight, S^1–invariant contact structure on $T(e_0)$.

Given two multicurves Γ and Γ' on a surface Σ, we say that they are *diffeomorphic* if there exists a diffeomorphism $\phi\colon \Sigma \to \Sigma$ such that $\phi|_{\partial\Sigma} = id$, and $\phi(\Gamma) = \Gamma'$. If we consider Γ and Γ' as abstract dividing sets, we require in addition that ϕ maps positive regions to positive regions and negative regions to negative regions.

Let Σ_0 be a pair of pants with $\partial\Sigma_0 = C_0 \cup C_1 \cup C_2$, and let Γ_{Σ_0} be an abstract dividing set on Σ_0 with $\#\Gamma_{\Sigma_0} \cap C_i \neq \emptyset$ for $i = 0, 1, 2$. If $\#\Gamma_{\Sigma_0} \cap C_2 = 2$, there is a canonical way up to isotopy to extend Γ_{Σ_0} to an abstract dividing set in $A = S^1 \times I$, namely by gluing a disc D to C_2 along the boundary and joining the endpoints of Γ_{Σ_0} on $\partial\Sigma_0$ with an arc contained in D. We will call the extension Γ_A, and will denote by ξ_{Γ_A} the contact structure on $T^2 \times I$ which is S^1–invariant over Γ_A. The contact manifold $(M', \xi_{\Gamma_{\Sigma_0}}(\eta))$ obtained by negative contact surgery on a vertical Legendrian curve with twisting number 0 in $(T^2 \times I, \xi_{\Gamma_A})$ is tight if and only if Γ_A is tight. In fact, if Γ_A is a tight abstract dividing set, we can take a tight abstract dividing set Γ_{T^2} on T^2 together with an embedding $\iota\colon A \to T^2$ so that $\iota(\Gamma_A) = \Gamma_{T^2} \cap \iota(A)$. By Giroux Tightness Criterion [22], Lemma 4.2, $(T^3, \xi_{\Gamma_{T^2}})$ is universally tight, therefore it is also symplectically fillable by Theorem 3.14. Since $(T^2 \times I, \xi_{\Gamma_A})$ contact embed into $(T^3, \xi_{\Gamma_{T^2}})$, then $(M', \xi_{\Gamma_{\Sigma_0}}(\eta))$ is tight because it can be embedded into a contact manifold obtained by negative contact surgery on $(T^3, \xi_{\Gamma_{T^2}})$.

On the contrary, if Γ_A is overtwisted, then Γ_{Σ_0} contains either a homotopically trivial closed curve, or a boundary parallel dividing arcs with endpoints on C_2. Then $(M', \xi_{\Sigma_0}(\eta))$ is overtwisted by the same argument as in the proof of Proposition 4.13 because $\#\Gamma_{\Sigma_0} > 1$.

Lemma 5.6 *Let Γ_{Σ_0} and Γ'_{Σ_0} be abstract dividing sets on the pair of pants Σ_0 such that their completions Γ_A and Γ'_A are tight, and let $\widehat{\Gamma}_A$, $\widehat{\Gamma}'_A$ be obtained from Γ_A, Γ'_A by throwing away every pair of closed curves bounding an annulus. If $\xi_{\Gamma_{\Sigma_0}}(\eta)$ is isotopic to $\xi_{\Gamma'_{\Sigma_0}}(\eta')$, then $\widehat{\Gamma}_A$ is diffeomorphic to $\widehat{\Gamma}'_A$ (as abstract dividing sets) and η is isotopic to η'.*

Proof Let $\xi_{\Gamma_{\Sigma_0}}(\eta)$ and $\xi_{\Gamma'_{\Sigma_0}}(\eta)$ be isotopic tight contact structures. By definition there exist neighbourhoods V and V' of the singular fibre and sections $\sigma\colon \Sigma_0 \to M' \setminus V$ and $\sigma'\colon \Sigma_0 \to M' \setminus V'$ coinciding on a neighbourhood of $\partial\Sigma_0$ such that:

(1) $-\partial(M \setminus V)$ and $-\partial(M \setminus V')$ have infinite boundary slope.

(2) $\xi_{\Gamma_{\Sigma_0}}(\eta)|_V$ is isotopic to η.

(3) $\xi_{\Gamma'_{\Sigma_0}}(\eta')|_{V'}$ is isotopic to η'.

(4) $\sigma(\Sigma_0)$ and $\sigma'(\Sigma_0)$ are convex with Legendrian boundary, $\Gamma_{\sigma(\Sigma_0)} = \Gamma_{\Sigma_0}$ and $\Gamma_{\sigma'(\Sigma_0)} = \Gamma'_{\Sigma_0}$.

If we glue a S^1–invariant tight contact structure on $T^2 \times I$ to either component of $\partial M'$, the result is tight if and only if Γ_A (respectively Γ'_A) glued to the dividing set on a convex horizontal annulus in $T^2 \times I$ produces no homotopically trivial curves. We will exploit this fact to prove the lemma using a technique called *Template Attaching*, first introduced by Honda in [22], section 5.3.2. In the following we will call *elementary template* a thickened torus $T^2 \times I$ carrying a tight contact structure which is S^1–invariant over a horizontal annulus with only one boundary parallel dividing arc and $2(n-1)$ dividing arcs with endpoints on different boundary components.

The set $\partial\Gamma_A = \partial\Gamma'_A$ consists of a finite collection of points with cardinality $2(\#\widehat{\Gamma}_A) = 2(\#\widehat{\Gamma}'_A)$. Given two points $p, q \in \partial\Gamma_A$ (respectively $\partial\Gamma'_A$) joined by an arc in the dividing set, we denote by (p, q) the arc in Γ_A (respectively Γ'_A) joining them. We partition $\partial\Gamma_A$ in the two subsets $\partial\Gamma_A \cap C_0 = \{p_0, \dots, p_n\}$ and $\partial\Gamma_A \cap C_1 = \{p'_0, \dots, p'_m\}$ and put a cyclic order on them.

We work by induction on the number of dividing arcs in Γ_A with both endpoints on the same boundary component. The base step is when there are no such curves, or when $\#\widehat{\Gamma}_A = 2$. In the first case, Γ_A coincides with $\widehat{\Gamma}_A$, and defines an order preserving bijection $\{p_0, \dots, p_n\} \to \{p'_0, \dots, p'_n\}$, which determines it up to diffeomorphism. We claim that Γ'_A has no boundary parallel dividing arcs either and induces the same bijection as Γ_A.

If $\widehat{\Gamma}_A$ contains no boundary parallel dividing arcs, then no single elementary template attaching produces a homotopically trivial closed curve. Suppose Γ'_A contains a boundary parallel arc (p_i, p_{i+1}). Then the attachment of an elementary template such that p_i and p_{i+1} are the endpoints of the boundary parallel dividing arc in its horizontal annulus produces an overtwisted disc, giving a contradiction. See Figure 5.1 case (b).

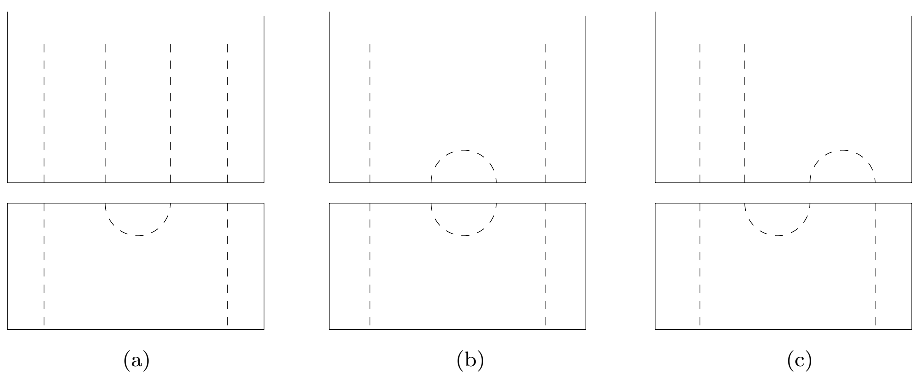

Figure 5.1: The attaching of an elementary template: Cases (a) and (c) preserve tightness, cases (b) produces an overtwisted disc.

Let T_0 and T_1 be the two components of $\partial M'$. If we attach elementary templates to T_0 and T_1 such that $\{p_0, p_1\}$ and $\{p'_i, p'_{i+1}\}$ are the endpoints of the boundary parallel dividing arcs, then the only case in which we get an overtwisted disc is when there are dividing arcs (p_0, p'_i) and (p_1, p'_{i+1}). This must be true for both Γ_A and Γ'_A, therefore the two dividing sets are isomorphic.

