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Motivic cell structures

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Abstract An object in motivic homotopy theory is called cellular if it can be built out of motivic spheres using homotopy colimit constructions. We explore some examples and consequences of cellularity. We explain why the algebraic K-theory and algebraic cobordism spectra are both cellular, and prove some Künneth theorems for cellular objects.

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1 Introduction

If $\mathcal{M}$ is a model category, and $\mathcal{A}$ is a set of objects in $\mathcal{M}$, one can consider the class of $\mathcal{A}$-*cellular objects*—things that can be built from the homotopy types in $\mathcal{A}$ by iterative homotopy colimit constructions. In the case of topological spaces such cellular classes have been studied by Farjoun [DF] and others. Another place these ideas have appeared is in the work of Dwyer, Greenlees, and Iyengar [DGI], who imported them into homological algebra. This paper is concerned with cellularity in motivic homotopy theory.

Recall that in the motivic context there is a bi-graded family of 'spheres' $S^{p,q}$; we will take this family as our set $\mathcal{A}$. One gets a slightly different theory depending on whether one works unstably or stably. In this paper we develop the basic theory concerning cellular objects in both contexts, and collect an assortment of results which we've found useful in applications. Specifically:

(1) We describe a collection of techniques for showing that schemes are cellular, and apply these to toric varieties, Grassmannians, Stiefel manifolds, and certain quadrics.

(2) We show that the algebraic K-theory spectrum KGL and the motivic cobordism spectrum MGL are stably cellular.

(3) For cellular objects, the usual collection of computational tools carries over from ordinary stable homotopy theory to motivic stable homotopy theory (see Section 7). If E is a motivic ring spectrum then we use these ideas to construct a convergent, tri-graded, Künneth spectral sequence for $E^{*,*}(X \times Y)$ as long as X or Y satisfies some kind of cellularity condition. See Theorems 8.6 and 8.12.

Our experience has been that this material is a good starting point for understanding some of the inner workings of motivic homotopy theory, and so we have tried to make the paper readable to people who only know the basic definitions from [MV].

1.1 Notions related to cellularity

Algebraic geometers have worked with the related notion of a scheme with an *algebraic cell decomposition* [F2, Ex. 1.9.1], and its generalization to that of a *linear variety* (introduced by Totaro [T]). A scheme X has an algebraic cell decomposition if it has a filtration by closed subschemes $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq \emptyset$ such that each complement $X_{i+1} - X_i$ is a disjoint union $\coprod_{n_{ij}} \mathbb{A}^{n_{ij}}$ of affines. The linear varieties constitute the smallest class which contains the affine spaces $\mathbb{A}^n$ and has the property that if $Z \hookrightarrow X$ is a closed inclusion and at least two of the varieties Z, X, and $X - Z$ are in the class, then so is the third.

These notions are useful when studying cohomology theories which have a *localization* (or *Gysin*) sequence, because the cohomology of a linear variety can be understood inductively. In the language of motivic homotopy theory these are the *algebraically oriented* cohomology theories, i.e., the ones that have Thom isomorphisms. This means that the cohomology of the Thom space of a bundle over Z is isomorphic to the cohomology of Z (up to a shift). If $Z \hookrightarrow X$ is a closed inclusion of smooth schemes then there is a homotopy cofiber sequence of the form $X - Z \to X \to \operatorname{Th} N_{X/Z}$ [MV, Thm. 3.2.23], where $\operatorname{Th} N_{X/Z}$ is the Thom space of the normal bundle of Z in X. One gets a long exact sequence relating the cohomology of $X - Z$, X, and Z.

Our class of stably cellular varieties is very close to the class of linear varieties. For every linear variety which we've tried to prove is stably cellular, we've been able to do so (and vice versa); however, proving that something is cellular is often much harder. This is true for the Grassmannians $\mathrm{Gr}_k(\mathbb{A}^n)$, for instance. The Schubert cells give an 'algebraic cell decomposition' showing that the Grassmannian is linear, but to show the variety is cellular it is not enough just to see the cells inside the variety: one has to produce an 'attaching map' showing explicitly how to build up the variety via homotopy colimits. In the Schubert cell approach one runs into some hairy problems in trying to make this work, which we have not been able to resolve. Our proof that Grassmannians are cellular follows a completely different strategy.

If one is only interested in cohomology theories with Thom isomorphism, then perhaps there is no reason for studying cellular varieties as opposed to linear ones. But the notion of 'cellular' seems more familiar and sensible to a topologist, and most of our techniques for understanding the classical stable homotopy category depend in some way on things being built from cells. As those techniques get imported into motivic homotopy theory, the notion of cellularity may become more useful.

1.2 Non-cellular varieties

Folklore says most schemes are not cellular. This should be a consequence of the theory of weights in the cohomology of algebraic varieties [De]. The spheres $S^{p,q}$ only have even weights in their cohomology, and so it should be impossible to construct something with odd weights (like an elliptic curve, for instance) from the spheres.

Unfortunately, to write down a careful proof that an elliptic curve is not cellular seems to require surmounting some obstacles. One possibility is to work over $\mathbb{C}$ and use mixed Hodge theory, but this requires showing that the mixed Hodge structures are well-defined invariants of the motivic stable homotopy category. This takes at least a little work, due to the presence of infinite objects (like infinite wedges of schemes, etc.) in the stable homotopy category.

Another possibility is to work over a number field k, and to use the weights coming from the Galois actions on l-adic cohomology (again, see [De]). Here, one should show that there is a realization functor from the motivic stable homotopy category to the derived category of $Gal(\bar{k}/k)$-modules. Proposition 9.4 shows that if an elliptic curve E is cellular then it can actually be built from spheres using a finite number of extensions and retracts, and so the argument with weights should work out. We have not pursued this further, however.

1.3 Pointed versus non-pointed

In this paper we are mostly interested in cellularity within the context of the stable motivic homotopy category. We briefly pursue the notion in the *pointed,* *unstable* motivic homotopy category as well, largely because it is convenient. One could also treat similar notions in the *unpointed* context, but in the present paper we have no need of this. It is perhaps worth remarking, though, that if a scheme is cellular in the unpointed motivic homotopy category then it must have a rational point. We leave this as an exercise for the interested reader.

1.4 Background

We assume a familiarity with model categories throughout this paper. Good references are [H], [Ho1], and [DS].

2 Cellular objects

Let $\mathcal{M}$ be a pointed model category, and let $\mathcal{A}$ be a set of objects in $\mathcal{M}$.

Definition 2.1 The class of $\mathcal{A}$*-cellular* objects is the smallest class of objects of $\mathcal{M}$ such that

(1) every object of $\mathcal{A}$ is $\mathcal{A}$-cellular;

(2) if X is weakly equivalent to an $\mathcal{A}$-cellular object, then X is $\mathcal{A}$-cellular;

(3) if $D\colon I \to \mathcal{M}$ is a diagram such that each D_i is $\mathcal{A}$-cellular, then so is $\operatorname{hocolim} D$.

The idea is that the $\mathcal{A}$-cellular objects are the ones that can, up to homotopy, be built out of objects in $\mathcal{A}$. This definition is precisely the same as the usual notion of cellularity for the category of pointed topological spaces [DF, Ex. 2.D.2.1].

We recall some useful results about cellularity in general. These properties apply to all possible $\mathcal{M}$ and $\mathcal{A}$. To start with, note that any contractible object of $\mathcal{M}$ (i.e., an object weakly equivalent to the initial and terminal object $*$) is $\mathcal{A}$-cellular because it is the homotopy colimit of an empty diagram.

Lemma 2.2 *If $X \to Y \to Z$ is a homotopy cofiber sequence in $\mathcal{M}$ such that X and Y are $\mathcal{A}$-cellular, then so is Z.*

Proof The object Z is the homotopy pushout of the diagram $* \leftarrow X \to Y$. □

Lemma 2.3 *If X is $\mathcal{A}$-cellular, then so is ΣX.*

Proof Apply Lemma 2.2 to the homotopy cofiber sequence $X \to * \to \Sigma X$. $\square$

Recall that a pointed model category is *stable* if the suspension and loops functors Σ and Ω are inverse self-equivalences of the homotopy category. Throughout the paper we will abuse notation and also write Σ (resp., Ω) for a chosen derived functor of suspension (resp., loops) on the model category level. For instance, to compute ΣX we can factor $X \to *$ as a cofibration $X \rightarrowtail CX$ followed by trivial fibration $CX \xrightarrow{\sim} *$ and then take the quotient CX/X.

Lemma 2.4 *Suppose that $\mathcal{M}$ is a stable model category. Also suppose that for every object A of $\mathcal{A}$, ΩA is weakly equivalent to an object of $\mathcal{A}$. Then an object X in $\mathcal{M}$ is $\mathcal{A}$-cellular if and only if ΣX is $\mathcal{A}$-cellular.*

Proof Consider the class $\mathcal{C}$ of objects X of $\mathcal{M}$ such that ΩX is cellular; we want to show that $\mathcal{C}$ coincides with the class of $\mathcal{A}$-cellular objects. By Lemma 2.3, $\mathcal{C}$ is contained in the class of $\mathcal{A}$-cellular objects. To show that $\mathcal{C}$ contains the class of $\mathcal{A}$-cellular objects, it suffices to check that $\mathcal{C}$ has the three properties listed in Definition 2.1.

Property (1) follows immediately from the hypothesis of the lemma, property (2) follows immediately from the fact that Ω respects weak equivalences, and property (3) follows from the fact that Ω respects homotopy colimits in a stable model category. $\square$

Lemma 2.5 *Suppose that $\mathcal{M}$ is a stable model category. Also suppose that for every object A of $\mathcal{A}$, ΩA is weakly equivalent to an object of $\mathcal{A}$. If $X \to Y \to Z$ is a homotopy cofiber sequence in $\mathcal{M}$ such that any two of X, Y, and Z are $\mathcal{A}$-cellular, then so is the third.*

Proof One case is Lemma 2.2. For the other two cases, first observe from Lemma 2.4 that ΩY is $\mathcal{A}$-cellular whenever Y is (and similarly for ΩZ). Now, the object Y is the homotopy pushout of a diagram $* \leftarrow \Omega Z \to X$, and the object X is the homotopy pushout of the diagram $* \leftarrow \Omega Y \to \Omega Z$. $\square$

2.6 The motivic setting

Let $\mathcal{MV}$ denote the Morel-Voevodsky model category on simplicial presheaves over Sm_k, the site of smooth schemes of finite type over some fixed ground field k. In fact, there are at least three versions of this model category: the injective [MV], [Ja, App. B], the flasque [I], and the projective [Bl]. The identity functors give Quillen equivalences between these model categories (which have the same class of weak equivalences), and this guarantees that it doesn't matter which model structure is considered for the purposes of cellularity. Thus, in each situation we can choose whichever structure is most convenient. Unless otherwise stated, we will use the injective version because it is convenient to have all objects cofibrant. Actually, we will work with the pointed category $\mathcal{MV}_*$, in which every object is equipped with a map from $\mathrm{Spec}(k)$.

We recall the following two important kinds of weak equivalences in $\mathcal{MV}$. First, if $\{U, V\}$ is a basic Nisnevich cover of X [MV, Defn. 3.1.3], then the map

$$U \amalg_{(U \times_X V)} V \to X$$

is an $\mathbb{A}^1$-weak equivalence. Second, if X is any object of Sm_k then the map $X \times \mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-weak equivalence. In a certain sense, these two kinds of maps generate all $\mathbb{A}^1$-weak equivalences—cf. [D, Sec. 8.1], especially the paragraph following the proof of (8.1). Every proof that an object is cellular must necessarily come down to using these two facts.

Actually, in this paper we will never have to explicitly use the Nisnevich topology—all our arguments only involve Zariski covers and facts from [MV]. Also, the $\mathbb{A}^1$-weak equivalence for two-fold covers given in the last paragraph implies a corresponding statement for Zariski covers of any size: the $\mathbb{A}^1$-homotopy type of a scheme can be recovered from a Zariski cover by an appropriate gluing construction. This is what we will mostly use (see Lemma 3.6 below for a precise statement).

