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Groups generated by positive multi-twists and the fake lantern problem

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Abstract Let Γ be a group generated by two positive multi-twists. We give some su cient conditions for Γ to be free or have no \unexpectedly reducible" elements. For a group Γ generated by two Dehn twists, we classify the elements in Γ which are multi-twists. As a consequence we are able to list all the lantern-like relations in the mapping class groups. We classify groups generated by powers of two Dehn twists which are free, or have no \unexpectedly reducible" elements. In the end we pose similar problems for groups generated by powers of n 3 twists and give a partial result.

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1 Introduction

In a meeting of the American Mathematical Society in Ann Arbor, MI in March 2002, John McCarthy posed the following question: Suppose a collection of simple closed curves satisfy the lantern relation (see Figure 3) algebraically. Is it true that they must form a lantern, as in the same gure, given the same commutativity conditions? In this article we consider groups that are generated by two multi-twists and give conditions that guarantee the group is free or does not contain an accidental multi-twist. This will, in particular, answer McCarthy's question to the a rmative (see Theorem 6.4).

To make this precise, let S be an oriented surface, possibly with punctures. For the isotopy class of[1] a simple closed curve c on S let T_c denote the right-handed Dehn twist about c. Let $(c_1; c_2)$ denote the minimum geometric intersection number of isotopy classes of 1-sub-manifolds $c_1; c_2$: By $M(S)$ we denote the

[1] We will usually drop the phrase \isotopy class of" in the rest of this paper for brevity, as all curves are considered up to isotopy.

mapping class group of S, i.e, the group of homeomorphisms of S which permute the punctures, up to isotopies fixing the punctures.

The free group on n generators will be denoted by $\mathbb{F}_n$.

Let $A = \{a_1, \cdots, a_k\}$ be a collection of non-parallel, non-trivial, pairwise disjoint simple closed curves. For any integers $m_1, \ldots, m_k$, we call $T_A = T_{a_k}^{m_k} \cdots T_{a_1}^{m_1}$ a multi-twist. If, furthermore, all $m_i > 0$, we call T_A a positive multi-twist. We will study the group generated by two positive multi-twists in detail. We will give explicit conditions which imply $\langle T_A, T_B \rangle = \mathbb{F}_2$ (see Theorem 3.2). For a group $\langle T_a, T_b \rangle$ generated by two Dehn twists, we give a complete description of elements: We determine which elements are multi-twists, and which elements are pseudo-Anosov restricted to the subsurface which is a regular neighborhood of $a \cup b$ (see Theorems 3.5, 3.9, and 3.10). A mapping class f is called pseudo-Anosov if $f^n(c) \neq c$ (up to isotopy) for all non-trivial simple closed curves c and $n > 0$. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of non-parallel, non-trivial simple closed curves on S. The surface filled by A, denoted by S_A, is a regular neighborhood N of $a_1 \cup \cdots \cup a_n$ together with the components of $S \setminus N$ which are discs with 0 or 1 puncture, assuming that a_i's are drawn as geodesics of some constant-curvature metric. S_A is well-defined independent of chosen metric [6]. We say that A fills up S if $S_A = S$.

Definition 1.1 A word $w = T_{c_1}^{n_1} \cdots T_{c_k}^{n_k}$ is called a cyclically-reduced word if $c_1 \neq c_k$. For such a word w, define $supp(w) = S_{\{c_1, \cdots, c_k\}}$. Then we say w is *relatively pseudo-Anosov* if the restriction of the map w is pseudo-Anosov in $M(U)$, for all components U of $supp(w)$ which are not annuli. If $g = hwh^{-1}$ (as words) with w cyclically-reduced, define $supp(g) = h(supp(w))$. Then define g to be relatively pseudo-Anosov in the same way as above.

In the above, the equation $g = hwh^{-1}$ is an equation of words, not elements, otherwise one can easily give examples where the definition breaks down. To show that the above definition is well-defined, note that if $w = T_{c_1}^{n_1} \cdots T_{c_k}^{n_k}$ is such that $c_1 = c_k$ but $n_1 \neq -n_k$, then one can write $w = T_{c_1}^{n_1} w' T_{c_1}^{-n_1} = T_{c_1}^{-n_k} w'' T_{c_1}^{n_k}$, where w', w'' are both cyclically reduced. Notice that $T_{c_1}^{n}(S_{\{c_1, \cdots, c_{k-1}\}}) = S_{\{c_1, \cdots, c_{k-1}\}}$ for all n and so $supp(w) = \{c_1, \cdots, c_{k-1}\}$.

Also note that a power of a Dehn twist $T_{c_i}^{n_i}$ is a relatively pseudo-Anosov word since its support is an annulus. Similarly a multi-twist is relatively pseudo-Anosov as well. A group Γ with given set of multi-twist generators is relatively pseudo-Anosov if every reduced word in generators of Γ is relatively pseudo-Anosov.

Intuitively, a group Γ generated by multi-twists is relatively pseudo-Anosov if no word in generators of Γ has "unexpected reducibility".

It should be noticed that in the case of two curves a, b filling up a closed surface this was done by Thurston as a method to construct pseudo-Anosov elements; i.e., he showed that $\langle T_a, T_b \rangle$ is free and consists of pseudo-Anosov elements besides powers of conjugates of the generators [4]. Our methods are completely different and elementary, and are only based on how the geometric intersection behaves under Dehn twists.

One surprising result that we find is a lantern-like relation:

$$(T_bT_a)^2 = T_{\partial_1}T_{\partial_2}T_y^{-4}T_{y'}^{-4},$$

where these curves are defined in Figures 2 and 4 (see Proposition 5.1). This relation is lantern-like in the sense that the left hand side is a word in two Dehn twists about intersecting curves and the right hand side is a multi-twist. We then prove that this relation and the lantern relation are the only lantern-like relations (Theorem 6.4).

In the case when $n \geq 3$, we give some sufficient conditions for $\Gamma = \langle T_{a_1}, \ldots, T_{a_n} \rangle$ to be isomorphic to F_n. To motivate our condition, look at the case $\Gamma = \langle T_{a_1}, T_{a_2}, T_{a_3} \rangle$, and assume $a_3 = T_{a_1}(a_2)$. Now $T_{a_3} = T_{a_1}T_{a_2}T_{a_1}^{-1}$, so $\Gamma \not\cong \mathbb{F}_3$. But notice that $(a_1, a_3) = (a_1, a_2)$ and $(a_2, a_3) = (a_1, a_2)^2$, by Lemma 2.1. This shows that the set $I = \{(a_i, a_j) \mid i \neq j\}$ is "spread around". It turns out that this is in a sense an obstruction for $\Gamma = \mathbb{F}_n$:

Theorem *Suppose* $\Gamma = \langle T_{a_1}, \ldots, T_{a_h} \rangle$, *and let* $m = \min I$ *and* $M = \max I$, *where* $I = \{(a_i, a_j) \mid i \neq j\}$. *Then* $\Gamma = \mathbb{F}_h$ *if* $M \leq m^2/6$.