When $\#\widehat{\Gamma}_A = 2$, we have to distinguish the cases when Γ_A consists of two non boundary parallel dividing arcs, or when it consists of one boundary parallel dividing arc on each side and a number of closed curves. In the first case, $(M', \xi_{\Gamma_{\Sigma_0}}(\eta))$ remains tight after gluing S^1–invariant tight contact structures with a boundary parallel dividing curve on its horizontal annulus in any possible way. In the second case some gluing produce an overtwisted disc. This forces $\widehat{\Gamma}'_A$ to be diffeomorphic to $\widehat{\Gamma}_A$. We observe that template attaching cannot detect multiple closed curves in Γ_A and Γ'_A. This is the reason why we work with $\widehat{\Gamma}$ instead of with Γ.

Now we suppose the lemma true when Γ_A has $k-1$ arcs with endpoints on the same boundary component. Let Γ_A have k of such arcs, and suppose that (p_i, p_{i+1}) is one of them. If we glue an elementary template to $(M', \xi_{\Gamma_{\Sigma_0}}(\eta))$ so that the boundary parallel dividing arc in its horizontal annulus matches with (p_i, p_{i+1}) to give a closed homotopically trivial curve, we produce an overtwisted disc. This fact implies that (p_i, p_{i+1}) is also a boundary parallel dividing arc in Γ'_A.

After slightly perturbing $\xi_{\Gamma_{\Sigma_0}}(\eta)$ and $\xi_{\Gamma'_{\Sigma_0}}(\eta')$, for both contact structures A contains a bypass along $\partial M'$ corresponding to the boundary parallel dividing arc (p_i, p_{i+1}). After attaching these bypasses to $\partial M'$, and removing the collars of ∂M in which the bypass attachment takes place, we obtain manifolds M'_1, M'_2 with tight contact structures $\xi_{\Gamma_{\Sigma_0}}(\eta))|_{M'_1} = \xi_{\Gamma_{\Sigma_0} \setminus (p_i, p_i+1)}(\eta)$ and $\xi_{\Gamma'_{\Sigma_0}}(\eta'))|_{M'_2} = \xi_{\Gamma'_{\Sigma_0} \setminus (p_i, p_i+1)}(\eta')$. If we prove that $\xi_{\Gamma_{\Sigma_0} \setminus (p_i, p_i+1)}(\eta)$ is isotopic to $\xi_{\Gamma'_{\Sigma_0} \setminus (p_i, p_i+1)}(\eta')$ we can use the inductive hypothesis to conclude that $\widehat{\Gamma}_A \setminus (p_i, p_i+1)$ is diffeomorphic to $\widehat{\Gamma}'_A \setminus (p_i, p_i+1)$. From this it follows that $\widehat{\Gamma}_A$ is diffeomorphic to $\widehat{\Gamma}'_A$.

To prove that $\xi_{\Gamma_{\Sigma_0} \setminus (p_i, p_i+1)}(\eta)$ is isotopic to $\xi_{\Gamma'_{\Sigma_0} \setminus (p_i, p_i+1)}(\eta')$, we glue an elementary template to M' so that the boundary parallel dividing arc on its horizontal annulus joins p_{i+1} to p_{i+2}. The resulting contact manifold M'' is tight and both $M'' \setminus M'_1$ and $M'' \setminus M'_2$ are contactomorphic to I–invariant thickened tori. See Figure 5.1 case (c).

If M decomposes as $(M \setminus V) \cup V$, consider the inclusions $\iota_V \colon V \hookrightarrow M$ and $\iota_{M \setminus V} \colon M \setminus V \hookrightarrow M$. From the obstruction theoretical definition of the Euler class it is immediate that

$$(\iota_V)_* PD(e(\xi|_V, s)) + (\iota_{M \setminus V})_* PD(e(\xi|_{M \setminus V}, s) = PDe(\xi)$$

for any section s of ξ on ∂V. If $\xi_{\Gamma_{\Sigma_0}}(\eta)$ and $\xi_{\Gamma_{\Sigma_0}}(\eta')$ are isotopic, then $e(\xi_{\Gamma_{\Sigma_0}}(\eta)) = e(\xi_{\Gamma_{\Sigma_0}}(\eta'))$. Moreover, $\xi_{\Gamma_{\Sigma_0}}(\eta)|_{M\setminus V} = \xi_{\Gamma_{\Sigma_0}}(\eta')|_{M\setminus V} = \xi_{\Gamma_{\Sigma_0}}$, then

$$(\iota_{M\setminus V})_* PD(e(\xi_{\Gamma_{\Sigma_0}}(\eta)|_{M\setminus V}, s)) = (\iota_{M\setminus V})_* PD(e(\xi_{\Gamma_{\Sigma_0}}(\eta')|_{M\setminus V}, s)).$$

By difference, $(\iota_V)_* PD(\eta, s) = (\iota_V)_* PD(\eta', s)$, therefore $e(\eta, s) = e(\eta', s)$ because $(\iota_V)_*$ is injective. By [22], Proposition 4.23, this proves that $(D^2 \times S^1, \eta)$ and $(D^2 \times S^1, \eta)$ are isotopic □

Given (M, ξ), and a neighbourhood V of the singular fibre F such that $-\partial(M \setminus V)$ is a standard torus with infinite slope, we can modify the Seifert fibration $\pi\colon M \to T^2$ by an isotopy so that $M \setminus V$ fibres onto $T^2 \setminus D = \Sigma$, where D is an embedded disc. Let $\sigma\colon \Sigma \to M \setminus V$ be a section such that $\sigma(\Sigma)$ is convex with Legendrian boundary and $\#\Gamma$–minimising in its isotopy class. With an abuse of notation, we will denote $\sigma(\Sigma)$ simply by Σ, and its dividing set by Γ_Σ. We will denote by Γ the extension of Γ_Σ to T^2 obtained by joining the endpoints of Γ_Σ with an arc in D.

Proposition 5.7 *Let $\gamma \subset T^2$ be a homotopically nontrivial simple closed curve disjoint from the image of the singular fibre, and let $\mathcal{T}_\gamma$ be the family of convex or pre-Lagrangian tori in $(M(e_0, r), \xi_{\Gamma_\Sigma}(\eta))$ isotopic to $\pi^{-1}(\gamma)$. If we define the division number of a pre-Lagrangian torus to be zero, then the equality*

$$\min_{T \in \mathcal{T}_\gamma} \operatorname{div}(T) = \frac{1}{2}|\gamma \cap \Gamma|$$

holds.

Remark We say that a convex vertical torus T in $M(e_0, r)$ has infinite slope if its dividing set is isotopic in $M(e_0, r)$ to regular fibres, otherwise we say that T has finite slope. In general we cannot give a well-defined value to the slope of T when it is finite.

Lemma 5.8 *$\mathcal{T}_\gamma$ contains a pre-Lagrangian torus if and only if it contains a convex torus which does not have infinite slope.*

Proof Suppose there is a pre-Lagrangian torus $T \in \mathcal{T}_\gamma$: then after a suitable choice of coordinates (x, y, z) in a neighbourhood of T so that $T = \{y = 0\}$, the contact structure has equation $dz - y\,dx$ in a neighbourhood of T. This local model shows that for some small $\epsilon \neq 0$, the torus T_ϵ is pre-Lagrangian and has rational slope different from the slope of T. Then we perturb T_ϵ and obtain a convex torus T' with finite slope by Lemma 3.4.

Suppose that $T \in \mathcal{T}_\gamma$ contains a convex torus T with finite slope. First, we show that $T \in \mathcal{T}_\gamma$ also contains a convex torus T' with infinite slope. In fact, by hypothesis there is a vertical Legendrian curve $L \subset M$ with twisting number 0, hence we obtain a convex torus with infinite slope by first isotoping T so that it becomes a convex torus T' with vertical ruling disjoint from L. Then, if T' has not infinite slope already, by attaching the bypasses along T' coming from a convex annulus between L and a Legendrian ruling curve of T'. This operation produces a convex torus T'' parallel to T with infinite slope. Once we have a convex torus with finite slope T and a convex torus with infinite slope T'', we can suppose by isotopy discretisation that they are disjoint, so they bound a tight thickened torus with different boundary slopes. By [22], Corollary 4.8, such thickened torus contains a pre-Lagrangian torus. □

Proof of Proposition 5.7 Take a curve $\gamma' \subset T^2 \setminus D = \Sigma$ isotopic to γ which realises the minimum of the intersection with the dividing set Γ. We can identify γ' with its image under the section σ and make it Legendrian. Since $(M \setminus V, \xi|_{M\setminus V}) \cong (\Sigma \times S^1, \xi_{\Gamma_\Sigma})$ is S^1–invariant, $T_0 = \pi^{-1}(\gamma')$ is a standard torus with division number $\frac{1}{2}|\gamma \cap \Gamma|$ if $\gamma' \cap \Gamma \neq \emptyset$, or a pre-Lagrangian torus if $\gamma' \cap \Gamma = \emptyset$, therefore

$$\min_{T \in \mathcal{T}_\gamma} \operatorname{div}(T) \leq \frac{1}{2}|\gamma \cap \Gamma|.$$

Suppose by contradiction that there exists a convex torus $T_1 \in \mathcal{T}_\gamma$ with either $\operatorname{div}(T_1) < \frac{1}{2}|\gamma \cap \Gamma|$ and $|\gamma \cap \Gamma| > 2$, or with slope different from infinity and $|\gamma \cap \Gamma| = 2$. By Isotopy Discretisation [25], Lemma 3.10, we can find a finite family of disjoint convex tori $T_0 = T^{(0)}, \dots, T^{(n)} = T_1$ such that, for any $i = 0, \dots, n-1$, $T^{(i+1)}$ is obtained from $T^{(i)}$ by the attachment of a single bypass. In particular, they bound a layer N_i diffeomorphic to $T^2 \times I$. If $T^{(n)} = T_1$ has finite slope, we can assume that it is the first torus in the family with that property.