Recall that $S^{1,1} = \mathbb{A}^1 - 0$, with the point 1 as basepoint; and $S^{1,0}$ is the constant simplicial presheaf whose sections are the simplicial set $\Delta^1/\partial\Delta^1$. For $p \geq q \geq 0$, one defines

$$S^{p,q} = (S^{1,0} \wedge \cdots \wedge S^{1,0}) \wedge (S^{1,1} \wedge \cdots \wedge S^{1,1})$$

where there are $p - q$ copies of $S^{1,0}$ and q copies of $S^{1,1}$.

Definition 2.7 Let $\mathcal{A} = \{S^{p,q} \,|\, p \geq q \geq 0\}$ be the set of spheres in $\mathcal{MV}_*$. An object X in $\mathcal{MV}_*$ is *unstably cellular* if X is $\mathcal{A}$-cellular.

If X is a pointed (possibly non-smooth) scheme, then the statement "X is unstably cellular" means that the object of $\mathcal{MV}_*$ represented by X is unstably cellular.

2.8 Motivic spectra

We let $\mathcal{S}pectra(\mathcal{MV})$ denote the category of symmetric $\mathbb{P}^1$-spectra. Starting with the injective model structure on $\mathcal{MV}_*$, we get an induced model structure on $\mathcal{S}pectra(\mathcal{MV})$ from [Ho2, Defn. 8.7]. This turns out to be identical to the one provided by [Ja, Thm. 4.15]. Note that there is a Quillen pair $\Sigma^\infty\colon \mathcal{MV}_* \rightleftarrows \mathcal{S}pectra(\mathcal{MV})\colon \Omega^\infty$. We may desuspend the objects $\Sigma^\infty(S^{p,q})$ in both variables, giving spectra which we will denote $S^{p,q}$ for all $p, q \in \mathbb{Z}$.

Remark 2.9 One can also use Bousfield-Friedlander $\mathbb{P}^1$-spectra rather than symmetric spectra. The model structure is provided by [Ho2, Defn. 3.3] or [Ja, Thm. 2.9]. Since this model category is Quillen equivalent to that of symmetric $\mathbb{P}^1$-spectra [Ho2, Sec. 10], all our basic results hold in either one. The smash product for symmetric spectra will be important in Sections 7 and 8, however.

Definition 2.10 Let $\mathcal{B} = \{S^{p,q} \,|\, p, q \in \mathbb{Z}\}$ be the class of all spheres in $\mathcal{S}pectra(\mathcal{MV})$. An object X of $\mathcal{S}pectra(\mathcal{MV})$ is *cellular* if X is $\mathcal{B}$-cellular. We say that an object X in $\mathcal{MV}_*$ is *stably cellular* if $\Sigma^\infty X$ is cellular.

Again, the statement that a (possibly non-smooth) *pointed scheme* is 'stably cellular' means that the object of $\mathcal{MV}_*$ represented by X is stably cellular.

Readers who find themselves worried about basepoints in the following examples should refer to Section 2.15.

Example 2.11 The scheme $\mathbb{A}^n - 0$ is unstably cellular, because after choosing any basepoint $\mathbb{A}^n - 0$ is weakly equivalent to the sphere $S^{2n-1,n}$. This fact is claimed in [MV, Ex. 3.2.20]. For the convenience of the reader, we give a detailed explanation.

For $n = 1$ this is the definition that $S^{1,1}$ equals $\mathbb{A}^1 - 0$. For $n = 2$, we cover $\mathbb{A}^2 - 0$ by the open sets $U = (\mathbb{A}^1 - 0) \times \mathbb{A}^1$ and $V = \mathbb{A}^1 \times (\mathbb{A}^1 - 0)$. Then $U \cap V = (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0)$, and we have an associated homotopy pushout square

$$\begin{array}{ccc} (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) & \longrightarrow & (\mathbb{A}^1 - 0) \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ (\mathbb{A}^1 - 0) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 - 0. \end{array}$$

By projecting away the $\mathbb{A}^1$ factors, we find that $\mathbb{A}^2 - 0$ is weakly equivalent to the homotopy pushout of the diagram

$$\mathbb{A}^1 - 0 \xleftarrow{\pi_1} (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \xrightarrow{\pi_2} (\mathbb{A}^1 - 0).$$

In order to compute this homotopy pushout, we look at the diagram

$$\begin{array}{ccccc} * & \longleftarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{A}^1 - 0 & \longleftarrow & (\mathbb{A}^1 - 0) \vee (\mathbb{A}^1 - 0) & \longrightarrow & \mathbb{A}^1 - 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^1 - 0 & \longleftarrow & (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) & \longrightarrow & \mathbb{A}^1 - 0 \end{array}$$

in which the two middle horizontal arrows collapse one summand to a point, and compute the homotopy colimit in two ways. If we first compute the homotopy colimits of the rows and then the homotopy colimit of this new diagram, we get the desired homotopy pushout because the homotopy colimit of either of the top two rows is contractible. On the other hand, if we first compute the homotopy colimits of the vertical columns, then we get

$$* \longleftarrow (\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0) \longrightarrow *,$$

and then the homotopy colimit of this new diagram is $\Sigma((\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0))$. Since we must get the same homotopy type no matter which way we go about computing the homotopy colimit, it must be that the desired homotopy pushout is $\Sigma[(\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0)]$, which is $S^{3,2}$.

For arbitrary n one proceeds by induction, covering $\mathbb{A}^n - 0$ by $(\mathbb{A}^{n-1} - 0) \times \mathbb{A}^1$ and $\mathbb{A}^n \times (\mathbb{A}^1 - 0)$ and using the same argument to see that $\mathbb{A}^n - 0$ is weakly equivalent to $\Sigma[(\mathbb{A}^{n-1} - 0) \wedge (\mathbb{A}^1 - 0)]$.

Example 2.12 According to [MV, Cor. 3.2.18], there is a cofiber sequence

$$\mathbb{P}^{n-1} \to \mathbb{P}^n \to S^{2n,n}$$

in $\mathcal{MV}_*$ after choosing any basepoint for $\mathbb{P}^{n-1}$. This shows inductively that $\mathbb{P}^n$ is stably cellular. It is a consequence of the following proposition that $\mathbb{P}^n$ is in fact unstably cellular.

For $n \geq 1$, there is a canonical projection map $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$ sending a point v to the line spanned by v. Also, there is a natural inclusion $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ coming from the inclusion $\mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^{n-1} \oplus \mathbb{A}^1 = \mathbb{A}^n$.

Proposition 2.13 *For $n \geq 1$, there is a homotopy cofiber sequence $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ in $\mathcal{MV}_*$ after choosing any basepoint in $\mathbb{A}^n - 0$. Thus each $\mathbb{P}^n$ is unstably cellular.*

Proof We decompose $\mathbb{A}^{n+1}$ into $\mathbb{A}^n \oplus \mathbb{A}^1$. Let l be the line spanned by the vector $(\mathbf{0}, 1)$ (where the notation is with respect to this decomposition), and let $U = \mathbb{P}^n - \{l\}$. There is an open embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ which sends $\mathbf{v}$ to the line spanned by $(\mathbf{v}, 1)$—call this open subset V. Then $U \cap V$ is isomorphic to $\mathbb{A}^n - 0$, and we have a homotopy pushout square

$$\begin{array}{ccc} \mathbb{A}^n - 0 & \longrightarrow & \mathbb{P}^n - \{l\} \\ \downarrow & & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{P}^n. \end{array}$$

Since $\mathbb{A}^n$ is contractible, this identifies $\mathbb{P}^n$ with the homotopy cofiber of the map $\mathbb{A}^n - 0 \to \mathbb{P}^n - \{l\}$.

Now, there is a projection map $\mathbb{P}^n - \{l\} \to \mathbb{P}^{n-1}$ induced by the obvious projection $\mathbb{A}^{n-1} \oplus \mathbb{A}^1 \to \mathbb{A}^{n-1}$. This is a line bundle, and is therefore a weak equivalence (in fact, an $\mathbb{A}^1$-homotopy equivalence). The composite $\mathbb{A}^n - 0 \hookrightarrow \mathbb{P}^n - \{l\} \to \mathbb{P}^{n-1}$ is precisely our projection map $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$. So the homotopy cofiber of $\mathbb{A}^n - 0 \to \mathbb{P}^n - \{l\}$ is weakly equivalent to the homotopy cofiber of $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$. □

Remark 2.14 Note that the homotopy cofiber sequence $\mathbb{A}^1 - 0 \to * \to \mathbb{P}^1$ identifies $\mathbb{P}^1$ with $\Sigma(\mathbb{A}^1 - 0) \simeq \Sigma(S^{1,1}) \simeq S^{2,1}$.

2.15 Basepoints

When working with unstable cellularity one has to be a little careful about basepoints. Here is one case where the issue disappears:

Proposition 2.16 *Suppose X is an object of $\mathcal{MV}$ and $a, b\colon * \to X$ are two choices of basepoint. If a and b are weakly homotopic in $\mathcal{MV}$, then (X, a) is weakly equivalent to (X, b) in $\mathcal{MV}_*$ (hence one is unstably cellular if and only if the other is).*

Proof First, one readily reduces to the case where X is fibrant. Let X' be the pushout in $\mathcal{MV}$ of the diagram

$$X \xleftarrow{a} * \xrightarrow{i_0} \mathbb{A}^1$$

where i_0 is the inclusion of $\{0\}$; the idea is that X' is X with an 'affine whisker' attached. The map $* \to \mathbb{A}^1$ is a trivial cofibration, so $X \to X'$ is also one. It follows that the map $X' \to X$ which collapses the $\mathbb{A}^1$ onto a is a weak equivalence in $\mathcal{MV}$. If we regard X' as pointed via the map $1 \to \mathbb{A}^1$, then the same map is a weak equivalence $(X', 1) \to (X, a)$ in $\mathcal{MV}_*$.

As a and b are weakly homotopic and X is fibrant, there is a map $H\colon \mathbb{A}^1 \to X$ such that $Hi_0 = a$ and $Hi_1 = b$. This induces a map $X' \to X$ sending 1 to b, which is readily checked to be a weak equivalence. So we have a zig-zag of weak equivalences in $\mathcal{MV}_*$ of the form $(X, a) \leftarrow (X', 1) \to (X, b)$. □

The applicability of the above result is limited by the fact that in motivic homotopy theory the set $\mathrm{Ho}(*, X)$ is often bigger than one would expect. For instance $\mathrm{Ho}(*, \mathbb{A}^1 - 0) = k^*$, which is extremely big if k is the complex numbers. For $\mathbb{A}^1 - 0$ the basepoint doesn't matter for another reason, namely because all choices are equivalent up to automorphism.

It turns out that we will almost always work with stable cellularity, and in this context the basepoint can be ignored:

Proposition 2.17 *If $X \in \mathcal{MV}_*$, then X is stably cellular if and only if X_+ is stably cellular. As a consequence, if $X \in \mathcal{MV}$ has two basepoints x_0 and x_1 then (X, x_0) is stably cellular if and only if (X, x_1) is stably cellular.*

Proof The first statement follows from considering the cofiber sequence $\Sigma^\infty S^{0,0} \to \Sigma^\infty(X_+) \to \Sigma^\infty X$ and applying Lemma 2.5. The second statement is true because either condition is equivalent to X_+ being stably cellular. □

Because of Proposition 2.17 we can now rephrase the definition of stable cellularity.

Definition 2.18 A (pointed or unpointed) object X of $\mathcal{MV}$ is *stably cellular* if $\Sigma^\infty(X_+)$ is stably cellular in $\mathcal{S}pectra(\mathcal{MV})$.

This is the definition that we will use from now on. It saves us the trouble of having to worry about basepoints.

3 Basic results

The notions of unstable cellularity and stable cellularity are related by the following lemma.

Lemma 3.1 *If X is an unstably cellular object of $\mathcal{MV}_*$, then it is also stably cellular.*

Proof The functor Σ^∞ is a left Quillen functor, so it respects weak equivalences and homotopy colimits. Thus, it suffices to show that $\Sigma^\infty S^{p,q}$ is stably cellular. But this is isomorphic to the stable sphere $S^{p,q}$, which is stably cellular by definition. □

We will now study the basic constructions that behave well with respect to cellularity.

Lemma 3.2 *If each X_i is a stably cellular object of $\mathcal{MV}$, then so is $X = \coprod_i X_i$.*

Proof This follows from the simple calculation that $\Sigma^\infty(X_+)$ is isomorphic to $\bigvee_i \Sigma^\infty(X_{i+})$. □

The set of spheres is closed under smash product. This implies that smash products preserve unstably cellular objects.