We will prove a more general version of this (see Theorem 7.2).

It should be noticed that similar arguments have been used to prove that certain groups generated by three 2×2 matrices are free [1, 13].

In Section 2 we go over basic facts about Dehn twists and geometric intersection pairing and different kinds of ping-pong arguments we are going to use. In Section 3 we prove our general theorems about groups generated by two positive multi-twists. In Section 4 we look at the specific case of a lantern formation. In Section 5 we look at a formation which produces a lantern-like relation. In Section 6 we prove that the only possible lantern-like relations are the ones given in Theorem 6.1. In Section 7 we prove a theorem on groups generated by n Dehn twists. In Section 8 we pose some questions that are of similar flavor.

Remark 1.2 After the completion of this work, the author learned that Dan Margalit has obtained some results on the subject of lantern relation using the action of the mapping class group on homology [12]. Also notice that Theorem 6.1 here answers the first question in [12, Section 7].

2 Basics

For two isotopy classes of closed 1-sub-manifolds a, b of S let $i(a, b)$ denote their geometric intersection number. For a set of closed 1-sub-manifolds $A = \{a_1, \ldots, a_n\}$ and a simple closed curve x put

$$||x||_A = \sum_{i=1}^{n} i(x, a_i).$$

For a non-trivial simple closed curve let T_a be the (right-handed) Dehn twist in curve a. The following lemma is proved in [4].

Lemma 2.1 *For simple closed curves a, x, b, and $n \ge 0$,*

$$|i(T_a{}^n(x), b) - n\, i(x, a)\, i(a, b)| \le i(x, b).$$

Let $a = \{a_1, \ldots, a_k\}$ be a collection of distinct, mutually disjoint non-trivial isotopy classes of simple closed curves. For integers $n_i > 0$, the mapping class $T_a = T_{a_1}^{n_1} \cdots T_{a_k}^{n_k}$ is called a positive multi-twist. We also have the following lemma:

Lemma 2.2 *For a positive multi-twist $T_a = T_{a_1}^{n_1} \cdots T_{a_k}^{n_k}$, 1-sub-manifolds x, b and $n \in \mathbb{Z}$,*

$$\left|i(T_a^n(x), b) - |n| \sum_{i=1}^{k} n_i\, i(x, a_i)\, i(a_i, b)\right| \le i(x, b).$$

For a proof see [8, Lemma 4.2]. The statement of that lemma has the expression $|n| - 2$ instead of $|n|$ above. Using the assumption that all n_i are positive, the same proof goes through to prove the improved statement given here. Alternatively, a proof can be found in [4, Exposé 4].

The classic ping-pong argument was used first by Klein [11]. We give two versions here which will be applied in Section 3. The group Γ can be a general group. The notation $\Gamma = \langle f_1, \ldots, f_n \rangle$ means that the group Γ is generated by elements $f_1, \ldots, f_n$.

Lemma 2.3 (Ping-pong) *Let $\Gamma = \langle f_1, \ldots, f_n \rangle$, $n \geq 2$. Suppose Γ acts on a set X. Assume that there are n non-empty mutually disjoint subsets $X_1, \ldots, X_n$ of X such that $f_i^{\pm k}(\cup_{j \neq i} X_j) \subset X_i$, for all $1 \leq i \leq n$ and $k > 0$. Then $\Gamma = \mathbb{F}_n$.*

Proof First notice that a non-empty reduced word of form $w = f_1^{*} f_i^{*} \cdots f_j^{*} f_1^{*}$ (*'s are non-zero integers) is not the identity because $w(X_2) \cap X_2 \subset X_1 \cap X_2 = \emptyset$. But any reduced word in $f_1^{\pm 1}, \ldots, f_n^{\pm 1}$ is conjugate to a w of the above form. □

Lemma 2.4 (Tower ping-pong) *Let Γ be a group generated by $f_1, \ldots, f_n$. Suppose Γ acts on a set X, and there is a function $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$, with the following properties: There are n non-empty mutually disjoint subsets $X_1, \ldots, X_n$ of X such that $f_i^{\pm k}(X \setminus X_i) \subset X_i$ and for any $x \in X \setminus X_i$, we have $\|f_i^{\pm k}(x)\| > \|x\|$ for all $k > 0$. Then $\Gamma = \mathbb{F}_n$. Moreover, the action of every $g \in \Gamma$ which is not conjugate to some power of some f_i on X has no periodic points.*

Proof Any non-empty reduced word in $f_1^{*}, \ldots, f_n^{*}$ (*'s denote non-zero integers) is conjugate to a reduced word $w = f_1^{*} \cdots f_1^{*}$. To show that $w \neq id$ notice that if $x_1 \in X \setminus X_1$, then $w(x_1) \in X_1$, therefore $w(x_1) \neq x_1$. To prove the last assertion, notice that it's enough to show the claim with "periodic points" replaced by "fixed points". Any element of Γ which is not conjugate to a power of some f_i is conjugate to some reduced word of the form $w = f_j^{*} \cdots f_i^{*}$ with $i \neq j$. Now suppose $w(x) = x$. First assume $x \in X \setminus X_i$. Then by assumption $\|w(x)\| > \|x\|$ which is impossible. If on the other hand, $x \in X_i$ and $w(x) = x$, then $w^{-1}(x) = f_i^{*} \cdots f_j^{*}(x) = x$. But again by assumption $\|w^{-1}(x)\| > \|x\|$, which is a contradiction. □

3 Groups generated by two positive multi-twists

Let $a = \{a_1, \ldots, a_k\}$ and $b = \{b_1, \ldots, b_l\}$ be two collections of isotopy classes of non-trivial, mutually disjoint simple closed curves on S, respectively, such that $i(a, b) > 0$. Let $m_1, \ldots, m_k$; and $n_1, \ldots, n_l$ be positive integers. In this section we will set

$T_a = T_{a_1}^{m_1} \cdots T_{a_k}^{m_k}$ and $T_b = T_{b_1}^{n_1} \cdots T_{b_l}^{n_l}$.

$A = \{a, b\}$.

$X = \{x \mid x$ is the isotopy class of a simple closed curve and $\|x\|_A > 0\}$.