For any i such that $T^{(i)}$ has infinite slope there is a Seifert fibration $\pi_i \colon M \setminus T^{(i)} \to S^1 \times I$ with one singular fibre, a neighbourhood $V_i \subset M \setminus N_i$ of the singular fibre such that $-\partial(M \setminus V_i)$ has infinite slope, and a collar $C_i = \pi_i(N_i)$ of a boundary component of Σ_0 such that

$$\begin{aligned} &\pi_i|_{N_i} \colon N_i \to C_i \\ &\pi_i|_{M\setminus(N_i \cup V_i)} \colon M \setminus (N_i \cup V_i) \to \Sigma_0 \setminus C_i \end{aligned}$$

are S^1–bundles. We choose sections $\sigma_i \colon \Sigma_0 \to M \setminus (T^{(i)} \cup V_i)$ so that:

(1) $\sigma_i(\Sigma_0)$ is a convex $\#\Gamma$–minimising surface with Legendrian boundary denoted by $\Sigma_{(i)}$.

(2) $\Sigma_{(i)} \cap T^{(i+1)}$ is a Legendrian curve.

(3) σ_i extends to a section $\overline{\sigma}_i \colon \Sigma \to M \setminus V_i$.

(4) $\overline{\sigma}_0 = \sigma$.

Define $\overline{\Sigma}_{(i)} = \overline{\sigma}_i(\Sigma)$ and identify $\Gamma_{\overline{\Sigma}_{(i)}}$ with a multicurve on Σ using π_i.

We claim that, for any i, $\Gamma_{\overline{\Sigma}_{(i)}}$ differs from Γ_Σ by a number of curves isotopic to γ or by Dehn twists around γ. The proof is by induction on i. If $i = 0$ the claim is true because $\Sigma = \overline{\Sigma}_{(0)}$. Now suppose the claim true for a fixed i. Let $\sigma'_{i+1} \colon \Sigma_0 \to M \setminus (V_i \cup T^{(i+1)})$ be the section which extends to the section $\overline{\sigma}_i \colon \Sigma \to M \setminus V_i$. We denote $\sigma'_{i+1}(\Sigma_0)$ by Σ'_{i+1} and $\overline{\sigma}'_{i+1}(\Sigma)$ by $\overline{\Sigma}'_{i+1}$. By properties (1) and (2) of σ_i, Σ'_{i+1} is a convex $\#\Gamma$–minimising surface with Legendrian boundary, then by [23], Lemma 4.1 the S^1–invariant contact manifolds $(M \setminus (T_{(i+1)} \cup V_i), \xi|_{M \setminus (T_{(i+1)} \cup V_i)})$ and $(\Sigma_0 \times S^1, \xi_{\Gamma_{\Sigma'_{i+1}}})$ are contactomorphic. Analogously, $(M \setminus (T_{(i+1)} \cup V_{i+1}), \xi|_{M \setminus (T_{(i+1)} \cup V_{i+1})})$ is contactomorphic to $(\Sigma_0 \times S^1, \xi_{\Gamma_{\Sigma_{i+1}}})$. These contactomorphisms give presentations of $(M \setminus T_{(i+1)}, \xi|_{M \setminus T_{(i+1)}})$ as negative contact surgery on $(T^2 \times I, \xi_{\Gamma_{\Sigma'_{i+1}}})$ and $(T^2 \times I, \xi_{\Gamma_{\Sigma_{i+1}}})$ respectively, therefore by Lemma 5.6, $\widehat{\Gamma}_{\Sigma_{i+1}}$ is diffeomorphic to $\widehat{\Gamma}_{\Sigma'_{i+1}}$. By construction, Γ_{Σ_i} and $\Gamma_{\Sigma'_{i+1}}$ extend to the same multicurve $\Gamma_{\overline{\Sigma}_i}$ on $\overline{\Sigma}$, so the claim is proved.

Suppose now that T_1 has infinite slope and $\operatorname{div}(T_1) < \operatorname{div}(T_0)$: then the geometric intersection $|\gamma \cap \Gamma_{\overline{\Sigma}_{(n)}}|$ is lesser than the the geometric intersection $|\gamma \cap \Gamma_{\Sigma_{(0)}}| = |\gamma \cap \Gamma|$. This is a contradiction because, by the claim, $\Gamma_{\overline{\Sigma}_{(0)}}$ and $\Gamma_{\overline{\Sigma}_{(n)}}$ differ only by Dehn twists along γ or by the number of curves isotopic to γ.

If the slope of T_1 is not infinity, attaching the bypasses coming from a vertical annulus $A \subset M \setminus N_{n-1}$ between $T_{(n)}$ and $T_{(n-1)}$ we find a layer $N_n \cong T^2 \times I$ so that $N = N_{n-1} \cup N_n$ has minimal boundary and infinite boundary slopes. N is rotative because it has infinite boundary slopes, but $T_{(n)} \subset N$ has finite slope, so by [22], Lemma 5.7, the dividing set of a $\#\Gamma$–minimising section of N contains no arcs with endpoints on different boundary components. We can complete the section in N to a section $\Sigma'_{(n-1)} = \sigma'_{n-1}(\Sigma_0)$ which has no dividing arcs with endpoints on different boundary components. By Lemma 5.6, $\widehat{\Gamma}_{\Sigma'_{(n-1)}}$ is isomorphic to $\widehat{\Gamma}_{\Sigma_{(n-1)}}$ and by the claim $\widehat{\Gamma}_{\Sigma_{(n-1)}}$ glues to the same dividing set on T^2 as $\widehat{\Gamma}_{\Sigma_{(0)}}$. This is a contradiction because $\widehat{\Gamma}_{\Sigma'_{(n-1)}}$ has a curve isotopic to γ and $\widehat{\Gamma}_{\Sigma_{(0)}}$ does not. $\square$

Theorem 5.9 *Let Γ_Σ and Γ'_Σ be two tight abstract dividing sets on the punctured torus Σ such that $\#\Gamma_\Sigma \cap \partial\Sigma = \#\Gamma'_\Sigma \cap \partial\Sigma = 2$ and without boundary parallel dividing arcs. Denote by Γ and Γ' their completion. The tight contact structures $\xi_{\Gamma_\Sigma}(\eta)$ and $\xi_{\Gamma'_\Sigma}(\eta')$ on $M(e_0, r)$ are isotopic if and only if Γ is isotopic to Γ', and η is isotopic to η'.*

Proof If Γ is isotopic to Γ' and η is isotopic to η', then $\xi_{\Gamma_\Sigma}(\eta)$ is isotopic to $\xi_{\Gamma'_\Sigma}(\eta')$ by Proposition 4.16.

Let now $\xi_{\Gamma_\Sigma}(\eta)$ and $\xi_{\Gamma'_\Sigma}(\eta')$ be isotopic tight contact structures. By Proposition 5.7 for any simple closed curve $\gamma \subset T^2$ we have $|\gamma \cap \Gamma| = |\gamma \cap \Gamma'|$, therefore Γ_Σ is isotopic to Γ'_Σ. Suppose now that there are tight contact structures η_0 and η_1 on $D^2 \times S^1$ such that $\xi_{\Gamma_\Sigma}(\eta)$ is isotopic to $\xi_{\Gamma_\Sigma}(\eta')$: then there exist isotopic convex vertical tori with infinite slope T_0, $T_1 \subset M$, and, for $i = 0, 1$, Seifert fibrations $\pi_i \colon M \setminus T_i \to S^1 \times I$, and neighbourhoods V_i of the singular fibre such that $\xi|_{V_i} \cong \eta_i$.

By isotopy discretisation [25], Lemma 3.10, there is a finite sequence of convex tori with infinite slope $T_0 = T^{(0)}, \dots, T^{(n)} = T_1$ such that, for $i = 0, \dots, n-1$, $T^{(i)}$ and $T^{(i+1)}$ bound N_i diffeomorphic to $T^2 \times I$. For any i, we can modify the Seifert fibration on M so that the singular fibre is contained in $M \setminus N_i$, and find a neighbourhood of the singular fibre V'_i contained in $M \setminus N_i$. By Lemma 5.6 applied to $M \setminus T^{(0)}$, $\eta_0 = \xi|_{V_0}$ is isotopic to $\xi|_{V'_0}$, by Lemma 5.6 applied to $M \setminus T^{(i)}$, $\xi|_{V'_i}$ is isotopic to $\xi|_{V'_{i+1}}$, and by Lemma 5.6 applied to $M \setminus T^{(n)}$, $\xi|_{V'_n}$ is isotopic to $\xi|_{V_1} \cong \eta_1$. □

6 Exceptional tight contact structures

In this section we prove tightness for the candidate tight contact structures with $\#\Gamma = 1$. The proof of tightness for this class of contact structures uses a purely topological and three dimensional technique known as *state traversal*, introduced by Honda in [23].