Lemma 3.3 *If X and Y are unstably cellular objects, then so is $X \wedge Y$.*

Proof The category $\mathcal{MV}_*$ with its injective, motivic model structure and smash product is a symmetric monoidal model category (the verification of the pushout-product axiom is a routine exercise). Since every object in $\mathcal{MV}_*$ is injective cofibrant, the functor $X \wedge (-)$ is a left Quillen functor from the pointed category $\mathcal{MV}_*$ to itself—so it respects homotopy colimits and weak equivalences. Thus, it suffices to show that $X \wedge S^{p,q}$ is unstably cellular for every p and q.

But the functor $(-) \wedge S^{p,q}$ also respects homotopy colimits and weak equivalences, so it suffices to show that $S^{s,t} \wedge S^{p,q}$ is unstably cellular. This is isomorphic to $S^{s+p,t+q}$, which is unstably cellular by definition. □

Note, in particular, that if X is a pointed unstably cellular object, then so is $\Sigma^{p,q}X$.

Lemma 3.4 *If X and Y are stably cellular objects of $\mathcal{MV}$, then so are $X \wedge Y$ and $X \times Y$.*

Proof The proof for $X \wedge Y$ works just as in Lemma 3.3, using the facts that $\Sigma^\infty(X \wedge Y)$ is weakly equivalent to $\Sigma^\infty X \wedge \Sigma^\infty Y$ and that every suspension spectrum is cofibrant.

For $X \times Y$, there is an unstable cofiber sequence $X \vee Y \to X \times Y \to X \wedge Y$, so we also have a stable cofiber sequence

$$\Sigma^\infty(X \vee Y) \to \Sigma^\infty(X \times Y) \to \Sigma^\infty(X \wedge Y).$$

We just showed that the third term is stably cellular. The first term is isomorphic to $\Sigma^\infty X \vee \Sigma^\infty Y$, which is a homotopy colimit of stably cellular spectra and thus also stably cellular. Hence, the second term is stably cellular as well. □

Example 3.5 By Lemma 3.4, the torus $(\mathbb{A}^1 - 0)^k$ is stably cellular for every k. We do not know if tori are unstably cellular, but we suspect not. Throughout the rest of the paper we will find that products arise all over the place, which is why we end up working primarily with stable cellularity.

Lemma 3.6 *If X is a scheme and $U_* \to X$ is a hypercover in $\mathcal{MV}$ in the sense of [SGA4, Defn. 7.3.1.2] (or [DHI, Defn. 4.2]) and each U_n is stably cellular, then X is also stably cellular. If the hypercover is in $\mathcal{MV}_*$ and each U_n is unstably cellular, then so is X.*

Proof This follows simply from the fact that the map $\operatorname{hocolim}_n U_n \to X$ is a weak equivalence in $\mathcal{MV}$ [DHI, Thm. 6.2]. Then $\operatorname{hocolim}_n \Sigma^\infty(U_{n+}) \to \Sigma^\infty(X_+)$ is also a weak equivalence. □

Note that if X is a scheme and $\{U_i\}$ a Zariski open cover of X, then the associated Čech complex is a hypercover in the above sense. It is not necessary that X be smooth here.

Definition 3.7 A Zariski cover $\{U_\alpha\}$ of a scheme X is *completely stably cellular* if each intersection $U_{\alpha_1 \cdots \alpha_n} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ is stably cellular.

A similar definition can be made with the notion of unstably cellular, but we will not bother with it.

Lemma 3.8 *If a variety X has a Zariski cover which is completely stably cellular, then X is also stably cellular.*

Proof Let $\{U_\alpha\}$ be the cover of X. Consider the Čech complex C_* of this cover, which is a simplicial scheme such that C_n is $\coprod U_{\alpha_0\alpha_1\cdots\alpha_n}$. Now C_* is obviously a hypercover of X in $\mathcal{MV}$ (cf. [DHI, 3.4, 4.2]). Each C_n is stably cellular by Lemma 3.2, so Lemma 3.6 applies. □

Recall that an algebraic fiber bundle with fiber F is a map $E \to B$ which in the Zariski topology on B is locally isomorphic to a projection $U \times F \to U$.

Lemma 3.9 *If $p\colon E \to B$ is an algebraic fiber bundle with fiber F such that F is stably cellular and B has a completely stably cellular cover that trivializes p, then E is also stably cellular.*

Proof Let $\{U_0, \dots, U_k\}$ be the completely stably cellular cover of B. Consider the cover $\{V_0, \dots V_k\}$ of E, where $V_i = p^{-1}U_i$. Each V_i is isomorphic to $F \times U_i$, so it is stably cellular by Lemma 3.4. Moreover, the intersections $V_{i_0\cdots i_n}$ are isomorphic to $F \times U_{i_0\cdots i_n}$, so this cover of E is completely stably cellular. Lemma 3.8 finishes the proof. □

Corollary 3.10 *If $p\colon E \to B$ is an algebraic vector bundle such that B has a completely stably cellular cover that trivializes p, then the Thom space $\mathrm{Th}(p)$ is also stably cellular.*

Proof Let $s\colon B \to E$ be the zero section of the vector bundle. From the definition of a Thom space [MV, Defn. 3.2.16], we have an unstable cofiber sequence

$$E - s(B) \to E \to \mathrm{Th}(p).$$

So we just have to show that the first two terms in this sequence are stably cellular. First, E is weakly equivalent to B (because E can be fiberwise linearly contracted onto its zero section), and B is stably cellular by Lemma 3.8.

Next, $E - s(B) \to B$ is an algebraic fiber bundle with fiber $\mathbb{A}^n - 0$. By Lemma 3.9, we know that $E - s(B)$ is stably cellular provided that $\mathbb{A}^n - 0$ is. Recall that $\mathbb{A}^n - 0$ is weakly equivalent to $S^{2n-1,n}$, so it is stably cellular. □

Lemma 3.11 *If x is a closed point of a smooth scheme X, then X is stably cellular if and only if $X - \{x\}$ is stably cellular.*

Proof The homotopy purity theorem [MV, Thm. 3.2.23] tells us that there is a cofiber sequence in $\mathcal{MV}$ of the form

$$X - \{x\} \to X \to \mathrm{Th}(p),$$

where p is the normal bundle of $\{x\}$ in X. Now $\mathrm{Th}(p)$ is just $\mathbb{A}^n/(\mathbb{A}^n - 0) \simeq S^{2n,n}$, where n is the dimension of X. Thus we have a cofiber sequence

$$X - \{x\} \to X \to S^{2n,n},$$

in which the third term is stably cellular. Lemma 2.5 finishes the proof. □

Remark 3.12 Suppose that $Z \hookrightarrow X$ is a closed inclusion of smooth schemes. The homotopy purity theorem gives a cofiber sequence $X - Z \to X \to \mathrm{Th}\, N_{X/Z}$. It is tempting to conclude that $\mathrm{Th}\, N_{X/Z}$ is cellular if Z is cellular, and so X is stably cellular if both Z and $X - Z$ are stably cellular. Unfortunately we haven't been able to prove the first implication, only the weaker version in Corollary 3.10. This weakness is the main reason that it often feels like more work than it should be to prove that a variety is cellular; in particular it is what causes trouble with Grassmannians in the next section.

4 Grassmannians and Stiefel varieties

Proposition 4.1 *The variety GL_n is stably cellular for every $n \geq 1$.*

Proof The proof is by induction on n. Let l be the line spanned by $(1, 0, \ldots, 0)$ in $\mathbb{A}^n$. There is a fiber bundle $GL_n \to \mathbb{P}^{n-1}$ that takes A to the line $A(l)$ (where GL_n acts on $\mathbb{A}^n$ from the left). The fiber over $[1, 0 \ldots, 0]$ is the parabolic subgroup P consisting of all invertible $n \times n$ matrices (a_{ij}) such that $a_{j1} = 0$ for $j \geq 2$. As a scheme (but not as a group), P is isomorphic to $(\mathbb{A}^1 - 0) \times \mathbb{A}^{n-1} \times GL_{n-1}$, which is stably cellular by induction and Lemma 3.4. The usual cover of $\mathbb{P}^n$ by affines $\mathbb{A}^n$ is a completely cellular trivializing cover for the bundle, so Lemma 3.8 applies. □

Recall that the Grassmannian $\mathrm{Gr}_k(\mathbb{A}^n)$ is the variety of k-planes in $\mathbb{A}^n$. Also, the Stiefel variety $V_k(\mathbb{A}^n)$ consists of all ordered sets of k linearly independent vectors in $\mathbb{A}^n$. These objects are connected by a fiber bundle $V_k(\mathbb{A}^n) \to \mathrm{Gr}_k(\mathbb{A}^n)$ that takes a set of k linearly independent vectors to the k-plane that it spans. The fiber of this bundle is GL_k.

Proposition 4.2 *For all $n \geq k \geq 0$, the Stiefel variety $V_k(\mathbb{A}^n)$ is stably cellular.*

Proof Consider the fiber bundle $p : V_k(\mathbb{A}^n) \to \mathbb{A}^n - 0$ that picks out the first vector in an ordered set of k linearly independent vectors. The fiber of this bundle (over the vector $(1, 0, \dots, 0)$, say) is $\mathbb{A}^{k-1} \times V_{k-1}(\mathbb{A}^{n-1})$, which we may assume by induction is stably cellular. Because of Lemma 3.9, it suffices to find a completely stably cellular cover of $\mathbb{A}^n - 0$ that trivializes p.

For $1 \leq i \leq n$, let U_i be the open set of $\mathbb{A}^n - 0$ consisting of all vectors $(x_1, \dots, x_n)$ such that $x_i \neq 0$. The intersections of these open sets are of the form $(\mathbb{A}^1 - 0)^k \times \mathbb{A}^{n-k}$, so they are stably cellular.

It remains to show that the bundle p is trivial over U_i. Without loss of generality, it suffices to consider U_1. We regard $\mathbb{A}^{n-1}$ as the subset of $\mathbb{A}^n$ consisting of vectors whose first coordinate is zero. Let $f \colon U_1 \times V_{k-1}(\mathbb{A}^{n-1}) \times \mathbb{A}^{k-1} \to p^{-1}U_1$ be the map

$$(\mathbf{x}, \mathbf{v}_1, \dots, \mathbf{v}_{k-1}, t_1, \dots, t_{k-1}) \mapsto (\mathbf{x}, t_1\mathbf{x} + \mathbf{v}_1, t_2\mathbf{x} + \mathbf{v}_2, \dots, t_{k-1}\mathbf{x} + \mathbf{v}_{k-1}).$$

One readily checks that this is an isomorphism. □

Let G be an algebraic group. Recall that a *principal G-bundle* is an algebraic fiber bundle $p \colon E \to B$ together with an action $E \times G \to E$, such that $p(eg) = p(e)$ and the induced map $E \times G \to E \times_B E$ sending (e, g) to (e, eg) is an isomorphism.

Proposition 4.3 *If $p : E \to B$ is a principal G-bundle where both E and G are stably cellular, then so is B.*

Proof Let C_* be the Čech complex of p. This means that C_m is the fiber product $E \times_B E \times_B \cdots \times_B E$ ($m+1$ copies of E). Because fiber bundles are locally split, C_* is a hypercover of B (cf. [DHI, 3.4, 4.2]). Using Lemma 3.6, we just need to show that each C_m is stably cellular.

From the definition of a principal bundle, C_m is isomorphic to $E \times G^m$, which is stably cellular by Lemma 3.4. □

Proposition 4.4 *For all $n \geq k \geq 0$, the Grassmannian variety $Gr_k(\mathbb{A}^n)$ is stably cellular.*

Proof Consider the fiber bundle $V_k(\mathbb{A}^n) \to \mathrm{Gr}_k(\mathbb{A}^n)$. This is a principal GL_k-bundle. Thus, Proposition 4.3 applies because of Propositions 4.1 and 4.2. □

Remark 4.5 One might also try to prove that Grassmannians are cellular by using the Schubert cell decomposition. There are various approaches to this, and as far as we know all of them run into unpleasant problems. One possibility, for instance, is to consider the standard open cover of $\mathrm{Gr}_k(\mathbb{A}^n)$ by affines $\mathbb{A}^{k(n-k)}$—these are precisely the top-dimensional open Schubert cells. If the finite intersections of these opens are all cellular, then so is $\mathrm{Gr}_k(\mathbb{A}^n)$ by Lemma 3.8. Unfortunately these finite intersections are complicated, and we have only managed to prove they are cellular for $\mathrm{Gr}_1(\mathbb{A}^n)$ and $\mathrm{Gr}_2(\mathbb{A}^n)$. The general case remains an intriguing open question.