For $\lambda \in (0, \infty)$ set

$$N_{a,\lambda} = \{x \in X \mid \iota(x,a) < \lambda\,\iota(x,b)\},$$
$$N_{b,\lambda^{-1}} = \{x \in X \mid \lambda\,\iota(x,b) < \iota(x,a)\}.$$

Notice that $a \in N_{a,\lambda}$ and $b \in N_{b,\lambda^{-1}}$, and $N_{a,\lambda} \cap N_{b,\lambda^{-1}} = \emptyset$. Moreover, $\langle T_a, T_b\rangle$ acts on X, and when λ is irrational $X = N_{a,\lambda} \cup N_{b,\lambda^{-1}}$.

Lemma 3.1 *With the above notation:*

(i) $T_a^{n}(N_{b,\lambda^{-1}}) \subset N_{a,\lambda}$ *if* $n m_i \iota(a_i,b) \geq 2\lambda^{-1}$ *for all* $1 \leq i \leq k$.

(ii) *If* $n m_i \iota(a_i,b) \geq 2\lambda^{-1}$ *for all* $1 \leq i \leq k$, *and* $x \in N_{b,\lambda^{-1}}$, *then* $\|T_a^{n}(x)\|_A > \|x\|_A$.

(iii) $T_b^{n}(N_{a,\lambda}) \subset N_{b,\lambda^{-1}}$ *if* $n n_j \iota(a,b_j) \geq 2\lambda$ *for all* $1 \leq j \leq l$.

(iv) *If* $n n_j \iota(a,b_j) \geq 2\lambda$ *for all* $1 \leq j \leq l$, *and* $x \in N_{a,\lambda}$, *then* $\|T_b^{n}(x)\|_A > \|x\|_A$.

Proof Suppose $x \in N_{b,\lambda^{-1}}$, and $n > 0$ such that $n m_i \iota(a_i,b) \geq 2\lambda^{-1}$. Then by Lemma 2.2,

$$\begin{aligned}
\iota(T_a^{n}(x),b) &\geq n \sum_i m_i \iota(x,a_i)\iota(a_i,b) - \iota(x,b) \\
&> n \sum_i m_i \iota(x,a_i)\iota(a_i,b) - \lambda^{-1} \sum_i \iota(x,a_i) \\
&= \sum_i (n m_i \iota(a_i,b) - \lambda^{-1})\iota(x,a_i) \\
&\geq \lambda^{-1} \sum_i \iota(x,a_i) \\
&= \lambda^{-1}\iota(x,a) \\
&= \lambda^{-1}\iota(T_a^{n}(x), T_a^{n}(a)) \\
&= \lambda^{-1}\iota(T_a^{n}(x), a).
\end{aligned}$$

This proves (i). By symmetry we immediately get (iii). Now notice that for

$x \in N_{b,\lambda^{-1}}$,

$$\begin{aligned} \|T_a^{\,n}(x)\|_A &= \iota(T_a^{\,n}(x),a) + \iota(T_a^{\,n}(x),b) \\ &\geq \iota(x,a) + n\sum_i m_i\,\iota(x,a_i)\iota(a_i,b) - \iota(x,b) \\ &> \sum_i (1 + nm_i\iota(a_i,b) - \lambda^{-1})\iota(x,a_i) \\ &= \sum_i (1 + nm_i\iota(a_i,b) - \lambda^{-1})(1+\lambda)^{-1}(\lambda^{-1}\iota(x,a_i) + \iota(x,a_i)). \end{aligned}$$

But $(1 + nm_i\iota(a_i,b) - \lambda^{-1})(1+\lambda)^{-1} \geq 1$ if and only if $nm_i\iota(a_i,b) \geq 2\lambda^{-1}$, which by assumption implies

$$\begin{aligned} \|T_a^{\,n}(x)\|_A &> \sum_i (\lambda^{-1}\iota(x,a_i) + \iota(x,a_i)) \\ &= \lambda^{-1}\iota(x,a) + \iota(x,a) \\ &> \iota(x,b) + \iota(x,a) \\ &= \|x\|_A. \end{aligned}$$

This proves (ii), and by symmetry (iv). □

Theorem 3.2 *For two positive multi-twists $T_a = T_{a_1}^{m_1}\cdots T_{a_k}^{m_k}$ and $T_b = T_{b_1}^{n_1}\cdots T_{b_l}^{n_l}$ on the surface S, the group $\langle T_a, T_b\rangle \cong \mathbb{F}_2$ if both of the following conditions are satisfied:*

(i) $m_i\iota(a_i,b) \geq 2$ *for all* $1 \leq i \leq k$.

(ii) $n_j\iota(a,b_j) \geq 2$ *for all* $1 \leq j \leq l$.

Proof The group $\langle T_a, T_b\rangle$ acts on $X = \{x \mid \|x\|_A > 0\}$, where $A = \{a,b\}$. Now use the sets $X_1 = N_{a,1}$ and $X_2 = N_{b,1}$ in Lemma 2.3 together with Lemma 3.1 (i), (iii). □

Let $\Gamma = \langle T_a, T_b\rangle$ as before. Consider $supp(\Gamma) = S_{a\cup b}$. If $supp(\Gamma)$ is not a connected surface, and U is one of its components, we can look at the group $\Gamma|_U$. Certainly if $\Gamma|_U \cong \mathbb{F}_2$ then $\Gamma \cong \mathbb{F}_2$ as well. Notice that an element $g|_U \in \Gamma|_U$ is obtained by dropping the twists in curves which can be isotoped off U from $g \in \Gamma$. So let us characterize the groups Γ such that $supp(\Gamma)$ is connected.

Remark 3.3 Let $\Gamma = \langle T_a, T_b\rangle$ where T_a, T_b are multi-twists. If $supp(\Gamma)$ is connected then $\iota(a_i,b) > 0$ and $\iota(a,b_j) > 0$ for all i,j.

Theorem 3.4 *For two positive multi-twists $T_a = T_{a_1}^{m_1} \cdots T_{a_k}^{m_k}$ and $T_b = T_{b_1}^{n_1} \cdots T_{b_l}^{n_l}$ on the surface S, let $\Gamma = \langle T_a, T_b \rangle$ and assume that $supp(\Gamma)$ is connected. Then $\Gamma = \mathbb{F}_2$ except possibly when either*

(i) *there is $1 \le i \le k$ such that $m_i(a_i,b) = 1$ and there is $1 \le j \le l$ such that $n_j(a,b_j) \le 3$, or*

(ii) *there is $1 \le j \le l$ such that $n_j(a,b_j) = 1$ and there is $1 \le i \le k$ such that $m_i(a_i,b) \le 3$.*

Proof Suppose that neither of the two cases happen. The group $\Gamma = \mathbb{F}_2$ if $m_i(a_i,b) \ge 2$ and $n_j(a,b_j) \ge 2$ for all i,j, by Theorem 3.2. To understand the other cases, without loss of generality assume that $m_1(a_1,b) = 1$. By Remark 3.3 $(a_i,b) > 0$ for all i and $(a,b_j) > 0$ for all j since $supp(\Gamma)$ is connected.