6.1 State traversal

Let (M, ξ) be a contact manifold and $W \subset M$ be a properly embedded incompressible surface. We will assume W is convex. The contact manifold $(M \setminus W, \xi|_{M \setminus W})$ will be called a *state*, and the surface W a *wall*. In general both W and $M \setminus W$ could be disconnected. A state is said tight if $\xi|_{M \setminus W}$ is tight. The boundary of $M \setminus W$ consists of two copies of W: W_+ and W_-. A *state transition* consists of detaching a collar of W_- and attaching it to W_+, or vice versa, so that Γ_W is changed by a bypass attachment. We observe that a state transition corresponds to moving W inside M by an isotopy.

Theorem 6.1 ([23], section 2.3.1 or [25], Theorem 3.5) *If the initial state* $(M \setminus W, \xi|_{M \setminus W})$ *is tight, and all the states reached from it in a finite number of state transitions are also tight, then the contact manifold* (M, ξ) *is tight.*

The set of states that can be reached from the initial state in a finite number of bypass attachments is a complete isotopy invariant of ξ in the following sense.

Theorem 6.2 (Corollary of [25], Theorem 3.1) *Let* ξ_1 *and* ξ_2 *be two tight contact structures on* M*, and let* $W \subset M$ *be a properly embedded incompressible convex surface. Let* $\mathcal{C}(\xi_i)$ *be the set of isotopy classes relative to the boundary of all the states reached from the initial state* $(M \setminus W, \xi|_{M \setminus W})$ *in a finite number of state transitions. Then* ξ_1 *is isotopic to* ξ_2 *if and only if* $\mathcal{C}(\xi_1) = \mathcal{C}(\xi_2)$.

Let Γ_Σ be an abstract dividing set on the one–punctured torus Σ with $\#\Gamma_\Sigma = 1$. In order to apply the state traversal to $(M(e_0, r), \xi_{\Gamma_\Sigma}(\eta))$, we give an alternative description of this contact structure. By ξ_l, with $l = 2$ or $l = -2$, we denote the tight contact structure with infinite boundary slopes and twisting π on $T^2 \times I$ such that ξ_l has relative Euler class $e(\xi_l) = \begin{pmatrix} 0 \\ l \end{pmatrix}$, ($\xi_1^\pm$ in the notations of [22], Lemma 5.2) and by $(M', \xi_l(\eta))$ we denote the contact manifold obtained by contact $(-\frac{1}{r})$–surgery along a vertical Legendrian curve in $(T^2 \times I, \xi_l)$ with twisting number 0.

Lemma 6.3 *After gluing* T_1 *to* T_0 *with the map* $\begin{pmatrix} 1 & 0 \\ -e_0 & 1 \end{pmatrix}$, *we obtain the contact manifold* $(M(e_0, r), \xi_{\Gamma_\Sigma}(\eta))$ *with* $\#\Gamma_\Sigma = 1$ *and* $\langle e(\xi_{\Gamma_\Sigma}), \Sigma \rangle = l$.

Proof To prove that we obtain a contact structure isotopic to $\xi_{\Gamma_\Sigma}(\eta)$ it is enough to show that we obtain a contact manifold whose background is isotopic to the background of $(M(e_0, r), \xi_{\Gamma_\Sigma}(\eta))$. We prove the isotopy between the backgrounds by showing that they induce isotopic dividing sets on convex $\#\Gamma$–minimising sections of $\Sigma \times S^1$, see [23] Lemma 4.1.

The background of $(M', \xi_l(\eta))$ is the contact manifold $(\Sigma_0 \times S^1, \xi_{\Gamma_{\Sigma_0}})$ where Σ_0 is a pair of pants and $\xi_{\Gamma_{\Sigma_0}}$ is the tight contact structure which is S^1–invariant

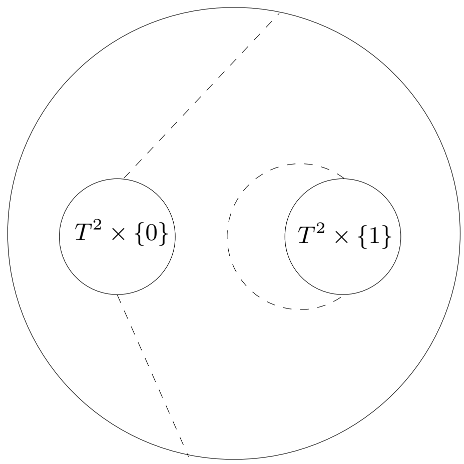

Figure 6.1: The dividing set Γ_{Σ_0}

over the abstract dividing set described in Figure 6.1. Γ_{Σ_0} consists of two arcs joining two boundary components of Σ_0 and a boundary parallel arc with both end-points on the third component of $\partial\Sigma_0$.

Consider a $\#\Gamma$–minimising section Σ_0' of $\Sigma_0 \times S^1$ so that, after gluing two boundary components of $\Sigma_0 \times S^1$ as prescribed by the statement, we obtain a section Σ' of $\Sigma \times S^1$. By [23] Lemma 4.5 $\Gamma_{\Sigma_0'}$ is isotopic to Γ_{Σ_0}, therefore $\Gamma_{\Sigma'}$ is isotopic to Γ_Σ.

□

For the rest of the section we fix the notation $M = M(e_0, r)$ and $\xi = \xi_l(\eta) = \xi_{\Gamma_\Sigma}(\eta)$.

Lemma 6.4 *Let $W_0, W_1 \subset (M, \xi)$ be vertical incompressible disjoint convex tori with finite slopes and let $N \subset M$ be the thickened torus bounded by W_0 and W_1. If $\partial(M \setminus N) = W_1 - W_0$, the slope of W_1 is $s_1 = \frac{p}{q}$, and the slope of W_0 is $s_0 \leq \frac{p}{q} - e_0$, then there is a vertical Legendrian curve $L \subset M \setminus N$ with twisting number $tb(L) = 0$.*

Proof Take a properly embedded convex vertical annulus with Legendrian boundary $A \subset (M \setminus N)$ and attach all the possible bypasses it carries to W_0 and W_1. If in the process we get a torus with infinite slope we are done, otherwise we end with two convex tori W_0' and W_1' parallel to W_0 and W_1 respectively with slopes $s_0' \leq s_0 \leq \frac{p}{q} - e_0$ and $s_1' \geq \frac{p}{q}$, and such that the vertical annulus A'

between W_i' and W_{i+1}' carries no bypasses. Let N' be the layer diffeomorphic to $T^2 \times I$ between W_0' and W_1'. If we cut $M \setminus N'$ along A' and round the edges, we get a solid torus with boundary slope $s \geq s_1' - s_0' - 1 \geq 0$. On the other hand, if we make the singular fibre F Legendrian with very low twisting number n, and remove a standard neighbourhood νF, we get slope $\frac{-n\beta-\beta'}{n\alpha+\alpha'}$ on $-\partial(M' \setminus \nu F)$. Taking the limit for n going to infinity, we see that this slope is negative for n small enough, therefore, by [22], Proposition 4.16, there is an intermediate torus in $M \setminus (N' \cup A' \cup \nu F)$ with infinite slope. $\square$

If $W \subset (M, \xi)$ is a vertical incompressible convex torus with finite slope, applying Lemma 6.4 with $W_0 = W_1 = W$, we find a vertical Legendrian curve L with twisting number zero in $M \setminus W$. Attaching the bypasses coming from L to W on either sides, we can engulf W in a rotative $T^2 \times [-1, 1]$ with infinite boundary slopes such that $W = T^2 \times \{0\}$. By [24], Theorem 2.2, we need to consider only transitions between states with minimal boundary, provided that the walls can be engulfed into rotative thickened tori. In the present situation, we have to consider transitions between states with finite boundary slope and minimal boundary, or between states with infinite boundary slopes.

6.2 Analysis of the states

Before performing the state traversal we analyse the possible states. We observe that the Seifert fibration on M can be isotoped so that W becomes a fibred torus. Consequently there is an induced Seifert fibration on $M \setminus W \cong M'$.