Our proof that Grassmannians are cellular generalizes easily to the case of flag varieties. Given integers $0 \leq d_1 < d_2 < \cdots < d_k \leq n$, let $\mathrm{Fl}(d_1, \ldots, d_k; n)$ denote the variety of flags $V_1 \subseteq \cdots \subseteq V_k \subseteq \mathbb{A}^n$ such that $\dim V_i = d_i$.

Proposition 4.6 *The flag variety* $Fl(d_1, \ldots, d_k; n)$ *is stably cellular.*

Proof $\mathrm{Fl} = \mathrm{Fl}(d_1, \ldots, d_k; n)$. There is an algebraic fiber bundle

$$V_{d_k}(\mathbb{A}^n) \to \mathrm{Fl}(d_1, \ldots, d_k; n)$$

taking an ordered set of k linearly independent vectors to the flag whose ith space is spanned by the first d_i vectors. This is a principal G-bundle, where G is the parabolic subgroup of GL_n consisting of matrices in block form

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,k+1} \\ 0 & A_{22} & \cdots & A_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{k+1,k+1} \end{bmatrix}$$

where $A_{11} \in GL_{d_1}$, $A_{ii} \in GL_{d_i - d_{i-1}}$, and $A_{k+1,k+1} \in GL_{n-d_k}$. As a variety G is isomorphic to $GL_{d_1} \times GL_{d_2 - d_1} \times \cdots \times GL_{d_k - d_{k-1}} \times GL_{n-d_k} \times \mathbb{A}^N$, where N is the (unimportant) number

$$d_1(d_2 - d_1) + \cdots + d_{k-1}(d_k - d_{k-1}) + d_k(n - d_k).$$

So G is stably cellular, and $\mathrm{Fl}(d_1, \ldots, d_k; n)$ is stably cellular by Proposition 4.3. ☐

5 Other examples of cellular varieties

5.1 Toric varieties

We will show that every toric variety is weakly equivalent in $\mathcal{MV}_*$ to a homotopy colimit of copies of tori $T^m = (\mathbb{A}^1 - 0)^m$ with disjoint basepoints. Since the tori are stably cellular by Lemma 3.4, so are toric varieties. This result is almost trivial in the smooth case, since all smooth affine toric varieties have the form $\mathbb{A}^n \times (\mathbb{A}^1 - 0)^m$. The singular case takes a tiny bit more work. Recall that a singular variety, when regarded as an element of $\mathcal{MV}$, is really the presheaf that it represents.

For background definitions and results, see [F1]. Let N denote an n-dimensional lattice; it is isomorphic to $\mathbb{Z}^n$, but we work in a coordinate-free context. Let V be the corresponding $\mathbb{R}$-vector space $N \otimes \mathbb{R}$. Recall that if σ is a strongly convex rational polyhedral cone in V, then one gets a finitely-generated semigroup $S_\sigma = \sigma^\vee \cap \mathrm{Hom}(N, \mathbb{Z}) \subseteq \mathrm{Hom}(N, \mathbb{R})$, where $\sigma^\vee$ is the dual cone of σ. The affine toric variety $X(\sigma)$ is defined to be $\mathrm{Spec}\, k[S_\sigma]$.

Lemma 5.2 *If σ generates V as an $\mathbb{R}$-vector space, then the corresponding toric variety $X(\sigma)$ is simplicially $\mathbb{A}^1$-contractible.*

Proof We need to construct a homotopy $H\colon X(\sigma) \times \mathbb{A}^1 \to X(\sigma)$ such that $H_0\colon X(\sigma) \times 0 \to X(\sigma)$ is constant and $H_1\colon X(\sigma) \times 1 \to X(\sigma)$ is the identity. First choose nonzero generators $u_1, \dots, u_t$ of S_σ. Then we can write $X(\sigma)$ as $\mathrm{Spec}\, k[Y_1, \dots, Y_t]/I$ where I is generated by all polynomials of the form

$$Y_1^{a_1} Y_2^{a_2} \cdots Y_t^{a_t} - Y_1^{b_1} Y_2^{b_2} \cdots Y_t^{b_t}$$

such that $a_1 u_1 + a_2 u_2 + \cdots + a_t u_t = b_1 u_1 + b_2 u_2 + \cdots + b_t u_t$ (see [F1, Exercise, p.19]).

We claim that because σ generates V, there is a vector w in V such that $\langle u_i, w \rangle > 0$ for all i (here $\langle -, - \rangle$ is the pairing between V^* and V). Accepting this for the moment, define

$$H(y_1, y_2, \dots, y_t, s) = (s^{\langle w, u_1 \rangle} y_1, s^{\langle w, u_2 \rangle} y_2, \dots, s^{\langle w, u_t \rangle} y_t).$$

To see that this map is well-defined, suppose that $u = \Sigma a_i u_i = \Sigma b_i u_i$ and that $(y_1, \dots, y_t)$ satisfies the equation $y_1^{a_1} y_2^{a_2} \cdots y_t^{a_t} - y_1^{b_1} y_2^{b_2} \cdots y_t^{b_t} = 0$. Then

$$(s^{\langle w, u_1 \rangle} y_1)^{a_1} \cdots (s^{\langle w, u_t \rangle} y_t)^{a_t} - (s^{\langle w, u_1 \rangle} y_1)^{b_1} \cdots (s^{\langle w, u_3 \rangle} y_t)^{b_t}$$

equals

$$s^{\langle w, u \rangle} y_1^{a_1} y_2^{a_2} \cdots y_t^{a_t} - s^{\langle w, u \rangle} y_1^{b_1} y_2^{b_2} \cdots y_t^{b_t},$$

which is still equal to zero. Note that H_0 is the constant map with value $(0, 0, \dots, 0)$, and that H_1 is the identity.

We have only left to produce the vector w. If we had $\langle u_i, v \rangle = 0$ for all $v \in \sigma \cap N$, then the fact that σ generates V and is rational would imply that

$u_i = 0$. Therefore, for each i there exists a $w_i \in \sigma \cap N$ such that $\langle u_i, w_i \rangle \neq 0$. Since $u_i \in \sigma^\vee$, we must in fact have $\langle u_i, w_i \rangle > 0$. Let $w = w_1 + \cdots + w_t$. Again using the fact that $u_i \in \sigma^\vee$ and $w_j \in \sigma$, we know that $\langle u_i, w_j \rangle \geq 0$ for $i \neq j$. Hence $\langle u_i, w \rangle > 0$ for all i. □

Proposition 5.3 *Let σ be a strongly convex polyhedral rational cone. Then $X(\sigma)$ is $\mathbb{A}^1$-homotopic to $(\mathbb{A}^1 - 0)^m$, where m is the codimension of $\mathbb{R} \cdot \sigma$ in V.*

Proof Split N as $N' \oplus N''$ so that $\mathbb{R} \cdot \sigma = V' = N' \otimes \mathbb{R}$ and $(\mathbb{R} \cdot \sigma) \cap V'' = 0$, where $V'' = N'' \otimes \mathbb{R}$ (cf. [F1, p. 29]). Then σ equals $\sigma' \times 0$ in $V' \times V''$, where σ' is the same cone as σ except that it lies in V'. Now $X(\sigma) \cong X(\sigma') \times (\mathbb{A}^1 - 0)^m$ by [F1, p. 5] and [F1, p. 19]. The above lemma shows that $X(\sigma')$ is simplicially $\mathbb{A}^1$-contractible, so $X(\sigma)$ is simplicially $\mathbb{A}^1$-homotopic to $(\mathbb{A}^1 - 0)^m$. □

Theorem 5.4 *Every toric variety is stably cellular.*

Proof Given a fan Δ, the toric variety $X(\Delta)$ has an open cover consisting of affine toric varieties $X(\sigma)$ for cones σ belonging to Δ. Since $X(\sigma) \cap X(\tau)$ is equal to $X(\sigma \cap \tau)$, this cover of $X(\Delta)$ has the property that every intersection of pieces of the cover is $\mathbb{A}^1$-homotopic to a torus. Now Lemma 3.8 finishes the argument. □

5.5 Quadrics

If $q(x_1, \ldots, x_n)$ is a quadratic form then one can look at the affine quadric $AQ(q) \hookrightarrow \mathbb{A}^n$ defined by $q(x_1, \ldots, x_n) = 0$, as well as the corresponding projective quadric $Q(q) \in \mathbb{P}^{n-1}$.

Recall that a quadratic form q on $\mathbb{A}^n$ is called hyperbolic if n is even and $q(a_1, b_1, \ldots, a_k, b_k) = a_1 b_1 + \cdots + a_k b_k$ (where $n = 2k$) up to a change of basis. Note that $AQ(q)$ is then singular (but only at the origin) whereas $Q(q)$ is nonsingular.

Proposition 5.6 *In the case where q is hyperbolic and non-degenerate, the schemes $\mathbb{A}^n - AQ(q)$, $AQ(q) - 0$, $Q(q)$, and $\mathbb{P}^{n-1} - Q(q)$ are all stably cellular.*

Proof We will abbreviate $AQ(q)$ and $Q(q)$ as just AQ and Q. The projection $\mathbb{A}^{2k} - AQ \to \mathbb{A}^k - 0$ given by $(a_i, b_i) \mapsto (a_i)$ is an algebraic fiber bundle with fiber $(\mathbb{A}^1 - 0) \times \mathbb{A}^{k-1}$. If $U_i \hookrightarrow \mathbb{A}^k - 0$ is the open subscheme of vectors whose ith coordinate is nonzero, then $\{U_i\}$ is a completely stably cellular trivializing cover for this bundle. Lemma 3.8 tells us that $\mathbb{A}^{2k} - AQ$ is stably cellular.

Now consider the closed subscheme $AQ - 0 \hookrightarrow \mathbb{A}^{2k} - 0$. The homotopy purity sequence has the form $\mathbb{A}^{2k} - AQ \hookrightarrow \mathbb{A}^{2k} - 0 \to \mathrm{Th}\, N$ where N is the normal bundle. But the normal bundle is trivial, so $\mathrm{Th}\, N \simeq S^{2,1} \wedge (AQ - 0)_+$. Since we already know that $\mathbb{A}^{2k} - AQ$ is stably cellular, the cofiber sequence shows us that $AQ - 0$ is also stably cellular.

For Q, we consider the principal $(\mathbb{A}^1 - 0)$-bundle $\mathbb{A}^1 - 0 \to AQ - 0 \to Q$ and apply Proposition 4.3. For $\mathbb{P}^{n-1} - Q$ we apply the same proposition to the principal bundle $\mathbb{A}^1 - 0 \to \mathbb{A}^{2k} - AQ \to \mathbb{P}^{n-1} - Q$. □

6 Algebraic K-theory and algebraic cobordism

We show that the motivic spectra KGL and MGL, representing algebraic K-theory and algebraic cobordism respectively, are stably cellular.

In this section it will be more convenient to work in the category of naive spectra (i.e., Bousfield-Friedlander spectra [BF]). We use the model structure on this category, induced by that on $\mathcal{MV}_*$, that is described in [Ho2, Defn. 3.3] and [Ja, Thm. 2.9] (the two turn out to be equal).

Several times in the following proofs we will use the fact that in $\mathcal{MV}_*$ filtered colimits are homotopy colimits. This is inherited from the corresponding property of $s\mathcal{S}et$, using the fact that homotopy colimits for simplicial presheaves can be computed objectwise.