Now put $\lambda = 2$ in Lemma 3.1. Clearly the condition $m_i(a_i,b) \ge 2\lambda^{-1} = 1$ is satisfied, so if $n_j(a,b_j) \ge 2\lambda = 4$ for all j, using Lemma 2.3 we get $\Gamma = \mathbb{F}_2$. □

One can completely answer the question "when is a group generated by powers of Dehn twists isomorphic to $\mathbb{F}_2$?", as follows:

Theorem 3.5 *Let $A = \{a,b\}$ be a set of two simple closed curves on a surface S and $m,n > 0$. Put $\Gamma = \langle T_a^m, T_b^n \rangle$. The following conditions are equivalent:*

(i) $\Gamma = \mathbb{F}_2$.

(ii) *Either $(a,b) \ge 2$, or $(a,b) = 1$ and*

$$\{m,n\} \notin \{\{1\},\{1,2\},\{1,3\}\}.$$

Proof By Theorem 3.4, (ii) implies (i). To prove (i) implies (ii), we must show that for $(a,b) = 1$, the groups $\langle T_a, T_b^n \rangle$ are not free for $n = 1,2,3$.

Let us denote T_a by a and T_b by b for brevity. We know that $(ab)^6$ commutes with both a and b, (see Figure 1; for a proof of this relation see [9].) so the case $n = 1$ is non-free. Also, notice the famous braid relation $aba = bab$ (see, for instance [9]). Now consider the case $n = 2$. Observe that

$$(ab^2)^2 = ab^2ab^2 = ab(bab)b = ab(aba)b = (ab)^3,$$

so $(ab^2)^4 = (ab)^6$ is in the center of $\langle a, b^2\rangle$. In the case $n = 3$, notice that

$$\begin{aligned}(ab^3)^3 &= ab^3ab^3ab^3\\ &= ab^2(bab)b(bab)b^2\\ &= ab^2abababab^2\\ &= ab(bab)(aba)(bab)b\\ &= ab(aba)(bab)(aba)b\\ &= (ab)^6.\end{aligned}$$

Therefore $(ab^3)^3$ is in the center of $\langle a, b^3\rangle$. □

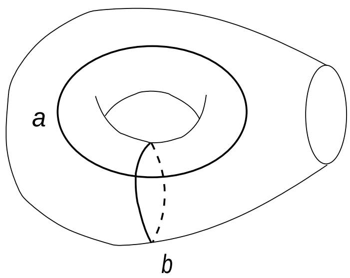

Figure 1: $(T_aT_b)^6 = T$

Remark 3.6 After the completion of this work the author learned that the isomorphism $\langle T_a, T_b\rangle = \mathbb{F}_2$ for $i(a,b) \geq 2$ was proved earlier by Ishida [7].

Let T_a, T_b be two positive multi-twists. In the rest of this section we answer the question "which words in $\langle T_a, T_b\rangle$ are relatively pseudo-Anosov?"(see Definition 1.1).

An element $f \in M(S)$ is called *pure* [8] if for any simple closed curve c, $f^n(c) = c$ implies $f(c) = c$. In other words, by Thurston classification [4], there is a finite (possibly empty) set $C = \{c_1, \ldots, c_k\}$ of disjoint simple closed curves c_i such that $f(c_i) = c_i$ and f keeps all components of $S \setminus (c_1 \cup \cdots \cup c_k)$ invariant, and is either identity or pseudo-Anosov on each such component. A subgroup of $M(S)$ is called pure if all elements of it are pure. Ivanov showed that $M(S)$ contains finite-index pure subgroups, namely, $ker(M(S) \to H_1(S; \mathbb{Z}/m\mathbb{Z}))$ for $m \geq 3$ [8]. A relatively pseudo-Anosov word induces a pure element of the mapping class group.

Theorem 3.7 *For two positive multi-twists $T_a = T_{a_1}^{m_1} \cdots T_{a_k}^{m_k}$ and $T_b = T_{b_1}^{n_1} \cdots T_{b_l}^{n_l}$ on the surface S, the group $\Gamma = \langle T_a, T_b\rangle$ is pure and relatively pseudo-Anosov if any of the following conditions holds:*

(i) *For all* i, $m_i(a_i;b)$ 2 *and for all* j, $n_j(a;b_j)$ 3.

(ii) *For all* i, $m_i(a_i;b)$ 3 *and for all* j, $n_j(a;b_j)$ 2.

(iii) *For all* i, $m_i(a_i;b)$ 1 *and for all* j, $n_j(a;b_j)$ 5.

(iv) *For all* i, $m_i(a_i;b)$ 5 *and for all* j, $n_j(a;b_j)$ 1.

Proof We use Lemma 2.4 together with Lemma 3.1 (ii),(iv). First assume that = 1 + where is a small irrational number. Notice that $X = N_{a;} \cup N_{b; -1}$. If all $m_i(a_i;b)$ 2 > 2 $^{-1}$ and $n_i(a;b_i)$ 3 > 2 , one can use Tower ping-pong to show that if a simple closed curve intersects $supp(\Gamma)$ then it cannot be mapped to itself by any element of Γ except conjugates of powers of T_a and T_b, which are already known to be pure and relatively pseudo-Anosov. This proves (i) (A relatively pseudo-Anosov word induces a pure element). Similarly by using = 1 − , $\in \mathbb{R} \setminus \mathbb{Q}$ in Lemma 3.1 (ii),(iv), we get (ii). To get parts (iii),(iv) we can set = 2 + and = 1/2 − respectively and argue similarly. □

This in particular proves:

Corollary 3.8 (Thurston [4]) *If* $a;b$ *are two simple closed curves, which ll up the closed surface* S *of genus* g 2, *then* $\langle T_a;T_b\rangle = \mathbb{F}_2$ *and all elements not conjugate to the powers of* T_a *and* T_b *are pseudo-Anosov.*

Proof If $a;b$ ll up S we must have $(a;b)$ 3. Now we can use Theorem 3.7. □

Theorem 3.9 *Let* $A = \{a;b\}$ *be a set of two simple closed curves on a surface* S *and* $m;n > 0$ *be integers and* $\Gamma = \langle T_a^m;T_b^n\rangle$. *The following conditions are equivalent:*

(i) Γ *is relatively pseudo-Anosov.*

(ii) *Either* $(a;b)$ 3, *or* $(a;b) = 2$ *and* $(m;n) \neq (1;1)$, *or* $(a;b) = 1$ *and*

$$\{m;n\} \notin \{\{1\};\{1;2\};\{1;3\};\{1;4\};\{2\}\}.$$

Proof If $(a;b)$ 3, then Γ is relatively pseudo-Anosov for all $m;n > 0$ by Theorem 3.7. If $(a;b) = 2$ then Γ is relatively pseudo-Anosov if $m > 1$ or $n > 1$ by Theorem 3.7. We prove that if $(a;b) = 2$ then $\Gamma = \langle T_a;T_b\rangle$ is not relatively pseudo-Anosov. We consider two cases.