Let ζ_l' be the minimally twisting tight contact structure on $T^2 \times [0, \frac{1}{2}]$ with boundary slopes $s_0 = \frac{p}{q} - e_0$, $s_{\frac{1}{2}} = \infty$, minimal boundary and relative Euler class $\pm \begin{pmatrix} -q \\ -1-p+e_0 q \end{pmatrix}$. Let ζ_l'' be the minimally twisting tight contact structure on $T^2 \times [\frac{1}{2}, 1]$ with boundary slopes $s_{\frac{1}{2}} = \infty$, $s_1 = \frac{p}{q}$, minimal boundary and relative Euler class $\pm \begin{pmatrix} q \\ 1+p \end{pmatrix}$. Here the signs of the relative Euler classes are chosen accordingly to the sign of l. All the basic slices in the decomposition of ζ_l' and ζ_l'' have the same sign. We denote by $(M'', \zeta_l'(\eta))$ the contact manifold constructed by contact $(-\frac{1}{r})$-surgery on a vertical Legendrian curve L with twisting number 0 in $(T^2 \times [0, \frac{1}{2}], \zeta_l')$, and by $(M', \xi_l'(\eta))$ the contact manifold obtained by gluing $(T^2 \times [\frac{1}{2}, 1], \zeta_{-l}'')$ to $(M'', \zeta_l'(\eta))$.

To the contact manifold $(D^2 \times S^1, \eta)$ we associate the set of numerical invariants $(r_0, \ldots, r_k)$ defined as

$$r_i = \#\{\text{positive basic slices in } N_i\} - \#\{\text{negative basic slices in } N_i\}$$

where N_i is the $(i+1)$–th continued fraction block in the basic slices decomposition of η. The classification of tight contact structures on solid tori [22, Theorem 2.3] implies the following proposition.

Proposition 6.5 *For any slope s, let Γ_s denote the tight abstract dividing set on T^2 with slope s and $\#\Gamma_s = 2$. Then the number of continued fraction blocks and the slopes of the borders between continued fraction blocks of any tight contact structure $\eta \in$ Tight$(D^2 \times S^1, \Gamma_s)$ depend only on s and are independent of η. Moreover, the map Tight$(D^2 \times S^1, \Gamma_s) \to \mathbb{Z}^{n+1}$ given by $\eta \mapsto (r_0, \dots, r_k)$ is injective.*

If $\Gamma_s = A(r)^{-1}\Gamma_\infty$, then the number of continued fraction blocks of $\eta \in$ Tight$(D^2 \times S^1, \Gamma_s)$ is equal to the number of the coefficient in the continued fraction expansion $-\frac{1}{r} = [d_0, \dots, d_n]$, moreover $|r_0| \leq |d_0|$, and $|r_i| \leq |d_i| - 1$ for $i > 0$. To classify the contact manifolds $(M', \xi_l'(\eta))$ we need the following lemmas.

Lemma 6.6 *Let $(T^2 \times [0, \frac{1}{2}], \zeta_l')$ be a basic slice with boundary slopes $-n$ and ∞, and let $(T^2 \times [\frac{1}{2}, 1], \zeta''_{-l})$ be a basic slice with boundary slopes ∞ and 1 with opposite sign. We call $(T^2 \times [0,1] \setminus V, \xi_l')$ the contact manifold obtained from $(T^2 \times [0, \frac{1}{2}], \zeta_l') \cup_{T^2 \times \{\frac{1}{2}\}} (T^2 \times [\frac{1}{2}, 1], \zeta''_{-l})$ by removing a standard neighbourhood V of a vertical Legendrian divide of $T^2 \times \{\frac{1}{2}\}$. Then $(T^2 \times [0,1] \setminus V, \xi_l')$ is isomorphic to $(T^2 \times [0, \frac{1}{2}] \setminus U, \zeta_l'|_{T^2 \times [0,\frac{1}{2}] \setminus U})$, where U is a standard neighbourhood of a vertical Legendrian ruling curve of a standard torus parallel to $T^2 \times \{0\}$ and contained in its invariant neighbourhood.*

Proof Up to isotopy we can assume that $T^2 \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ is an invariant neighbourhood of $T^2 \times \{\frac{1}{2}\}$ in $(T^2 \times [0, \frac{1}{2}], \zeta_l') \cup (T^2 \times [\frac{1}{2}, 1], \zeta''_{-l})$, and that V is contained in it. Clearly $\xi_l'|_{T^2 \times [0, \frac{1}{2} - \epsilon]}$ is isomorphic to ζ_l' and $\xi_l'|_{T^2 \times [\frac{1}{2}+\epsilon, 1]}$ is isomorphic to ζ''_{-l}. By [23, Lemma 4.1] $\xi_l'|_{T^2 \times [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon] \setminus V}$ is S^1–invariant, and the dividing set of a convex $\#\Gamma$–minimising horizontal section Σ with Legendrian boundary is as in Figure 6.2.

We decompose $(T^2 \times [0, \frac{1}{2}] \setminus U, \zeta_l'|_{T^2 \times [0, \frac{1}{2}] \setminus U})$ into three pieces which are isomorphic to $(T^2 \times [0, \frac{1}{2} - \epsilon], \xi_l'|_{T^2 \times [0, \frac{1}{2} - \epsilon]})$, $(T^2 \times [\frac{1}{2} + \epsilon, 1], \xi_l'|_{T^2 \times [\frac{1}{2}+\epsilon, 1]})$, and $(T^2 \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \setminus V, \xi_l'|_{T^2 \times [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon] \setminus V})$ respectively. Thicken $U \in (T^2 \times [0, \frac{1}{2}], \zeta_l')$ to U' by attaching the bypasses coming from a vertical annulus between a Legendrian ruling curve of ∂U and a Legendrian divide of $T^2 \times \{\frac{1}{2}\}$ so that $\partial U'$ has infinite slope. In a similar way find a collar C of $T^2 \times \{0\}$ in

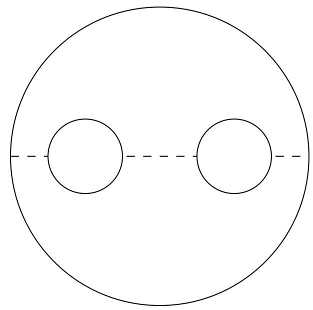

Figure 6.2: The dividing set on Σ and Σ'

$(T^2 \times [0, \frac{1}{2}] \setminus U', \zeta'_l|_{T^2\times[0,\frac{1}{2}]\setminus U'})$ so that C has boundary slopes $-n$ and ∞. The isomorphism between $(T^2 \times [0,1] \setminus V, \xi'_l)$ and $(T^2 \times [0, \frac{1}{2}] \setminus U, \zeta'_l|_{T^2\times[0,\frac{1}{2}]\setminus U})$ identifies $T^2 \times [0, \frac{1}{2} - \epsilon]$ to C, $T^2 \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \setminus V$ to $T^2 \times [0, \frac{1}{2}] \setminus (U' \cup C)$, and $T^2 \times [\frac{1}{2} + \epsilon, 1]$ to $U' \setminus U$.

$(T^2 \times [0, \frac{1}{2}] \setminus (U' \cup C), \zeta'_l|_{T^2\times[0,\frac{1}{2}]\setminus(U'\cup C)})$ has infinite boundary slopes, therefore by [23, Lemma 4.1] it is S^1–invariant. Let Σ' be a convex $\#\Gamma$–minimising horizontal section with Legendrian boundary. The dividing sets of Σ' cannot contain any boundary parallel dividing arc, otherwise such an arc would produce a bypass attached horizontally to $T^2 \times [0, \frac{1}{2}] \setminus (U' \cup C)$. The attachment of this bypass would give a convex torus with slope zero in $(T^2 \times [0, \frac{1}{2}], \zeta'_l)$, contradicting either tightness or minimal twisting. Thus the dividing sets of Σ' is forced to be as in Figure 6.2, therefore $(T^2 \times [0, \frac{1}{2}] \setminus (U' \cup C), \zeta'_l|_{T^2\times[0,\frac{1}{2}]\setminus(U'\cup C)})$ is isomorphic to $(T^2 \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \setminus V, \xi'_l|_{T^2\times[\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon]\setminus V})$ by [23, Lemma 4.1] because they induce diffeomorphic dividing sets on the convex $\#\Gamma$–minimising horizontal sections Σ and Σ'.

$\zeta'_l|_C$ is isomorphic to $\zeta'_l \cong \xi'_l|_{T^2\times[0,\frac{1}{2}-\epsilon]}$ because their relative Euler classes have the same evaluation on a vertical annulus. In a similar way, the relative Euler class of $\zeta'_l|_{U'\setminus U}$ evaluates on a vertical annulus as the relative Euler class of ζ'_l, therefore it evaluates as the opposite of the relative Euler class of $\zeta''_{-l} \cong \xi'_l|_{T^2\times[\frac{1}{2}+\epsilon,1]}$. The change of sign is due to the fact that, in evaluating the relative Euler class of $\zeta'_l|_{U'\setminus U}$, the boundary component ∂U is oriented as $T^2 \times \{0\}$, i. e. by the inward normal. On the other hand, the isomorphism maps ∂U to $T^2 \times \{1\}$, which is oriented by the outward normal. This change of orientation on the boundary forces the orientation of the vertical annulus to change too, in order to keep the global orientation unchanged. □

The contact manifold $(T^2 \times [0, \frac{1}{2}], \zeta'_l) \cup (T^2 \times [\frac{1}{2}, 1], \zeta''_{-l})$ is overtwisted because it does not satisfy the conditions of the Gluing Theorem [22], Theorem 4.25. However, it become tight when we remove V, as Lemma 6.6 shows.