First recall a standard idea from stable homotopy theory:

Lemma 6.1 *Let $\{E_n, \Sigma^{2,1}E_n \to E_{n+1}\}$ be a motivic spectrum. Then E is weakly equivalent to the homotopy colimit of the diagram*

$$\Sigma^\infty E_0 \to \Sigma^{-2,-1}\Sigma^\infty E_1 \to \Sigma^{-4,-2}\Sigma^\infty E_2 \to \cdots .$$

Proof One model for $\Sigma^{-2n,-n}\Sigma^\infty E_n$ is given by the formulas

$$(\Sigma^{-2n,-n}\Sigma^\infty E_n)_k = \Sigma^{2(k-n),k-n} E_n$$

if $k \geq n$ and $(\Sigma^{-2n,-n}\Sigma^\infty E_n)_k = E_k$ otherwise. Thus, for every k, $(\Sigma^{-2n,-n}\Sigma^\infty E_n)_k = E_k$ for sufficiently large n. Since homotopy colimits of spectra are computed degreewise in k, this shows that the kth term of the homotopy colimit is E_k, as desired. □

Theorem 6.2 *The algebraic K-theory spectrum KGL is stably cellular.*

Proof Recall that KGL_n equals the object $\mathbb{Z}\times BGL$ of $\mathcal{MV}_*$ [V1, Ex. 2.8]. By Lemma 6.1, it suffices to show that $\mathbb{Z}\times BGL$ is stably cellular, or equivalently by Lemma 3.2 that BGL is stably cellular. Now BGL is weakly equivalent to $\operatorname{colim}_k \operatorname{colim}_n \operatorname{Gr}_k(\mathbb{A}^n)$ by [V1, Ex. 2.8]. Proposition 4.4 finishes the proof because these colimits are filtered colimits and hence homotopy colimits. □

Remark 6.3 In the above result one can avoid the use of Grassmannians by observing that BGL is the homotopy colimit of the usual bar construction—i.e., of the simplicial diagram $[n]\mapsto GL^n$. Here $GL = \operatorname{colim}_k GL_k$, as usual; note that this is a filtered colimit, hence a homotopy colimit. Proposition 4.1 shows that each GL_k is stably cellular, and so GL is as well. Then so is each GL^n, hence BGL is stably cellular.

Theorem 6.4 *The algebraic cobordism spectrum MGL is stably cellular.*

Proof By Lemma 6.1, we need to show that each MGL_k is stably cellular. Let $p_{n,k}\colon E_{n,k}\to \operatorname{Gr}_k(\mathbb{A}^n)$ be the tautological k-dimensional bundle, and let $E^0_{n,k}$ be the complement of the zero section. From [V1, Ex. 2.10], the object MGL_k is $\operatorname{colim}_n \operatorname{Th}(p_{n,k})$, which is equal to $\operatorname{colim}_n E_{n,k}/E^0_{n,k}$. Since the colimit is a filtered colimit, it is also a homotopy colimit. Therefore, we only have to show that $E_{n,k}/E^0_{n,k}$ is stably cellular. By Lemma 2.5, this reduces to showing that $E_{n,k}$ and $E^0_{n,k}$ are stably cellular. The first is weakly equivalent to $\operatorname{Gr}_k(\mathbb{A}^n)$, so it is stably cellular by Proposition 4.4. The following proposition shows that the second is also stably cellular. □

Proposition 6.5 *Let $p_{n,k}\colon E_{n,k}\to \operatorname{Gr}_k(\mathbb{A}^n)$ be the tautological k-dimensional bundle, and let $E^0_{n,k}$ be the complement of the zero section. The variety $E^0_{n,k}$ is stably cellular.*

Proof To simplify notation, let $V=V_k(\mathbb{A}^n)$, $X=\operatorname{Gr}_k(\mathbb{A}^n)$, and $E^0=E^0_{n,k}$. The projection map $E^0\times_X V\to E^0$ is a principal bundle with group GL_k. By Proposition 4.3, we just need to show that $E^0\times_X V$ is stably cellular. Now $E^0\times_X V$ is the variety of ordered sets of k linearly independent vectors together with a non-zero vector in the span of these vectors, which is isomorphic to $V\times(\mathbb{A}^k-0)$. This variety is stably cellular by Proposition 4.2 and Lemma 3.4. □

Remark 6.6 It is known, at least over fields of characteristic 0, that the Lazard ring $\mathbb{Z}[x_1, x_2, \ldots]$ sits inside of $MGL_{2*,*}$ as a retract (here x_i has degree $(2i, i)$). Hopkins and Morel have announced a proof that these two rings are actually equal. In this case the x_i's form a regular sequence, and one can inductively start forming homotopy cofibers of MGL-module spectra: $MGL/(x_1)$ is defined to be the homotopy cofiber of $\Sigma^{2,1}MGL \to MGL$, then $MGL/(x_1, x_2)$ is the homotopy cofiber of a map $\Sigma^{4,2}MGL/(x_1) \to MGL/(x_1)$, etc. According to Hopkins and Morel, the spectrum $MGL/(x_1, x_2, \ldots)$ is weakly equivalent to the motivic cohomology spectrum $H\mathbb{Z}$, just as happens in classical topology. From this it would follow that $H\mathbb{Z}$ is cellular: we already know MGL is cellular, and $H\mathbb{Z}$ is built inductively from suspensions of MGL via homotopy cofibers. We don't formally claim that $H\mathbb{Z}$ is cellular because the work of Hopkins and Morel is not yet available.

7 Cellularity and stable homotopy

The material in this section is completely formal, but at the same time surprisingly powerful. If E is a motivic spectrum, write $\pi_{p,q}E$ for the set of maps $\mathrm{Ho}\,(S^{p,q}, E)$ in the stable motivic homotopy category. First we will prove that stable $\mathbb{A}^1$-weak equivalences between cellular objects are detected by $\pi_{p,q}$. Then we'll produce a pair of spectral sequences for computing $\pi_{p,q}$ of smash products and function spectra that are applicable only under certain cellularity assumptions.

Proposition 7.1 *If E is cellular and $\pi_{p,q}E = 0$ for all p and q in $\mathbb{Z}$, then E is contractible.*

Proof We may as well assume that E is both cofibrant and fibrant. Consider the class of all motivic spectra A such that the pointed simplicial set $\mathrm{Map}(\tilde{A}, E)$ is contractible, where $\tilde{A} \to A$ is a cofibrant replacement for A. This class is closed under weak equivalences and homotopy colimits, and our assumptions imply that it contains the spheres $S^{p,q}$. Therefore the class contains every cellular object; in particular, it contains E. But if $\mathrm{Map}(E, E)$ is contractible, then the identity map is null in the stable motivic homotopy category (because E is cofibrant and fibrant), and this implies that E is contractible. □

Corollary 7.2 *Let $E \to F$ be a map between cellular motivic spectra, and assume it induces isomorphisms on $\pi_{p,q}$ for all p and q in $\mathbb{Z}$. Then the map is a weak equivalence.*

This corollary is proved in [H, Thm. 5.1.5], but we include a proof for completeness and because it's short.

Proof Let C be the homotopy cofiber of $E \to F$. Since we are in a stable category, it is enough to prove that C is contractible. Our assumptions imply that C is cellular, and that $\pi_{*,*}C = 0$. So Corollary 7.2 gives $C \simeq *$. □

Proposition 7.3 *If E is any motivic spectrum, there is a zig-zag $A \to \hat{E} \leftarrow E$ in which $E \to \hat{E}$ is a weak equivalence, $A \to \hat{E}$ induces isomorphisms on bigraded homotopy groups, and A is cellular.*

The following proof is an adaptation of the usual construction in ordinary topology of cellular approximations to any space.

Proof First let $E \to \hat{E}$ be a fibrant replacement. Consider all possible maps $f\colon S^{p,q} \to \hat{E}$ as p and q range over all integers, and let $C_0 = \vee_f S^{p,q}$. There is a canonical map $C_0 \to \hat{E}$.

Factor this map as $C_0 \overset{\sim}{\rightarrowtail} \hat{C}_0 \to \hat{E}$, where the first map is a trivial cofibration and the second is a fibration. Next consider all possible maps $f\colon S^{p,q} \to \hat{C}_0$ which become zero in $\pi_{p,q}(\hat{E})$. We get a map $\vee_f S^{p,q} \to \hat{C}_0$, and let C_1 be the mapping cone. There exists a commutative triangle of the form:

$$\begin{array}{ccc} \hat{C}_0 & \longrightarrow & \hat{E} \\ \downarrow & \nearrow & \\ C_1 & & \end{array}$$

We again factor $C_1 \to \hat{E}$ as $C_1 \overset{\sim}{\rightarrowtail} \hat{C}_1 \to \hat{E}$, and repeat the procedure to get C_2. Continuing, we get a sequence of cofibrations between fibrant objects

$$\hat{C}_0 \rightarrowtail \hat{C}_1 \rightarrowtail \hat{C}_2 \to \cdots$$

all with maps to $\hat{E}$. Let C denote the homotopy colimit, and note that there is a natural map $C \to \hat{E}$. The map $C \to \hat{E}$ is surjective on homotopy groups because of the way in which $\hat{C}_0$ was defined. To show that $C \to \hat{E}$ is *injective* on homotopy groups, we need Proposition 9.3 which tells us that $\pi_{p,q}C \cong \operatorname{colim}_n \pi_{p,q}\hat{C}_n$. From this observation, injectivity follows immediately. Finally, note that each $\hat{C}_i$ is cellular and therefore C is cellular. □

Remark 7.4 The above proof actually shows that the full subcategory of $\mathrm{Ho}\,(\mathcal{S}pectra(\mathcal{MV}))$ consisting of the cellular spectra is the same as the smallest full triangulated subcategory which contains the spheres and is closed under infinite direct sums. It might seem like we also need to include mapping telescopes in this statement, but we get these for free—see [BN].

Definition 7.5 Given any motivic spectrum E, let $\mathrm{Cell}(E)$ be the corresponding cellular spectrum constructed in Proposition 7.3.

It is easy to see from the proof of Proposition 7.3 that Cell is a functor. It's slightly inconvenient that we don't obtain a natural map $\mathrm{Cell}(E) \to E$. However, because $E \to \hat{E}$ is a weak equivalence, we do obtain a natural weak homotopy class $\mathrm{Cell}(E) \to E$.

The following simple lemma will be important later in the proof of Theorem 8.12.

Lemma 7.6 *The functor* Cell *takes homotopy cofiber sequences to homotopy cofiber sequences.*

Proof Let $A \to B \to C$ be a homotopy cofiber sequence, and let D denote the homotopy cofiber of $\mathrm{Cell}(A) \to \mathrm{Cell}(B)$. Then D is cellular, and the induced homotopy class $D \to C$ is an isomorphism on $\pi_{*,*}$ by the five-lemma. Now the map $\mathrm{Cell}(B) \to \mathrm{Cell}(C)$ induces a map $D \to \mathrm{Cell}(C)$; the latter is an isomorphism on $\pi_{*,*}$ because the same is true of $\mathrm{Cell}(C) \to C$ and $D \to C$. By Corollary 7.2, $D \to \mathrm{Cell}(C)$ is a weak equivalence. □

If E is a motivic ring spectrum, one can consider E-modules, smash products over E (denoted $\wedge_E$), and function spectra $F_E(-,-)$. The definitions are formal, given a symmetric monoidal model category of spectra (see [EKMM, Ch. III], for example). As in [EKMM] we will blur the distinction between these constructions and their derived versions— it will always be clear which we mean (and it's almost always the derived one).

We will need the following basic tool from the algebra of ring spectra:

Proposition 7.7 *Let E be a motivic ring spectrum, M be a right E-module, and N be a left E-module. Assume that E and M are cellular. Then there there is a strongly convergent tri-graded spectral sequence of the form*

$$E^2_{a,(b,c)} = \mathrm{Tor}^{\pi_{*,*}E}_{a,(b,c)}(\pi_{*,*}M, \pi_{*,*}N) \Rightarrow \pi_{a+b,c}(M \wedge_E N),$$

and a conditionally convergent tri-graded spectral sequence of the form

$$E_2^{a,(b,c)} = \mathrm{Ext}^{a,(b,c)}_{\pi_{*,*}E}(\pi_{*,*}M, \pi_{*,*}N) \Rightarrow \pi_{-a-b,-c}F_E(M,N).$$

Some kind of cellularity hypothesis is essential for this proposition in order to guarantee convergence.