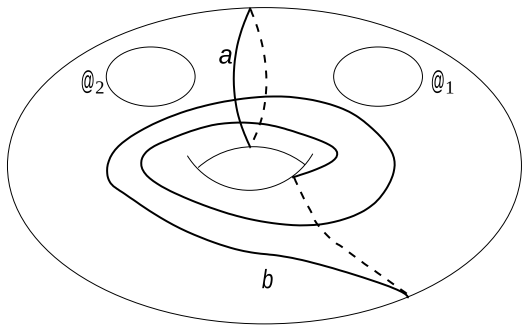

Figure 2: $S_{1,2,0}$ and the curves a and b

Case 1 $(a, b) = 2$ and the algebraic intersection number of a, b is 2.

In this case both a, b can be embedded in a twice punctured torus subsurface of S (see Figure 2). We will prove in Proposition 5.1 that $(T_b T_a)^2$ is in fact a multi-twist.

Case 2 $(a, b) = 2$ but the algebraic intersection number of a, b is 0.

In this case a, b can be embedded in a 4-punctured sphere. According to the lantern relation [9] (see Figure 3), $T_b T_a$ is a multi-twist.

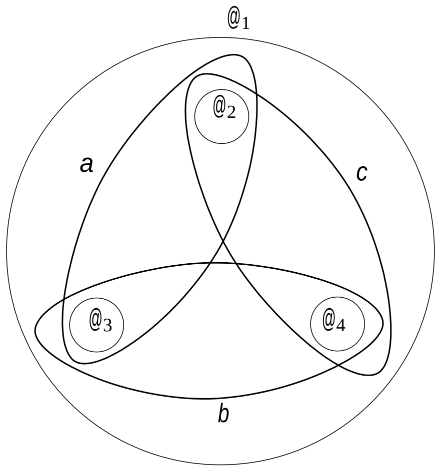

Figure 3: $T_a T_b T_c = T_{\partial_1} T_{\partial_2} T_{\partial_3} T_{\partial_4}$

This proves that when $(a, b) = 2$, $\langle T_a, T_b \rangle$ is not relatively pseudo-Anosov.

If $(a,b) = 1$, the group Γ is relatively pseudo-Anosov except possibly when $\{m,n\}$ is one of $\{1,i\}$, $i = 1,2,3,4$, or $\{m,n\} = \{2,2\}$, by Theorem 3.7. The groups $\langle T_a, T_b\rangle$, $\langle T_a, T_b^2\rangle$ and $\langle T_a, T_b^3\rangle$ are not relatively pseudo-Anosov because the map $(T_aT_b)^6 = (T_aT_b^2)^4 = (T_aT_b^3)^3$ is in fact a Dehn twist in the boundary of the surface defined by a,b (see Figure 1) hence they induce the identity on $S_{a,b}$.

If $(a,b) = 1$, then $\Gamma = \langle T_a^2, T_b^2\rangle$ is not relatively pseudo-Anosov. This is because $T_b^2T_a^2$ has a trace of -2 and hence is reducible (see Remark 6.2). Similarly when $(a,b) = 1$, the maps $T_aT_b^4$ and $T_a^4T_b$ both have a trace of -2 and hence are reducible. □

We saw that if a,b are two simple closed curves with $(a,b) \geq 2$, a word $w(T_a,T_b) \in \langle T_a,T_b\rangle$ is relatively pseudo-Anosov except possibly when $(a,b) = 2$. In the following theorem we narrow down the search for words which are not relatively pseudo-Anosov in this case.

Theorem 3.10 *Let a,b be two simple closed curves on a surface S with $(a,b) = 2$. Then a word w in T_a,T_b representing an element of $\langle T_a,T_b\rangle$ is a pure and relatively pseudo-Anosov unless possibly when w is cyclically reducible to a power of either $T_bT_a^{-1}$ or T_bT_a.*

Proof The proof is based on repeated application of Lemma 2.1. Clearly T_a and T_b are both pure and relatively pseudo-Anosov. So in what follows we assume that w is a cyclically reduced word of length > 1. Let $X, A, N_{a,1}, N_{b,1}$ be defined as in the beginning of this section. Let

$$Y = \{x \in X \mid (x,a) = (x,b)\}.$$

Hence X is a disjoint union $N_{a,1} \cup N_{b,1} \cup Y$.

By Lemma 3.1, we have $T_a^{\pm n}(N_{b,1}) \subset N_{a,1}$ and $T_b^{\pm n}(N_{a,1}) \subset N_{b,1}$ for all $n > 0$. Moreover, for $x \in N_{b,1}$, we have $\|T_a^{\pm n}(x)\| > \|x\|$ and for $x \in N_{a,1}$, $\|T_b^{\pm n}(x)\| > \|x\|$. By the same lemma, $T_a^{\pm n}(Y) \subset N_{a,1}$ and $T_b^{\pm n}(Y) \subset N_{b,1}$ for all $n \geq 2$ (This follows by applying the lemma to $\lambda = 1 + \epsilon$ and $\lambda = 1 - \epsilon$, where ϵ is a small positive number).

Let $w = T_b^{n_k}T_a^{m_k} \cdots T_b^{n_1}T_a^{m_1}$ be a cyclically reduced word, where $m_i, n_i \neq 0$ and $k \geq 1$. If any of m_i is greater that 1 in absolute value, we can assume without loss of generality that $|m_1| > 1$, by conjugation. Therefore if $x \in Y \cup N_{b,1}$, then $T_a^{m_1}(x) \in N_{a,1}$ and hence $\|w^n(x)\| > \|x\|$ for all $n > 0$. Hence $w^n(x) \neq x$ for all integers n. If $x \in N_{a,1}$, then $\|w^{-n}(x)\| > \|x\|$ for $n > 0$, and so $w^n(x) \neq x$

for all n. This shows that w is relatively pseudo-Anosov. The case where some $|n_i| > 1$ follows by symmetry by replacing w with w^{-1}.

So let us assume that for all $1 \le i \le k$, we have $m_i, n_i = \pm 1$. If w is not conjugate to a power of T_bT_a or $T_bT_a^{-1}$, by conjugating w we can assume either $m_1 \neq m_2$, or $n_k \neq n_{k-1}$. We assume the former. The latter can be dealt with similarly by symmetry and replacing w with w^{-1}. In this case the word w could have any of the following forms:

(i) $w = T_b^{n_k} T_a^{m_k} \cdots T_aT_bT_a^{-1}$,

(ii) $w = T_b^{n_k} T_a^{m_k} \cdots T_a^{-1}T_bT_a$,

(iii) $w = T_b^{n_k} T_a^{m_k} \cdots T_aT_b^{-1}T_a^{-1}$,

(iv) $w = T_b^{n_k} T_a^{m_k} \cdots T_a^{-1}T_b^{-1}T_a$.