Lemma 6.7 *Let $(T^2 \times [0,1] \setminus V, \xi'_l)$ be a contact manifold as in Lemma 6.6. Then there is a convex annulus $A \subset T^2 \times I \setminus V$ whose boundary consists of vertical Legendrian ruling curves of $T^2 \times \{0\}$ and $T^2 \times \{1\}$ such that its dividing set Γ_A has no boundary parallel dividing curves, and $\xi'_l|_{T^2 \times I \setminus (V \cup A)}$ is isotopic to ζ'_l.*

Proof The annulus A can be easily found in the contact manifold $(T^2 \times [0, \frac{1}{2}], \zeta'_l)$, which is isomorphic to $(T^2 \times [0,1] \setminus V, \xi'_l)$ by Lemma 6.6. It is a convex vertical annulus with Legendrian boundary between $T^2 \times \{0\}$ and ∂U contained in an invariant collar of $T^2 \times \{0\}$. The invariance of the collar implies that the annulus can carry no bypass. □

Lemma 6.8 *If $e_0 \geq 2$, then $(M', \xi'_l(\eta))$ is tight for any η and any l. If $e_0 = 1$, then $(M', \xi'_l(\eta))$ is tight if and only if $r_0 = \frac{ld_0}{2}$. Moreover, $\xi'_{l'}(\eta')$ is isotopic to $\xi'_l(\eta)$ if and only if either $l = l'$ and η is isotopic to η' or $s_0 \in \mathbb{Z}$ and*

(1) *$l' = -l$ and $r'_0 = r_0 + l$ when $e_0 = 2$,*

(2) *$l' = -l$, $r_0 = -r'_0$ and $r'_1 = r_1 + l$ when $e_0 = 1$.*

Proof After acting, if necessary, on M' by a self-diffeomorphism supported outside the surgery and preserving the Seifert fibration, we can assume $s_1 = \frac{p}{q} \in (0,1]$. There is a unique universally tight contact structure on $D^2 \times S^1$ with boundary slope $-s_1$ which can be glued to $T^2 \times [\frac{1}{2}, 1]$ along $-T^2 \times \{1\}$ with the identity map to give the tight contact structure with infinite boundary slope on $T^2 \times [\frac{1}{2}, 1] \cup D^2 \times S^1 \cong D^2 \times S^1$. After filling $T^2 \times \{1\}$ in $(M', \xi'_l(\eta))$ in this way, we obtain a contact structure on $D^2 \times S^1$ still denoted by $\xi'_l(\eta)$ which can be decomposed as $(D^2 \times S^1, \xi'_l(\eta)) \cong (T^2 \times [0, \frac{1}{2}], \zeta''_l) \cup_{A(r)} (D^2 \times S^1, \eta)$.

We start studying tightness for this bigger contact manifold. Since both $(T^2 \times [0, \frac{1}{2}], \zeta''_l)$ and $(D^2 \times S^1, \eta)$ are tight, the Gluing Theorem [22], Theorem 4.25, gives necessary and sufficient conditions for $\xi_l(\eta)$ to be tight. The application of the Gluing Theorem for thickened tori to solid tori is possible because [22], Propositions 4.15, 4.17, and 4.18 give an identification between isotopy classes of tight contact structures on $D^2 \times S^1$ and isotopy classes of tight minimally twisting contact structures on $T^2 \times I$.

Let $\frac{p}{q} - e_0 = s_0 > s_{\frac{1}{2n}} > \ldots > s_{\frac{n-1}{2n}} > s_{\frac{1}{2}} = \infty$ be the sequence of the boundary slopes of a minimal basic slices decomposition of $(T^2 \times [0, \frac{1}{2}], \zeta_l')$, and let $\infty = s'_{\frac{1}{2}} > s'_{\frac{m-1}{2m}} > \ldots > s'_{\frac{1}{2m}} > s'_0$ be the sequence of the boundary slopes of a minimal basic slices decomposition of $(D^2 \times S^1, \eta)$ computed with respect to the basis of $T^2 \times [0, \frac{1}{2}]$. The contact structure $\xi_l(\eta)$ is tight if and only if either $s_0 > \ldots > s_{\frac{n-1}{2n}} > \infty > s'_{\frac{m-1}{2m}} > \ldots > s'_0$ is the shortest sequence of slopes between s_0 and s'_0, or there exist $s_{\frac{k}{2n}} < \infty < s'_{\frac{k'}{2m}}$ joined by an edge in the Farey tessellation of $\mathbb{H}^2$ such that the basic slices between them have all the same signs.

If $\frac{p}{q} - e_0 < 0$, then the shortest sequence of slopes between s_0 and s'_0 needs to go through ∞, because all the $s_{\frac{i}{2n}}$ are negative and all the $s'_{\frac{j}{2m}}$ are positive, therefore $(D^2 \times S^1, \xi_l'(\eta))$ is tight for any l and any η. If $\frac{p}{q} - e_0 = 0$, which means $\frac{p}{q} = 1$, and $e_0 = 1$, there is an edge in the Farey Tessellation joining 0 with $-\frac{1}{d_0+1}$, so ∞ is not a border between basic slices. In this case $(D^2 \times S^1, \xi_l'(\eta))$ is tight if and only if $(T^2 \times [0, \frac{1}{2}], \zeta_l'')$ with boundary slopes $s_0 = 0$ and $s_1 = \infty$ glues with the first continued fraction block of η with boundary slopes $s'_{\frac{1}{2}} = \infty$ and $s'_{\frac{m-d_0-1}{2m}} = -\frac{1}{d_0+1}$ to give a basic slice. This happens if and only if $l = 2$ and $r_0 = d_0$, or $l = -2$ and $r_0 = -d_0$. When $(D^2 \times S^1, \xi_l'(\eta))$ is not tight, then $(M', \xi_l'(\eta))$ is not tight either, because Lemma 6.7 gives a contact embedding of $(D^2 \times S^1, \xi_l'(\eta))$ into $(M', \xi_l'(\eta))$.

In order to determine whether $(M', \xi_l'(\eta))$ is isotopic to $(M', \xi_{l'}'(\eta'))$, we again study the problem in $D^2 \times S^1$ first. In fact, if $(M', \xi_l'(\eta))$ is isotopic to $(M', \xi_{l'}'(\eta'))$, then $(D^2 \times S^1, \xi_l'(\eta))$ is isotopic to $(D^2 \times S^1, \xi_{l'}'(\eta'))$. We have three cases here: when ∞ is a border between continued fraction blocks, when ∞ is a border between basic slices but not between continued fraction blocks, and when ∞ is not a border between basic slices.

Case 1 When ∞ is a border between continued fraction blocks in the sequence $s_0 > \ldots > s_{\frac{n-1}{2n}} > \infty > s'_{\frac{m-1}{2m}} > \ldots > s'_0$, it follows from the classification theorem for tight contact structures on solid tori that $(D^2 \times S^1, \xi_l'(\eta))$ is isotopic to $(D^2 \times S^1, \xi_{l'}'(\eta'))$ if and only if $l = l'$, and η is isotopic to η'.

Case 2 The condition for ∞ to be a border between basic slices but not a border between continued fraction blocks in the minimal basic slices decomposition of $(D^2 \times S^1, \xi_l'(\eta))$ is that the slopes $s_{\frac{n-1}{2n}}$ and $s'_{\frac{m-1}{2m}}$ are represented by shortest integer vectors v_{-1}, v_0, v_1 such that (v_{-1}, v_0) and (v_0, v_1) are integer bases, and $|\det(v_{-1}, v_1)| = 2$. This condition is satisfied if and only if $e_0 = 2$ and $\frac{p}{q} = 1$. In this case the basic slices belonging to the outermost continued fraction block of η, which has boundary slopes ∞ and $-\frac{1}{d_0+1}$, and the basic slice $(T^2 \times [0, \frac{1}{2}], \zeta_l'')$, with boundary slopes -1 and ∞ form a unique continued fraction block in $(D^2 \times S^1, \xi_l'(\eta))$, therefore their signs can be shuffled. The shuffle can occur in $(M', \xi_l'(\eta))$ as well, because by Lemma 6.7 there is a contact embedding of $(D^2 \times S^1, \xi_l'(\eta))$ in $(M', \xi_l'(\eta))$. We conclude that, when M' has boundary slopes -1 and 1, $(M', \xi_l(\eta))$ is isotopic to $(M', \xi_{l'}'(\eta'))$ if and only if one of the following holds: either $l = l'$ and η is isotopic to η', or $l' = -l$ and $r_0' = r_0 + l$.