The proof of this result is almost exactly the same as the one of [EKMM, Thm. IV.4.1]; we will record some consequences before spelling out exactly what changes need to be made. In the notation, a is the homological grading on Tor and (b,c) is the internal grading coming from the bi-graded motivic stable homotopy groups. The differentials in the first spectral sequence have the form $d_r\colon E^r_{a,(b,c)} \to E^r_{a-r,(b+r-1,c)}$. The edge homomorphism of the spectral sequence is the obvious map

$$[\pi_{*,*}M \otimes_{\pi_{*,*}E} \pi_{*,*}N]_{(b,c)} \to \pi_{b,c}(M \wedge_E N).$$

Similar remarks apply to the Ext spectral sequence. Now the differentials have the form $d_r\colon E_r^{a,(b,c)} \to E_r^{a+r,(b-r+1,c)}$.

Proof We follow the method explained in [EKMM, IV.5]. First, we set $K_{-1} = M$ and inductively build a sequence of homotopy cofiber sequences $K_i \to F_i \to K_{i-1}$ with the property that F_i is a free right E-module and $\pi_{*,*}F_i \to \pi_{*,*}K_{i-1}$ is surjective. The resulting chain complex

$$\cdots \to \pi_{*,*}F_2 \to \pi_{*,*}F_1 \to \pi_{*,*}F_0 \to \pi_{*,*}M$$

is a free resolution of $\pi_{*,*}M$ over $\pi_{*,*}E$.

We now have a tower of homotopy cofiber sequences of the form

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & M & \longrightarrow & M & \longrightarrow & \Sigma^{1,0}K_0 & \longrightarrow & \Sigma^{2,0}K_1 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & & * & & F_0 & & \Sigma^{1,0}F_1 & & \Sigma^{2,0}F_2 & & \cdots \end{array} \tag{7.8}$$

(the tower is trivial as it extends to the left). We apply the functor $(-)\wedge_E N$ to this tower, and consider the resulting homotopy spectral sequence. Note that for each fixed q we get a homotopy spectral sequence for $\pi_{*,q}(-)$, so we really have a family of spectral sequences, one for each q. We are free to think of this as a 'tri-graded' spectral sequence.

Note that F_i is a wedge of various suspensions of E, indexed by a set of free generators for $\pi_{*,*}F_i$ as a $\pi_{*,*}E$-module. So $F_i \wedge_E N$ is a wedge of suspensions of N indexed by the same set. Therefore $\pi_{*,*}(F_i \wedge_E N)$ is a direct sum of copies of $\pi_{*,*}N$ (with the grading shifted appropriately), and the identification of the E_2-term as Tor falls out immediately. This is all the same as the argument in [EKMM].

The place where we have to be careful is in convergence. By [Bd, Thm. 6.1(b)] we only have to show that $\operatorname{colim} \pi_{*,*}(\Sigma^{n,0}K_n \wedge_E N) = 0$. Let K_∞ be the homotopy colimit of the sequence $M \to \Sigma^{1,0}K_0 \to \Sigma^{2,0}K_1 \to \cdots$. From [I], we have $\pi_{p,q}(K_\infty) = \operatorname{colim}_n \pi_{p,q}\Sigma^{n+1,0}K_n$. The tower was constructed in such a way that each $K_i \to \Sigma^{1,0}K_{i+1}$ induces the zero map on $\pi_{*,*}$; so $\pi_{*,*}K_\infty = 0$. In ordinary topology this would tell us that K_∞ is contractible, and therefore that $K_\infty \wedge_E N$ is contractible—and we would be done. In our case the conclusion that K_∞ is contractible is not quite automatic. However, our assumptions imply inductively that all the K_i are cellular—and therefore so is K_∞. By Proposition 7.1 the vanishing of $\pi_{*,*}$ gives that K_∞ is indeed contractible.

The proof for the Ext case follows the same ideas. For conditional convergence [Bd, Defn. 5.10], we need to show that $\lim \pi_{*,*}F_E(\Sigma^{n,0}K_n, N)$ and $\lim^1 \pi_{*,*}F_E(\Sigma^{n,0}K_n, N)$ are both zero. This follows from the usual short exact sequence for homotopy groups of the homotopy limit of a tower [BK, IX.3.1] and the fact that $\operatorname{holim} F_E(\Sigma^{n,0}K_n, N)$ is weakly equivalent to $F_E(K_\infty, N)$, which is contractible. □

7.9 Cellularity for E-modules

If E is a motivic ring spectrum, we can consider the model category of right E-modules [SS]. We define the *E-cellular modules* to be the smallest class which contains the modules $S^{p,q} \wedge E$ and is closed under weak equivalence and homotopy colimits.

Most of the results of the previous section carry over to E-cellularity. The key observation is that $\operatorname{Ho}_E(S^{p,q} \wedge E, X)$ is isomorphic to $\pi_{p,q}(X)$.

It follows as in Corollary 7.2 that an E-module map between E-cellular spectra is a weak equivalence if it induces isomorphisms on $\pi_{p,q}$ for all p and q in $\mathbb{Z}$. For any E-module X, the construction of Proposition 7.3 gives us a zig-zag $\operatorname{Cell}_E(X) \to \hat{X} \leftarrow X$ of E-module maps in which $X \to \hat{X}$ is a weak equivalence, $\operatorname{Cell}_E(X) \to \hat{X}$ induces isomorphisms on bi-graded homotopy groups, and $\operatorname{Cell}_E(X)$ is E-cellular. As in Definition 7.5, we obtain natural weak homotopy classes $\operatorname{Cell}_E(X) \to X$, but not actual maps. We can also prove that Cell_E takes homotopy cofiber sequences of E-modules to homotopy cofiber sequences.

We will need the following improvement on Proposition 7.7.

Proposition 7.10 *The spectral sequences of Proposition 7.7 have the indicated convergence properties as long as M is E-cellular (but without any other assumptions on E and M).*

Proof The basic setup is the same as in the proof of Proposition 7.7; for the Tor-spectral sequence we again only need to show that

$$\operatorname*{colim}_n \pi_{*,*}(\Sigma^{n,0} K_n \wedge_E N) = 0.$$

Because M is E-cellular, so is each K_i and so is K_∞. We know that $\pi_{*,*}K_\infty = 0$. From the E-cellular version of Corollary 7.2, we conclude that K_∞ is contractible.

The Ext case is again similar to the classical setting; see the end of the proof of Proposition 7.7 for an outline of the differences. □

If X is a cellular spectrum, then $X \wedge E$ is E-cellular. This lets us apply the spectral sequences in the case where M has the form $X \wedge E$, but without any assumptions on the spectrum E.

8 Finite cell complexes and Künneth theorems

Definition 8.1 The *finite cell complexes* are the smallest class of objects in $\mathcal{S}pectra(\mathcal{MV})$ with the following properties:

1. the class contains the spheres $S^{p,q}$;
2. the class is closed under weak equivalence;
3. if $X \to Y \to Z$ is a homotopy cofiber sequence and two of the objects are in the class, then so is the third.

The full subcategory of $\mathrm{Ho}\,(\mathcal{S}pectra(\mathcal{MV}))$ consisting of the finite cell complexes is the smallest full triangulated subcategory containing all the spheres $S^{p,q}$ (which is necessarily closed under finite direct sums, but not infinite ones).

Remark 8.2 It is worth mentioning that a finite homotopy colimit of finite cell complexes need not be a finite cell complex. This happens even in ordinary topology: for example, the homotopy co-invariants of $\mathbb{Z}/2$ acting on a point is $\mathbb{R}P^\infty$.

Remark 8.3 If a scheme X has a finite Zariski cover $\{U_i\}$ such that each intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is a finite cell complex, then so is X. The point is that the Čech complex consider in the proof of Lemma 3.8 is finite. Our arguments from Section 4 therefore show that GL_n and $V_k(\mathbb{A}^n)$ are finite cell complexes. The argument from Proposition 4.4 does *not* show that Grassmannians are finite cell complexes, however. It turns out that they are, but the proof is much more elaborate. We have omitted it because for us the *linear* spectra (see Definition 8.9 below) are almost as good as finite cell complexes, and Grassmannians are obviously linear.

For any two motivic spectra A and B and any motivic ring spectrum E, there is a natural map

$$F(A,E) \wedge F(B,E) \to F(A \wedge B, E \wedge E) \to F(A \wedge B, E). \tag{8.4}$$

In particular, using the identification $F(S^0, E) \cong E$ one finds that $F(X,E)$ is a bimodule over the ring spectrum E, and that the above map factors as

$$F(A,E) \wedge F(B,E) \to F(A,E) \wedge_E F(B,E) \xrightarrow{\eta_{A,B}} F(A \wedge B, E).$$

Proposition 8.5 *If A (or B) is a finite cell complex, then the map $\eta_{A,B}$ induces isomorphisms on all $\pi_{p,q}(-)$.*

Proof Fix B, and consider the class of objects A such that $\eta_{A,B}$ induces isomorphisms on bi-graded homotopy groups. One easily checks that this class is closed under weak equivalences, and has the two-out-of-three property for homotopy cofiber sequences. To see that the class contains $S^{p,q}$, use the fact that $F(S^{p,q}, E) \simeq S^{-p,-q} \wedge E$ as E-modules. □

Theorem 8.6 *Suppose that A and B are two motivic spectra such that A is a finite cell complex. Let E be a motivic ring spectrum. Then there exists a strongly-convergent tri-graded Künneth spectral sequence of the form*

$$\left[\mathrm{Tor}_a^{E^{*,*}}(E^{*,*}A, E^{*,*}B)\right]^{(b,c)} \Rightarrow E^{b-a,c}(A \wedge B).$$

Proof First note that $F(S^{p,q}, E)$ is E-cellular (being weakly equivalent to $S^{-p,-q} \wedge E$). Using this together with the fact that A is a finite cell complex, it follows that $F(A,E)$ is E-cellular. We now apply Proposition 7.10 with $M = F(A,E)$ and $N = F(B,E)$. The groups $\pi_{*,*}(M \wedge_E N)$ are identified with $\pi_{*,*}F(A \wedge B, E)$ by Proposition 8.5. □

The necessity of some kind of finiteness hypothesis in the above result is well known in ordinary topology—see [A, Lect. 1]. The result is often applied when $A = \Sigma^\infty X_+$ and $B = \Sigma^\infty Y_+$, where X and Y are schemes, in which case $A \wedge B = \Sigma^\infty (X \times Y)_+$.

Remark 8.7 Note that the higher Tor's vanish if $E^{*,*}(A)$ is free as a module over $E^{*,*}$, in which case we obtain a Künneth isomorphism

$$E^{*,*}(A) \otimes_{E^{*,*}} E^{*,*}(B) \cong E^{*,*}(A \wedge B).$$

Remark 8.8 In [J], Joshua produced a similar Künneth spectral sequence. His result was stated only for algebraic K-theory and motivic cohomology, and assumed that A and B were *schemes* (rather than arbitrary objects of $\mathcal{MV}$). Also, his spectral sequence was only bi-graded rather than tri-graded: this is essentially because he was applying the results of [EKMM] rather than reproving them in the bi-graded context, and so his motivic cohomology rings were graded by total degree.

Our proof is essentially the same as Joshua's (which in turn is the same as the modern proof in stable homotopy theory) although we were able to streamline things by using the language of motivic spectra.

Joshua's result assumes that one of the schemes is *linear*, as opposed to being a finite cell complex in our sense. If one assumes the ring spectrum E satisfies a Thom isomorphism theorem, then one can make our result encompass Joshua's by expanding the class of finite cell complexes so as to be closed under the process of 'taking Thom spaces':

Definition 8.9 The *linear* motivic spectra are the smallest class of objects in $\mathcal{S}pectra(\mathcal{MV})$ with the following properties:

(1) the class contains the spheres $S^{p,q}$;
(2) the class is closed under weak equivalence;
(3) if $X \to Y \to Z$ is a homotopy cofiber sequence and two of the objects are in the class, then so is the third;
(4) if $\xi\colon E \to X$ is an algebraic vector bundle over a smooth scheme X, then $\Sigma^\infty \operatorname{Th}\xi$ belongs to the class if and only if $\Sigma^\infty X_+$ belongs to the class.

Remark 8.10 If $Z \hookrightarrow X$ is a closed inclusion of smooth schemes, recall that there is a stable homotopy cofiber sequence $X - Z \to X \to \operatorname{Th} N_{X/Z}$ [MV, 3.2.23]. It follows that if two of the three objects $\Sigma^\infty Z_+$, $\Sigma^\infty X_+$, and $\Sigma^\infty (X-Z)_+$ are linear, then so is the third.