Suppose, for example, that $w = T_b^{n_k} T_a^{m_k} \cdots T_aT_bT_a^{-1}$. As before, if $x \in N_{a,1} \cup N_{b,1}$, we get that $w^n(x) \neq x$ for all $n > 0$. So let us assume that $x \in Y$. Then, by definition of Y, $i(x, a) = i(x, b) = p > 0$. Then we have $i(T_a^{-1}(x), a) = p$ and by Lemma 2.1,

$$|i(T_a^{-1}(x), b) - i(a, b)i(x, a)| \le i(x, b);$$

which implies $p \le i(T_a^{-1}(x), b) \le 3p$. If $p < i(T_a^{-1}(x), b)$, then $T_a^{-1}(x) \in N_{a,1}$ and so $w^n(x) \neq x$, for all $n > 0$. So let us assume $i(T_a^{-1}(x), b) = p$. Notice that this implies $i(T_bT_a^{-1}(x), b) = p$. Again by Lemma 2.1,

$$|i(T_b(T_a^{-1}(x)), a) - i(a, b)i(b, T_a^{-1}(x))| \le i(T_a^{-1}(x), a);$$

which gives $p \le i(T_bT_a^{-1}(x), a) \le 3p$. Again, if $p < i(T_bT_a^{-1}(x), a)$, then $T_bT_a^{-1}(x) \in N_{b,1}$ which implies $w^n(x) \neq x$ for $n > 0$. Otherwise, we can further assume that $i(T_bT_a^{-1}(x), a) = p$. Notice that this gives $i(T_aT_bT_a^{-1}(x), a) = p$. At this point it looks like the argument is going to go on forever, but here is a new ingredient. For any mapping class f, we have the following well-known equation: $fT_bf^{-1} = T_{f(b)}$. In particular: $T_aT_bT_a^{-1} = T_{T_a(b)}$.

Claim $i(T_a(b), b) = 4$

This follows from Lemma 2.1: $|i(T_a(b), b) - i(b, a)i(a, b)| \le i(b, b)$.

Now by the same lemma,

$$|i(T_{T_a(b)}(x), b) - i(T_a(b), x)i(T_a(b), b)| \le i(x, b);$$

which gives $3p \le i(T_aT_bT_a^{-1}(x), b) \le 5p$, i.e., $T_aT_bT_a^{-1}(x) \in N_{a,1}$, and so $w^n(x) \neq x$ for all $n \ge 0$. The other cases (ii),(iii) and (iv) follow similarly. □

4 The case of two simple closed curves with intersection number 2 filling a 4-punctured sphere

Let a, b be two simple closed curves such that $(a, b) = 2$ and $S_{\{a,b\}}$ is a four-holed sphere. (Figure 3).

The relation $T_a T_b T_c = T_{\partial_1} T_{\partial_2} T_{\partial_3} T_{\partial_4}$ was discovered by Dehn [3] and later on by Johnson [10]. A proof of the lantern relation can be found in [9]. Note the commutativity between the various twists.

Proposition 4.1 *In the group $\langle T_a, T_b \rangle$ all words are pure. All words are relatively pseudo-Anosov except precisely words that are cyclically reducible to a non-zero power of $T_b T_a$.*

Proof The lantern relation implies:

$$T_a T_b = T_c^{-1} T_{\partial_1} T_{\partial_2} T_{\partial_3} T_{\partial_4}.$$

This shows that $T_a T_b$ (and hence its conjugate $T_b T_a$) is a multi-twist. Notice that

$$T_a^{-1} T_b = T_a^{-2} T_c^{-1} T_{\partial_1} T_{\partial_2} T_{\partial_3} T_{\partial_4}.$$

Hence restricted to $S_{a,b}$, $T_a^{-1} T_b = T_a^{-2} T_c^{-1}$. But the group $\langle T_a^2, T_c \rangle$ is pure and relatively pseudo-Anosov by Theorem 3.9, which shows that $T_a^{-1} T_b$ (and hence its conjugate $T_b T_a^{-1}$) is pure relatively pseudo-Anosov. Moreover, a, c fill the same surface as a, b. Finally, we invoke Theorem 3.10. □

5 The case of two simple closed curves with intersection number 2 filling a twice-punctured torus

Let $S_{g,b,n}$ denote a surface of genus g with b boundary components and n punctures. Let a and b be two simple closed curves such that $(a, b) = 2$ and assume both intersections have the same sign. In this case a and b are both non-separating. One can therefore assume, up to diffeomorphism that they are as given in Figure 2. Since the regular neighborhood of $a \cup b$ is homeomorphic to $S_{1,2,0}$, the surface filled by a, b is $S_{1,i,j}$, for $i, j = 0, 1, 2$, $i + j \quad 2$.

Assume that $S_{\{a,b\}} = S_{1,2,0}$. Let γ and γ' be the curves defined in Figure 4. By following Figure 4 one can see that $(T_b T_a)^2(\gamma) = \gamma$, preserving the orientation. Since by definition $T_b T_a(\gamma) = \gamma'$, one also gets $(T_b T_a)^2(\gamma') = \gamma'$. Now notice that γ and γ' cut up $S_{1,2,0}$ into two pairs of pants. Hence $(T_b T_a)^2$ is a multi-twist in curves γ, γ', ∂_1 and ∂_2.