Case 3 If $e_0 = 1$, $s_1 = \frac{p}{q} = 1$ and $\xi_l'(\eta)$ is tight, then $(T^2 \times [0, \frac{1}{2}], \zeta_l'')$ glues with the outermost continued fraction block of η to give a basic slice with boundary slopes 0 and $-\frac{1}{d_0+1}$. This basic slice forms a continued fraction block with the basic slices belonging to the second outermost continued fraction block in η, which has boundary slopes $-\frac{1}{d_0+1}$ and $-\frac{d_1+1}{d_0 d_1 - 1}$, therefore their signs can be shuffled. Again, by Lemma 6.7, the same result holds on M', so we conclude that $(M', \xi_l'(\eta))$ is isotopic to $(M', \xi_{l'}'(\eta'))$, when M' has boundary slopes 0 and 1, if and only if one of the following holds: either $l = l'$ and η is isotopic to η', or $l' = -l$, $r_0' = -r_0$, and $r_1' = r_1 + l$. □

Now we analyse the states with infinite boundary slopes. Let $A = S^1 \times [0, 1]$ be an annulus and let Γ_A be a multicurve on A which closes to a homotopically trivial closed curve in T^2 if we identify $S^1 \times \{0\}$ to $S^1 \times \{1\}$. Let $(M', \xi_{\Gamma_A}(\eta))$ be the contact manifold obtained by contact $(-\frac{1}{r})$–surgery on a vertical Legendrian curve with twisting number 0 in $(T^2 \times I, \xi_{\Gamma_A})$.

Lemma 6.9 *Let Γ_A be an abstract dividing set on $A = S^1 \times I$ without homotopically trivial closed curves. Suppose that Γ_A closes to an abstract dividing set on T^2 consisting of a unique homotopically trivial closed curve if we identify $S^1 \times \{0\}$ and $S^1 \times \{1\}$. If Γ_A' is another multicurve on A and η' a tight contact structure on $D^2 \times S^1$ such that $(M', \xi_{\Gamma_A}(\eta))$ is isotopic to $(M', \xi_{\Gamma_A'}(\eta'))$, then η is isotopic to η' and Γ_A is diffeomorphic to Γ_A'.*

Proof By Lemma 5.6, η is isotopic to η' and $\widehat{\Gamma}_A$ is diffeomorphic to $\widehat{\Gamma}_A'$. If Γ_A contains an arc with endpoints on different boundary components, then $\widehat{\Gamma}_A = \Gamma_A$ and $\widehat{\Gamma}_A' = \Gamma_A'$, so we are done. If this is not the case, Γ_A consists of arcs with both endpoints on the same side, so $\widehat{\Gamma}_A = \Gamma_A$ and Γ_A' differs from Γ_A by a number of closed curves.

We take a multicurve Γ_B on $B = S^1 \times [1,2]$ consisting of arcs with both endpoints on the same side so that, after identifying $S^1 \times \{0\}$ with $S^1 \times \{2\}$, $\Gamma_A \cup \Gamma_B$ closes to some homotopically nontrivial curves in T^2. Then, gluing $(T^2 \times [1,2], \xi_{\Gamma_B})$ to $(M', \xi_{\Gamma_A}(\eta)) \cong (M', \xi_{\Gamma'_A}(\eta'))$ yields a generic tight contact structure on $M(e_0, r)$. By Theorem 5.9, the closure of $\Gamma_A \cup \Gamma_B$ is isotopic to the closure $\Gamma'_A \cup \Gamma_B$, in particular they have the same number of components. This implies that Γ_A and Γ'_A contain the same number of closed curves, then Γ_A is diffeomorphic to Γ'_A by Lemma 5.6. $\square$

6.3 Analysis of the transitions

Theorem 6.10 *If $e_0 \geq 2$, then $(M(e_0, r), \xi_l(\eta))$ is tight for any η and any l. If $e_0 = 1$, then $(M(e_0, r), \xi_l(\eta))$ is tight if and only if $r_0 = \frac{ld_0}{2}$. Moreover, $\xi_{l'}(\eta')$ is isotopic to $\xi_l(\eta)$ if and only if either $l = l'$ and η is isotopic to η', or*

- *$l' = -l$ and $r'_0 = r_0 + l$ when $e_0 = 2$,*
- *$l' = -l$, $r'_0 = -r_0$ and $r'_1 = r_1 + l$ when $e_0 = 1$.*

Proof Let $W_0 \subset M$ be a convex incompressible vertical torus with infinite slope and $\#\Gamma_{W_0} = 2$ such that the initial state $(M \setminus W_0, \xi_l(\eta)|_{M \setminus W_0}) \cong (M', \xi'_{l_0}(\eta_0))$ is contactomorphic to $(M', \xi'_l(\eta))$. If $e_0 = 1$ and $r_0 \neq \frac{ld_0}{2}$, then there is a transition from W_0 to W_1 which brings us to a state $(M \setminus W_1, \xi_l(\eta)|_{M \setminus W_1}) \cong (M', \xi'_{l_1}(\eta_1))$ with boundary slopes 0 and 1 which is overtwisted by Lemma 6.8, therefore $(M(e_0, r), \xi_l(\eta))$ is overtwisted. In the rest of the proof we will suppose either $e_0 > 1$ or $r_0 = \frac{ld_0}{2}$.

By induction, we assume we have reached a state $(M \setminus W_i, \xi_l(\eta)|_{M \setminus W_i})$ of one of the following kinds.

(1) $(M \setminus W_i, \xi_l(\eta)|_{M \setminus W_i})$ is contactomorphic to $(M', \xi_{l_i}(\eta_i))$ with boundary slopes $\frac{p_i}{q_i} - e_0$ and $\frac{p_i}{q_i}$. If $e_0 > 2$, then $l_i = l$ and η_i is isotopic to η. If $e_0 = 2$, then, either $l_i = l$ and η_i is isotopic to η, or $l_i = -l$ and $r_0^i = r_0 + l$. If $e_0 = 1$, then, either $l_i = l$ and η_i is isotopic to η, or $l_i = -l$, $r_0^i = -r_0$ and $r_1^i = r_1 + l$. Here $(r_0^i, \dots, r_n^i)$ denote the invariants determining η_i and $(r_0, \dots, r_n)$ denote the invariants determining η.

(2) $(M \setminus W_i, \xi_l(\eta)|_{M \setminus W_i})$ is contactomorphic to the tight contact manifold $(M', \xi_{\Gamma_{\Sigma_i}}(\eta))$ obtained by contact $(-\frac{1}{r})$–surgery on the S^1–invariant contact manifold $(T^2 \times I, \xi_{\Gamma_{\Sigma_i}})$. Here Γ_{Σ_i} is a multicurve on $S^1 \times I$ which closes to a multicuve $\Gamma_{\overline{\Sigma}_i}$ on T^2 consisting of a unique homotopically trivial closed curve if we identify $S^1 \times \{0\}$ to $S^1 \times \{1\}$.

We denote by $\chi_+(T^2 \setminus \Gamma_{\overline{\Sigma}_i})$ the Euler characteristic of the positive region of $T^2 \setminus \Gamma_{\overline{\Sigma}_i}$ and by $\chi_-(T^2 \setminus \Gamma_{\overline{\Sigma}_i})$ the Euler characteristic of the negative region of $T^2 \setminus \Gamma_{\overline{\Sigma}_i}$. We have here three cases.

- If $e_0 > 2$, then $\chi_+(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) - \chi_-(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) = l$ and η_i is isotopic to η.
- If $e_0 = 2$, then, either $\chi_+(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) - \chi_-(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) = l$ and η_i is isotopic to η, or $\chi_+(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) - \chi_-(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) = -l$ and $r_0^i = r_0 + l$.
- If $e_0 = 1$, then, either $\chi_+(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) - \chi_-(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) = l$ and η_i is isotopic to η, or $\chi_+(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) - \chi_-(T^2 \setminus \Gamma_{\overline{\Sigma}_i}) = -l$, $r_0^i = -r_0$ and $r_1^i = r_1 + l$.

A transition from the state $M \setminus W_i$ to the state $M \setminus W_{i+1}$ consists of taking a layer $N_i \cong T^2 \times [\frac{1}{2}, 1] \subset M \setminus W_i$ with boundary $W_{i+1} \cup W_i$, and moving it from the front to the back, or vice versa. We only consider the case when N_i is a front layer. When N_i is a back layer the proof is completely analogous. There are two cases, depending on whether the transition changes the boundary slopes or the division number of the boundary.