Let E be a motivic ring spectrum, and let $\xi \to X$ be an algebraic vector bundle of rank n over a smooth scheme X. We'll say that E *satisfies Thom isomorphism for* ξ if there is a class $u \in E^{2n,n}(\mathrm{Th}\,\xi)$ such that multiplication by u gives an isomorphism $E^{*,*}(X) \to \tilde{E}^{*+2n,*+n}(\mathrm{Th}\,\xi)$. To be more precise, note that we have a map of motivic spaces

$$\frac{\xi}{\xi - 0} \xrightarrow{\Delta} \frac{\xi \times \xi}{(\xi - 0) \times \xi} \cong \frac{\xi}{\xi - 0} \wedge \xi_+ \simeq \frac{\xi}{\xi - 0} \wedge X_+.$$

This map induces $\alpha\colon F(\mathrm{Th}\,\xi \wedge X_+, E) \to F(\mathrm{Th}\,\xi, E)$. If we write u as a homotopy class $S^{-2n,-n} \to F(\mathrm{Th}\,\xi, E)$, we can consider the composite

$$\begin{array}{c} S^{-2n,-n} \wedge F(X_+, E) \to F(\mathrm{Th}\,\xi, E) \wedge F(X_+, E) \to F(\mathrm{Th}\,\xi \wedge X_+, E) \\ \downarrow \alpha \\ F(\mathrm{Th}\,\xi, E), \end{array} \tag{8.11}$$

where the second map is the same as in (8.4). The requirement that E satisfy Thom isomorphism for ξ is that this composite is an isomorphism on $\pi_{*,*}$.

We say that E *satisfies Thom isomorphism* if it does so for every algebraic vector bundle over a smooth scheme. The spectra $H\mathbb{Z}$, KGL, and MGL are all known to satisfy Thom isomorphism. The reader may wish to compare the above discussion to the notion of algebraically orientable spectrum from [HK].

Theorem 8.12 *Suppose E is a ring spectrum satisfying Thom isomorphism. If X and Y are motivic spectra such that X is linear, then there is a strongly convergent Künneth spectral sequence as in Theorem 8.6.*

In the following proof, we continue our sloppiness about distinguishing various functors and their derived versions. It should be clear from context that we almost always mean the derived version.

Proof The proof requires a little more care than the similar things we've done so far. If Z is a pointed motivic space, we abbreviate $F(\Sigma^\infty Z, E)$ as just $F(Z, E)$.

Recall from Section 7.9 that Cell_E is a functorial E-cellular approximation. Let $\mathcal{C}$ denote the class of all motivic spectra A such that for all motivic spectra Y, the composite

$$\begin{array}{c} \pi_{p,q}\Big(\big[\mathrm{Cell}_E(F(A,E))\big] \wedge_E F(Y,E)\Big) \longrightarrow \pi_{p,q}\Big(F(A,E) \wedge_E F(Y,E)\Big) \\ \downarrow \\ \pi_{p,q} F(A \wedge Y, E) \end{array}$$

is an isomorphism for all p and q. The first map above makes sense because we have a homotopy class $\mathrm{Cell}_E(F(A,E)) \to F(A,E)$ even though we don't have an actual map.

The class $\mathcal{C}$ clearly is closed under weak equivalences and contains the spheres $S^{p,q}$. The E-cellular version of Lemma 7.6 and the five-lemma show that it also has property (3) of Definition 8.9. To show that $\mathcal{C}$ contains every linear spectrum, we must check that if ξ is a vector bundle over a smooth scheme X, then $\mathrm{Th}\,\xi$ belongs to $\mathcal{C}$ if and only if $\Sigma^\infty X_+$ belongs to $\mathcal{C}$.

However, from (8.11) we have the map $u\colon F(X_+,E) \to S^{2n,n} \wedge F(\mathrm{Th}\,\xi, E)$, which is an isomorphism on $\pi_{*,*}$. It follows that $\mathrm{Cell}_E(F(X_+,E))$ is weakly equivalent to $S^{2n,n} \wedge \mathrm{Cell}_E(F(\mathrm{Th}\,\xi,E))$. We now look at the diagram

$$\begin{array}{ccc}
\pi_{p,q}\Big(\mathrm{Cell}_E(F(X_+,E)) \wedge_E F(Y,E)\Big) & \longrightarrow & \pi_{p,q}F(X_+ \wedge Y, E) \\
\downarrow & & \downarrow \eta_Y \\
\pi_{p,q}\Big(S^{2n,n} \wedge \big[\mathrm{Cell}_E(F(\mathrm{Th}\,\xi,E))\big] \wedge_E F(Y,E)\Big) & \longrightarrow & \pi_{p,q}(S^{2n,n} \wedge F(\mathrm{Th}\,\xi \wedge Y, E))
\end{array}$$

where η_Y is defined similarly to the map in (8.11). Both vertical maps induce isomorphisms on $\pi_{*,*}$—the one on the left is a weak equivalence, and for the one on the right this is the 'generalized Thom isomorphism' from Lemma 8.13 below. So the top horizontal map is a $\pi_{*,*}$-isomorphism if and only if the bottom map is one. This is equivalent to property (4) of Definition 8.9.

We have now shown that for every linear spectrum X and every motivic spectrum Y, the map

$$\pi_{p,q}\big(\mathrm{Cell}_E(F(X,E)) \wedge_E F(Y,E)\big) \to \pi_{p,q}F(X \wedge Y, E)$$

is an isomorphism. By Proposition 7.10 there is a strongly convergent spectral sequence of the form

$$\mathrm{Tor}^{E^{*,*}}_{a,(b,c)}(\pi_{*,*}\,\mathrm{Cell}_E(F(X,E)), E^{*,*}(Y)) \to E^{*,*}(X \wedge Y).$$

But $\mathrm{Cell}_E(F(X,E)) \to F(X,E)$ is an isomorphism on bi-graded homotopy groups, so this completes the proof. □

Lemma 8.13 *Let E be a ring spectrum satisfying Thom isomorphism, and let ξ be a vector bundle of rank n over a smooth scheme X. Then for every motivic spectrum Y, the map*

$$\eta_Y\colon F(X_+ \wedge Y, E) \to S^{2n,n} \wedge F(\mathrm{Th}\,\xi \wedge Y, E)$$

induces isomorphisms on $\pi_{*,*}$. In other words, multiplication by the Thom class gives an isomorphism

$$E^{*,*}(X_+ \wedge Y) \cong E^{*+2n,*+n}(\mathrm{Th}\,\xi \wedge Y).$$

The construction of the map η_Y is analogous to the discussion preceding Theorem 8.12.

Proof Fix X and ξ. Let $\mathcal{C}$ be the full subcategory of $\mathrm{Ho}\,(\mathcal{S}pectra(\mathcal{MV}))$ consisting of all spectra Y such that η_Y induces isomorphisms on $\pi_{*,*}$. Clearly the class is closed under weak equivalences, $S^{p,q}$-suspensions, homotopy cofibers, and infinite direct sums. It also contains $\Sigma^\infty Z_+$ for every smooth scheme Z, as this is just the Thom isomorphism for the bundle $\pi^*\xi$, where $\pi\colon X \times Z \to X$ is the projection. We conclude from Theorem 9.2 below that $\mathcal{C}$ is the entire category $\mathrm{Ho}\,(\mathcal{S}pectra(\mathcal{MV}))$. □

9 Compact generators of the stable homotopy category

In this section we establish some basic properties of the motivic stable homotopy category. These are used at various points throughout the paper.

Let $\mathcal{T}$ be a triangulated category with infinite direct sums, as in [BN, Def. 1.2]. An object $X \in \mathcal{T}$ is called *compact* if $\mathcal{T}(X, \oplus_\alpha E_\alpha) \cong \oplus_\alpha \mathcal{T}(X, E_\alpha)$ for every collection of objects $\{E_\alpha\}$. The full subcategory of $\mathcal{T}$ consisting of compact objects is readily seen to be a triangulated subcategory (and therefore closed under finite sums).

Recall that a set of objects S in $\mathcal{T}$ forms a set of *weak generators* if the only full triangulated subcategory containing S and closed under infinite direct sums is $\mathcal{T}$ itself. In the case where the objects in S are compact one drops the 'weak' adjective.

If $W \in \mathcal{MV}_*$, we abbreviate $\Sigma^{2n,n}(\Sigma^\infty W)$ to $\Sigma^{(2n,n)+\infty} W$. Note that this object is a cofibrant spectrum by [Ho2, Prop. 1.14]. We will prove the following results:

Theorem 9.1 *If X is any pointed smooth scheme and $n \in \mathbb{Z}$, then $\Sigma^{(2n,n)+\infty} X$ is a compact object of the motivic stable homotopy category.*

Theorem 9.2 *The set* $\{\Sigma^{(2n,n)+\infty}X_+ | X$ *a smooth scheme and* $n \in \mathbb{Z}\}$ *is a set of compact generators for* $Ho(\mathcal{S}pectra(\mathcal{MV}))$.

The phrase 'directed system', as used in the present section, refers to sequences $E_0 \to E_1 \to \cdots$ indexed by some (possibly transfinite) ordinal. The following result falls out immediately from the proof of Theorem 9.1; a separate proof is given in [I].

Proposition 9.3 *Let* $\alpha \mapsto E_\alpha$ *be a directed system of motivic spectra. Then* $\operatorname{colim}_\alpha \pi_{p,q}E_\alpha \to \pi_{p,q}(\operatorname{hocolim} E_\alpha)$ *is an isomorphism for all* $p, q \in \mathbb{Z}$.

Recall from Definition 8.1 the notion of a finite cell complex. One consequence of Theorem 9.1 is that we never really need infinite homotopy colimits to build stably cellular schemes:

Proposition 9.4 *Let* X *be a smooth scheme, and suppose that* X *is stably cellular. Then* $\Sigma^\infty X_+$ *is a retract, in* $Ho(\mathcal{S}pectra(\mathcal{MV}))$, *of a finite cell complex.*

Proof Let $\mathcal{T}$ be the full subcategory of $\operatorname{Ho}(\mathcal{S}pectra(\mathcal{MV}))$ consisting of the cellular objects. Then the spheres $S^{p,q}$ are a set of weak generators for $\mathcal{T}$ (see Remark 7.4). A result of Neeman, recounted in [K, 5.3], shows that any compact object in $\mathcal{T}$ is a retract of something that can be built from the generators via finitely many extensions. The fact that $\Sigma^\infty X_+$ is compact therefore finishes the proof. □

9.5 The flasque model structure

A fundamental difficulty with the injective model structure on $\mathcal{MV}$ (cf. [MV] or [Ja, App. B]) is that a directed colimit of fibrant objects need not be fibrant. The projective structure [Bl] doesn't have this problem, but in this structure the map $* \to \mathbb{P}^1$ is not a cofibration—one therefore has difficulties when dealing with $\mathbb{P}^1$-spectra. The flasque model structure of [I] avoids both these problems, while maintaining the same class of weak equivalences.

The reader is referred to [I] for a complete account of the flasque model structure on $\mathcal{MV}$. Here we will only need to know that the representable presheaves are cofibrant, and that an object $F \in \mathcal{MV}$ is *(motivic) flasque-fibrant* if and only if it satisfies the following properties:

(1) It is flasque, in the sense of [Ja, Section 1.4];

(2) It is objectwise-fibrant;

(3) For every elementary Nisnevich cover $\{U, V \to X\}$, the natural map of simplicial sets $F(X) \to F(U) \times_{F(U \times_X V)} F(V)$ is a weak equivalence;

(4) For every smooth scheme X the map $F(X) \to F(X \times \mathbb{A}^1)$ is a weak equivalence.

One readily observes, as in [I], that a directed colimit of motivic flasque-fibrant objects is again motivic flasque-fibrant.