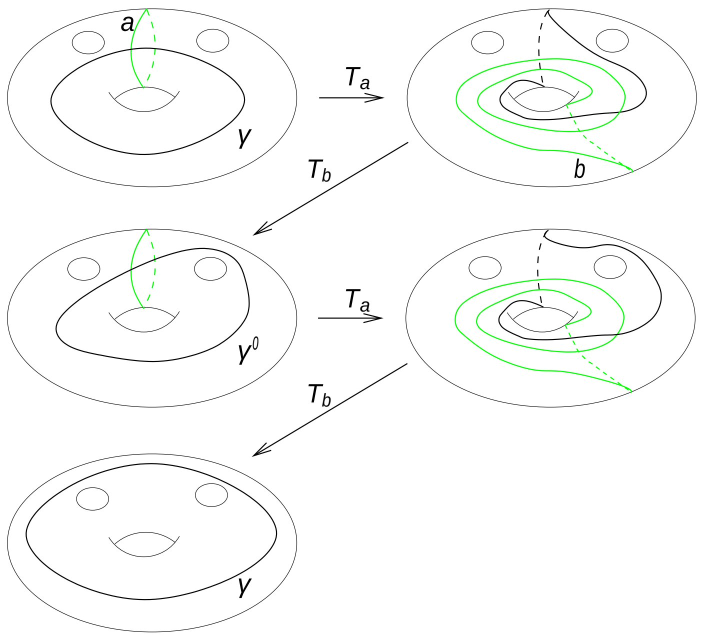

Figure 4: $(T_bT_a)^2(\gamma) = \gamma$ and $(T_bT_a)(\gamma) = \gamma'$

Proposition 5.1 *With the notation in Figures 2 and 4, we have*

$$(T_bT_a)^2 = T_{\partial_1}T_{\partial_2}T_{\gamma}^{-4}T_{\gamma'}^{-4}.$$

Proof Since $(T_bT_a)^2$ fixes $\partial_1, \partial_2, \gamma$ and γ', it has to be a multi-twist in these curves. We consider an arc joining ∂_1 to ∂_2 crossing γ once as in Figure 5. We apply $(T_bT_a)^2$ to l, and the result is the same as applying $T_{\partial_1}T_{\partial_2}T_{\gamma}^{-4}$ to l (again see Figure 5). Hence

$$(T_bT_a)^2 = T_{\partial_1}T_{\partial_2}T_{\gamma}^{-4}T_{\gamma'}^{n},$$

where n is to be found. One can argue by drawing another arc joining ∂_1 to ∂_2 passing through γ' once, but here is a simpler way: We know that $(T_bT_a)(\gamma) = \gamma'$ and $(T_bT_a)(\gamma') = \gamma$ (see Figure 4), so if we conjugate the above equation by T_bT_a, we get:

$$(T_bT_a)^2 = T_{\partial_1}T_{\partial_2}T_{\gamma'}^{-4}T_{\gamma}^{n},$$

which shows that $n = -4$. □

Proposition 5.2 *The word $T_bT_a^{-1}$ is relatively pseudo-Anosov.*

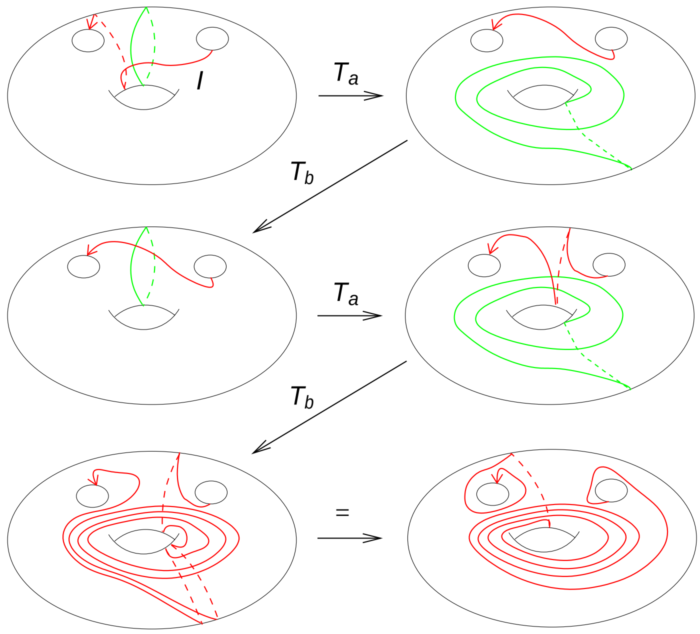

Figure 5: The arc l, and $(T_bT_a)^2(l)$

Proof We use a \brute force" method to show that restricted to $S = S_{1;0;2}$ (the boundary components $@_1;@_2$ shrunk to punctures $p_1;p_2$, respectively), the word $T_bT_a^{-1}$ induces a pseudo-Anosov map. It is enough to show that the word $T_b^{-1}T_a$ is relatively pseudo-Anosov. Let f be the mapping class induced by the word $T_b^{-1}T_a$. We will nd measured laminations $F_1;F_2$ and > 0 such that $f(F_1) = F_1$ and $f(F_2) = {}^{-1}F_2$. To this end, we use the theory of measured train-tracks. For a review of these methods and the theory, see for example [5].

Consider the polygon R obtained by cutting S open as in Figure 6. Identifying parallel sides of R yields back the surface S. Consider the measured train-track $= (x;y;z)\ (x;y;z \quad 0)$ on S de ned as in Figure 7. We can calculate the image $f(\ )$ in two steps as in Figures 8 and 9. (Remember that Dehn twists are right-handed.) Luckily the action of f on the space of measures on is linear, so we can easily nd xed laminations carried on : The matrix representing f on the space of measured laminations carried on is

$$\begin{pmatrix} 2 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix}:$$

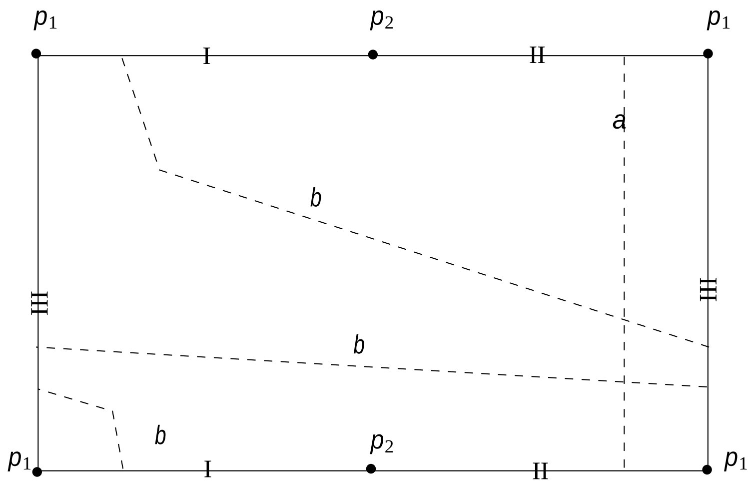

Figure 6: The surface $S_{1,0,2}$ cut-open into a polygon R

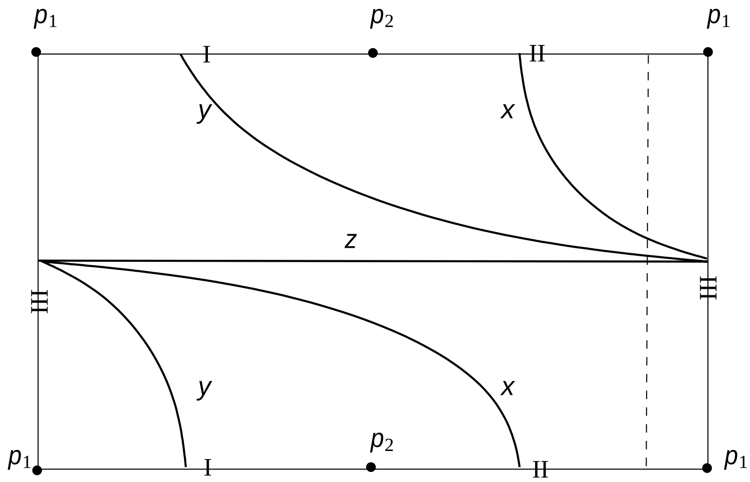

Figure 7: The measured train-track (x, y, z)