Case 1 This case corresponds to state transitions from $(M\setminus W_i,\, \xi_l(\eta)|_{M\setminus W_i})$ to $(M\setminus W_{i+1},\, \xi_l(\eta)|_{M\setminus W_{i+1}})$ such that W_i and W_{i+1} are minimal and at least one of them has finite slope. Suppose that $(M \setminus W_i,\, \xi_l(\eta)|_{M\setminus W_i})$ is contactomorphic to $(M', \xi'_{l_i}(\eta_i))$ and the transition changes the boundary slopes of the state from $\frac{p_i}{q_i} - e_0$ and $\frac{p_i}{q_i}$ to $\frac{p_{i+1}}{q_{i+1}} - e_0$ and $\frac{p_{i+1}}{q_{i+1}}$. We can assume that the interior of N_i contains no tori with infinite slope. If this is not the case, we split the transition in two parts, therefore we can assume that $W_{i+1} \subset M \setminus W_i$ has slope $\frac{p_{i+1}}{q_{i+1}} \in (\frac{p_i}{q_i}, \infty]$. We isotope the Seifert fibration on $M \setminus W_i$ so that W_{i+1} is a fibred torus and the singular fibre F_i is contained in $M \setminus N_i$.

By Lemma 6.4 there is a vertical Legendrian curve with twisting number 0 in $M\setminus N_i$. Using such curve we can find a neighbourhood $V_{i+1} \subset M\setminus N_i$ of the singular fibre such that $-\partial(M\setminus V_{i+1})$ has infinite slope by arguing as in the proof of Lemma 4.1. The contact structures $\xi'_{l_i}(\eta_i)|_{V_{i+1}} = \eta_{i+1}$ and $\xi'_{l_i}(\eta_i)|_{M\setminus(W_i \cup V_{i+1})} = \xi'_{l_{i+1}}$ are determined by η_i and l_i as described in Lemma 6.8. In particular, $\xi'_{l_i}(\eta_i)|_{N_i} = \xi'_{l_{i+1}}|_{N_i}$ is a minimally twisting tight contact structure with relative Euler class $\pm \begin{pmatrix} q_i - q_{i+1} \\ p_i - p_{i+1} \end{pmatrix}$, with the sign depending on l_{i+1}. Moving N_i to the back, its relative Euler class becomes $\pm \begin{pmatrix} q_i - q_{i+1} \\ p_i - p_{i+1} - e_0(q_i - q_{i+1}) \end{pmatrix}$, so $\xi_l(\eta)|_{(M\setminus W_{i+1} \cup V)}$ is contactomorphic to $\xi'_{l_{i+1}}(\eta_{i+1})$ with boundary slopes

$\frac{p_{i+1}}{q_{i+1}} - e_0$ and $\frac{p_{i+1}}{q_{i+1}}$. This proves that any admissible transition transforms a state of the type described in the inductive assumption to another state of the same type.

Case 2 This case corresponds to state transitions from $(M \setminus W_i, \xi_l(\eta)|_{M\setminus W_i})$ to $(M \setminus W_{i+1}, \xi_l(\eta)|_{M\setminus W_{i+1}})$ such that W_i and W_{i+1} have both infinite slope. Suppose that $(M \setminus W_i, \xi_l(\eta)|_{M\setminus W_i})$ is contactomorphic to $(M', \xi_{\Gamma_{\Sigma_i}}(\eta_i))$ and $\#\Gamma_{W_{i+1}} = \#\Gamma_{W_i} \pm 1$. We isotope the Seifert fibration $M' \to S^1 \times I$ so that the singular fibre is contained in $M' \setminus N_i$ and, fixed a neighbourhood V_{i+1} of the singular fibre so that $-\partial(M \setminus V_{i+1})$ has infinite slope, the restrictions of the fibration to N_i and $M' \setminus (N_i \cup V_{i+1})$ are S^1–bundles. Let Σ be a pair of pants and let $\overline{\Sigma}$ be a punctured torus obtained by identifying two of the boundary components of Σ. Let $\sigma_i \colon \overline{\Sigma} \to M \setminus V_{i+1}$ be a section so that $\Sigma_i' = \sigma_i(\Sigma) \subset M'$ is a convex, $\#\Gamma$–minimising surface with Legendrian boundary, and $\Sigma_i' \cap W_{i+1}$ is a Legendrian curve. We call $\overline{\Sigma}_i' = \sigma_i(\overline{\Sigma}) \subset M \setminus V_{i+1}$. $(M \setminus W_i, \xi_l(\eta)|_{M\setminus W_i})$ is contactomorphic to $(M', \xi_{\Gamma_{\Sigma_i'}}(\eta_{i+1}))$, therefore, by Lemma 6.9, η_{i+1} is isotopic to η_i and $\Gamma_{\overline{\Sigma}_i'}$ is isotopic to $\Gamma_{\overline{\Sigma}_i}$. If we define $\Sigma_{i+1} \subset (M \setminus (W_{i+1} \cup V_{i+1})$ as $\Sigma_{i+1} = \overline{\Sigma}_i' \setminus W_{i+1}$, then $\Gamma_{\overline{\Sigma}_{i+1}} = \Gamma_{\overline{\Sigma}_i'}$ and $(M \setminus W_{i+1}, \xi_l(\eta)|_{M\setminus W_{i+1}})$ is contactomorphic to $(M', \xi_{\Gamma_{\Sigma_{i+1}}}(\eta_{i+1}))$. □

As proved in Theorem 4.18, the tight contact structures considered in this section become overtwisted after lifting to any finite covering of $M(e_0, r)$ induced by a covering of T^2, so they are the exceptional tight contact structure of Theorem 2.3. The following corollary gives the number of the exceptional tight contact structures on $M(e_0, r)$.

Corollary 6.11 *When $e_0 > 0$ the number of exceptional tight contact structures on $M(e_0, r)$ is finite and positive. It is*

- $2|d_0(d_1+1)\dots(d_k+1)|$ *if* $e_0 > 2$,
- $|(d_0-1)(d_1+1)\dots(d_k+1))|$ *if* $e_0 = 2$,
- $|d_1(d_2+1)\dots(d_k+1)|$ *if* $e_0 = 1$.

The last expression has to be interpreted as 2 when $-\frac{1}{r} = d_0 \in \mathbb{Z}$.

Proof By Theorem 6.10 and 2.4, when $e_0 > 2$ for any l there are $|d_0(d_1+1)\dots(d_k+1)|$ choices for η and 2 choices for the background, and all choices give distinct tight contact structures, therefore the total number of exceptional tight contact structures on $M(e_0, r)$ is $2|d_0(d_1+1)\dots(d_k+1)|$.

When $e_0 = 2$, there are $|d_0(d_1+1)\dots(d_k+1)|$ choices for η for any choice of the background, but not all choices give distinct tight contact structures. In fact, any exceptional tight contact structure with $l = -2$ and $r_0 > d_0$ is isotopic to a tight contact structure with $l = 2$, therefore we count $|d_0(d_1+1)\dots(d_k+1)|$ distinct exceptional tight contact structures with $l = 2$ and $|(d_1+1)\dots(d_k+1)|$ distinct tight contact structures with $l = -2$, namely those obtained from η with $r_0 = d_0$. The total number of distinct exceptional tight contact structures up to isotopy on $M(e_0, r)$ with $e_0 = 2$ is therefore $|(d_0-1)(d_1+1)\dots(d_k+1)|$.

If $e_0 = 1$ and $\frac{1}{d} \notin \mathbb{Z}$, then for any choice of the background there are $|(d_1 + 1)\dots(d_k + 1)|$ choices for η, but not all choices give distinct tight contact structures. In fact, we have $|(d_1 + 1)\dots(d_k + 1)|$ distinct exceptional tight contact structures with $l = 2$ up to isotopy, and $|(d_2 + 1)\dots(d_k + 1)|$ distinct exceptional tight contact structures with $l = -2$ which have not already been counted, namely the ones with $r_i = d_i$. The total number of exceptional tight contact structures on $M(e_0, r)$ with $e_0 = 1$ is therefore $|d_1(d_2+1)\dots(d_k+1)|$. If $e_0 = 1$ and $\frac{1}{r} \in \mathbb{Z}$, for any l there is only one possibility for η, and different choices for the background produce non isotopic tight contact structures, therefore the total of exceptional tight contact structures on $M(e_0, r)$ with $e_0 = 1$ and $\frac{1}{r} \in \mathbb{Z}$ is 2. □

The exceptional tight contact structures on $M(e_0, r)$ are negative contact surgeries on the exceptional tight contact structures on $T(e_0)$ when $e_0 > 1$. On the contrary, there are no exceptional tight contact structures on $T(1)$, and the exceptional tight contact structures on $M(1, r)$ are negative contact surgeries on an overtwisted contact structure on $T(1)$. On $T(2)$ there is only one exceptional tight contact structure up to isotopy, therefore the two backgrounds extend to isotopic tight contact structure on $T(2)$. This reflects the fact that $T(2)$ with the exceptional tight contact structure contains two non Legendrian isotopic vertical Legendrian curves with twisting number 0, and negative contact surgeries with the same surgery data on such curves yield different contact manifolds. The shuffling between the background and the surgery data when $e_0 = 2$ shows that suitably stabilisations of the two non isotopic vertical Legendrian curves with twisting number 0 become Legendrian isotopic.

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