Since the model categories of motivic symmetric spectra and naive spectra are Quillen equivalent [Ho2, Sec. 10], we can prove our theorems in either setting. It is easier to work with naive spectra. A (naive) motivic spectrum E is *flasque-fibrant* if each E_i is motivic flasque-fibrant and the maps $E_i \to \Omega^{2,1}E_{i+1}$ are all weak equivalences. A standard fact about the Bousfield-Friedlander model for spectra (see [Ho2]) is that a map $E \to F$ between flasque-fibrant motivic spectra is a stable weak equivalence if and only if each map $E_i \to F_i$ is a weak equivalence in $\mathcal{MV}_*$. Moreover, since each E_i and F_i are flasque-fibrant in $\mathcal{MV}_*$, the map $E \to F$ is a stable weak equivalence if and only if each map $E_i \to F_i$ is an *objectwise* weak equivalence of simplicial presheaves.

Proposition 9.6 *If $\alpha \mapsto E_\alpha$ is a directed system of flasque-fibrant motivic spectra, then the natural map* $\operatorname{hocolim} E_\alpha \to \operatorname{colim} E_\alpha$ *is a weak equivalence.*

Proof It suffices to show that if $\{E_\alpha\}$ and $\{F_\alpha\}$ are directed systems of flasque-fibrant objects and each $E_\alpha \to F_\alpha$ is a stable weak equivalence, then the map $\operatorname{colim} E_\alpha \to \operatorname{colim} F_\alpha$ is a stable weak equivalence. For each α and each n, the map $[E_\alpha]_n \to [F_\alpha]_n$ of nth spaces is an objectwise weak equivalence in $\mathcal{MV}_*$. It follows that $[\operatorname{colim}_\alpha E_\alpha]_n \to [\operatorname{colim}_\alpha F_\alpha]_n$ is still an objectwise weak equivalence in $\mathcal{MV}_*$, and this implies that $\operatorname{colim} E_\alpha \to \operatorname{colim} F_\alpha$ is a stable weak equivalence. □

Recall that naive spectra form a *simplicial* model category, with the simplicial action being the levelwise one inherited from $\mathcal{MV}_*$. We write $\operatorname{Map}(-,-)$ for the simplicial mapping space, both for motivic spectra and in $\mathcal{MV}_*$.

Lemma 9.7 *If E is a flasque-fibrant motivic spectrum then the set of homotopy classes $Ho\,(\Sigma^{(2n,n)+\infty}X, E)$ is isomorphic to $\pi_0 \operatorname{Map}(\Sigma^{(2n,n)+\infty}X, E)$, for any pointed smooth scheme X.*

Proof This follows immediately from the fact that there is a model structure for motivic stable homotopy theory in which the fibrant objects are the flasque-fibrant spectra and $\Sigma^{(2n,n)+\infty}X$ is cofibrant for any pointed smooth scheme X. See [I]. ☐

Lemma 9.8 *Let $\alpha \mapsto F_\alpha$ be a directed system in $\mathcal{MV}_*$, and let X be a pointed smooth scheme. Then $\mathrm{Map}_{\mathcal{MV}_*}(X, \mathrm{colim}_\alpha F_\alpha) \cong \mathrm{colim}_\alpha \mathrm{Map}_{\mathcal{MV}_*}(X, F_\alpha)$.*

Proof Note that the result is obvious for unpointed mapping spaces, since $\mathrm{Map}_{\mathcal{MV}}(X,F) \cong F(X)$. In the pointed case $\mathrm{Map}_{\mathcal{MV}_*}(X, \mathrm{colim}\, F_\alpha)$ is the pullback of $* \to \mathrm{Map}_{\mathcal{MV}}(*, \mathrm{colim}\, F_\alpha) \leftarrow \mathrm{Map}_{\mathcal{MV}}(X, \mathrm{colim}\, F_\alpha)$; this is the same as the colimit of the pullbacks of $* \to \mathrm{Map}_{\mathcal{MV}}(*, F_\alpha) \leftarrow \mathrm{Map}_{\mathcal{MV}}(X, F_\alpha)$. ☐

Recall that if $F \in \mathcal{MV}_*$ then $\Omega^{2,1}F$ is the simplicial presheaf whose value at X is the fiber of $F(X \times \mathbb{P}^1) \to F(X)$. Note that each $F(X)$ has a basepoint because F is a pointed presheaf. We will also need the following:

Proposition 9.9 *If $\alpha \mapsto F_\alpha$ is a directed system in $\mathcal{MV}_*$, then the canonical map $\mathrm{colim}_\alpha \Omega^{2,1}F_\alpha \to \Omega^{2,1}(\mathrm{colim}_\alpha F_\alpha)$ is an isomorphism.*

Proof This is immediate from the definitions. ☐

9.10 Proofs of the main results

We begin with the following auxilliary result. Recall the definition of λ-*sequence* from [H, Section 10.2].

Proposition 9.11 *Let $\alpha \mapsto E_\alpha$ be a λ-sequence in $\mathcal{S}pectra(\mathcal{MV})$ for some ordinal λ. Then for any pointed smooth scheme X, the natural map*

$$\mathrm{colim}\, Ho\left(\Sigma^{(2n,n)+\infty}X, E_\alpha\right) \to Ho\left(\Sigma^{(2n,n)+\infty}X, \mathrm{hocolim}\, E_\alpha\right)$$

is an isomorphism.

Proof By taking a functorial flasque-fibrant replacement for each E_α, we can assume that the E_α's are flasque-fibrant. But then it follows from Proposition 9.6 that the map $\mathrm{hocolim}_\alpha E_\alpha \to \mathrm{colim}_\alpha E_\alpha$ is a stable weak equivalence. Thus, we have

$$\mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, \mathrm{hocolim}_\alpha E_\alpha) = \mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, \mathrm{colim}_\alpha E_\alpha).$$

This in turn is isomorphic to $\pi_0 \operatorname{Map}(\Sigma^{(2n,n)+\infty} X, \operatorname{colim}_\alpha E_\alpha)$ by Lemma 9.7 because the spectrum $\operatorname{colim}_\alpha E_\alpha$ is flasque-fibrant; note that filtered colimits preserve flasque-fibrant spectra [I]. So we are reduced to showing that

$$\operatorname{colim} \pi_0 \operatorname{Map}(\Sigma^{(2n,n)+\infty} X, E_\alpha) \to \pi_0 \operatorname{Map}(\Sigma^{(2n,n)+\infty} X, \operatorname{colim} E_\alpha)$$

is an isomorphism. The idea is to prove that the mapping spaces themselves are isomorphic, using adjointness to reduce to mapping spaces in $\mathcal{MV}_*$.

When $n < 0$ the mapping space $\operatorname{Map}(\Sigma^{(2n,n)+\infty} X, \operatorname{colim} E_\alpha)$ is equal to $\operatorname{Map}_{\mathcal{MV}_*}(X, \operatorname{colim}[E_\alpha]_{-n})$. By Lemma 9.8 we can pull the colimit outside, and then adjointness gives us $\operatorname{colim} \operatorname{Map}(\Sigma^{(2n,n)+\infty} X, E_\alpha)$.

When $n > 0$ one has

$$\operatorname{Map}(\Sigma^{(2n,n)+\infty} X, \operatorname{colim} E_\alpha) \cong \operatorname{Map}_{\mathcal{MV}_*}(X, \Omega^{2n,n}(\operatorname{colim}[E_\alpha]_0)),$$

where $\Omega^{2n,n}(-)$ is shorthand for $\Omega^{2,1} \cdots \Omega^{2,1}(-)$. By Proposition 9.9 we can commute the $\Omega^{2n,n}$ past the colimit, and then Lemma 9.8 lets us take the colimit outside. Using adjointness again, we get $\operatorname{colim} \operatorname{Map}(\Sigma^{(2n,n)+\infty} X, E_\alpha)$. This completes the proof. □

Finally we can prove our main results:

Proof of Theorem 9.1 Let $\{E_\alpha\}$ be a collection of motivic spectra, indexed by some set S. Without loss of generality we may assume the E_α's are cofibrant. We need to show that $\oplus_\alpha \operatorname{Ho}(\Sigma^{(2n,n)+\infty} X, E_\alpha) \to \operatorname{Ho}(\Sigma^{(2n,n)+\infty} X, \vee_\alpha E_\alpha)$ is an isomorphism. It is immediate that the map is injective, so we are only interested in proving surjectivity.

Choose a well-ordering of S, and let λ denote the associated ordinal. Consider the λ-sequence

$$E_0 \rightarrowtail E_0 \vee E_1 \rightarrowtail E_0 \vee E_1 \vee E_2 \rightarrowtail \cdots$$

as in [H, Prop. 10.2.7]. In any model category, the homotopy colimit of a directed sequence of cofibrations between cofibrant objects is weakly equivalent to the colimit. So the homotopy colimit of the above sequence is $\vee_\alpha E_\alpha$. The result now follows by an application of Proposition 9.11, together with the observation that in the homotopy category of a stable model category finite coproducts are the same as finite products. □

Proof of Proposition 9.3 Note that $\pi_{p,q}(W) = \pi_{2q,q}(\Sigma^{2q-p,0} W)$ for any motivic spectrum W. So by replacing each E_α with $\Sigma^{2q-p,0} E_\alpha$—and noting that the suspension commutes across the hocolim—one reduces to the case where $p = 2q$. So we are looking at the group $\operatorname{Ho}(\Sigma^{(2q,q)+\infty} S^{0,0}, \operatorname{hocolim} E_\alpha)$, and therefore we are in a special case of Proposition 9.11. □

Proof of Theorem 9.2 Note that the compactness is already taken care of by Theorem 9.1. Let $\mathcal{T}$ be a full, triangulated subcategory of the stable homotopy category $\mathrm{Ho}(\mathcal{S}pectra(\mathcal{MV}))$ such that $\mathcal{T}$ is closed under infinite direct sums. Assume as well that $\mathcal{T}$ contains the objects $\Sigma^{(2n,n)+\infty}X_+$ for all smooth schemes X and all $n \in \mathbb{Z}$. We must show that $\mathcal{T}$ contains every motivic spectrum.

By the main observations of [BN], $\mathcal{T}$ is closed under countable, directed homotopy colimits. This is because if $X_0 \to X_1 \to X_2 \to \cdots$ is such a directed sequence then one can model the homotopy colimit as the mapping telescope; one therefore gets a homotopy cofiber sequence of the form $\oplus_i X_i \to \mathrm{hocolim}\, X_i \to \oplus_i \Sigma X_i$ (where $\Sigma = \Sigma^{1,0}$ is the triangulated-category suspension). So if $\mathcal{T}$ contains all the X_i's, it also contains the homotopy colimit.

Using Lemma 6.1 and the above observation, we are reduced to showing that $\mathcal{T}$ contains all objects $\Sigma^{(2n,n)+\infty}F$ where $F \in \mathcal{MV}_*$ is a cofibrant object and $n \in \mathbb{Z}$.

If $F \in \mathcal{MV}_*$, then by [D, Prop. 2.8] F is weakly equivalent to a pointed simplicial presheaf Q_* in which every level consists of a coproduct of representables together with a disjoint basepoint. Consider the skeletal filtration

$$* = \mathrm{sk}_{-1} Q \rightarrowtail \mathrm{sk}_0 Q \rightarrowtail \mathrm{sk}_1 Q \rightarrowtail \mathrm{sk}_2 Q \cdots$$

The colimit (and therefore homotopy colimit) of this sequence is just Q, and each cofiber $\mathrm{sk}_k Q/\mathrm{sk}_{k-1} Q$ has the form $\bigvee_\alpha S^{k,0} \wedge (X_\alpha)_+$ where $\{X_\alpha\}$ is some collection of smooth schemes.

Applying the functor $\Sigma^{(2n,n)+\infty}$ gives a directed sequence of motivic spectra whose homotopy colimit is weakly equivalent to $\Sigma^{(2n,n)+\infty}F$. The homotopy cofiber of each map in the sequence has the form $\bigvee_\alpha \left[S^{k,0} \wedge \Sigma^{(2n,n)+\infty}(X_\alpha)_+ \right]$, and therefore belongs to $\mathcal{T}$ (using that $\mathcal{T}$ is closed under the triangulated-category suspension and also closed under infinite direct sums). Induction then shows that $\Sigma^{(2n,n)+\infty}(\mathrm{sk}_k Q)$ belongs to $\mathcal{T}$, for every $k \geq -1$. Using that $\mathcal{T}$ is closed under directed homotopy colimits, we have $\Sigma^{(2n,n)+\infty}F \in \mathcal{T}$. This finishes the proof. □

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