This matrix has eigenvalues $1, 3+\sqrt{10}, 3-\sqrt{10}$. The only eigenvalue that has a non-negative eigenvector is $\lambda = 3+\sqrt{10}$ and the eigenvector corresponds to the measured train-track $(2+\sqrt{10}, 2+\sqrt{10}, 2)$ (up to a positive factor). If we "fatten up" this measured train-track, we get a lamination F_1 as in Figure 10 with the property $f(F_1) = \lambda F_1$. Notice that, geometrically, all leaves have slope -1. One can see that there are no closed loops of leaves (if there were they would have been caught as eigenvectors already). Also, there is no leaf in F_1

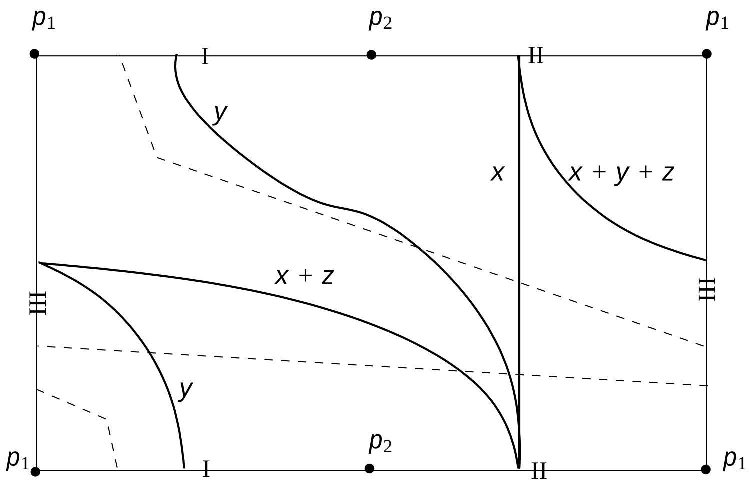

Figure 8: The measured train-track $T_a(\ (x;y;z))$

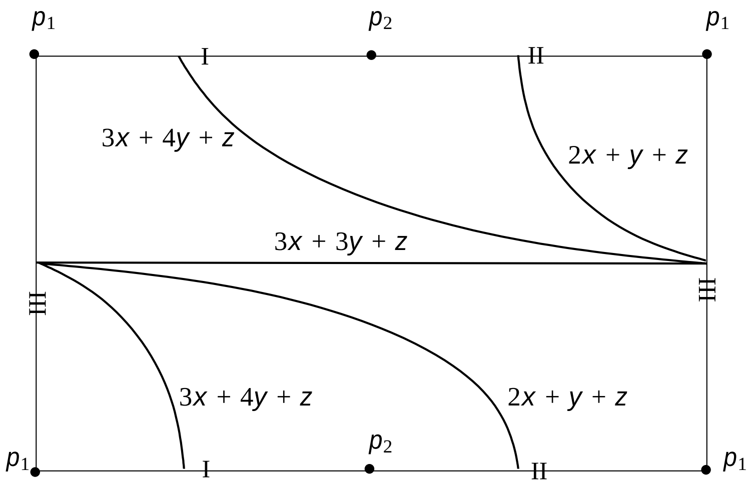

Figure 9: The measured train-track $f(\ (x;y;z)) = T_b^{-1}T_a(\ (x;y;z))$

connecting a puncture to a puncture, since if it were so $\sqrt{10}$ would be rational. Similarly, one can find a lamination F_2 which satisfies $f(F_2) = \ ^{-1}F_2$. But establishing such F_1 is already enough to show that f is pseudo-Anosov on S, hence proving the proposition. □

Corollary 5.3 *All words in $\langle T_a; T_b\rangle$ are pure except precisely the ones conjugate to the odd powers of T_bT_a. All words are relatively pseudo-Anosov except*

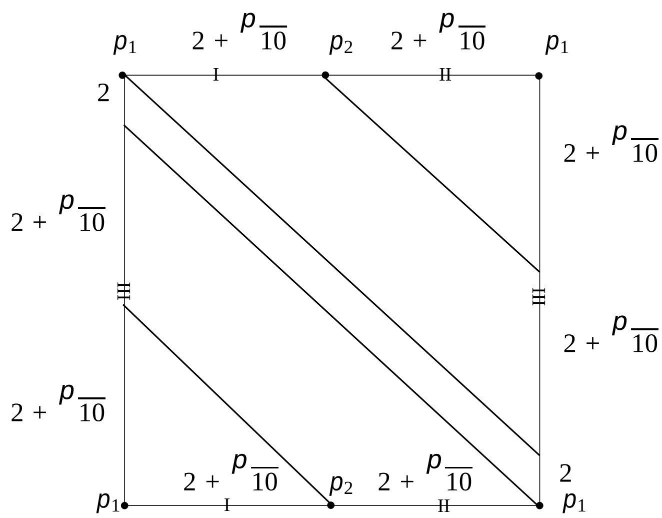

Figure 10: The measured lamination F_1 satisfies $f(F_1) = \lambda F_1$

precisely the ones cyclically reducible to $(T_b T_a)^n$ *for some non-zero integer* n.

Proof Notice that the words conjugate to powers of $T_b T_a^{-1}$ are relatively pseudo-Anosov and hence pure. Now the claim follows from Theorem 3.10 and Proposition 5.2. □

6 Application to Lantern-type relations

Theorem 6.1 *Let* a, b *be two simple closed curves on a surface* S *such that* $(a, b) \geq 2$. *Let* w *be a word in* T_a, T_b *which is not cyclically reducible to a power of* T_a *or* T_b, *but representing an element in* $M(S)$ *which is a multi-twist. Then* $(a, b) = 2$ *and exactly one of the following conditions hold:*

(i) *The curves* a, b *have algebraic intersection number* 0, *the word* w *can be cyclically reduced to* $(T_a T_b)^n$ *for some* $n \in \mathbb{Z}$, *and*

$$T_a T_b = T_{\partial_1} T_{\partial_2} T_{\partial_3} T_{\partial_4} T_c^{-1}.$$

(See Figure 3).

(ii) *The curves* a, b *have algebraic intersection number* 2, *the word* w *can be cyclically reduced to* $(T_b T_a)^{2n}$ *for some* $n \in \mathbb{Z}$, *and*

$$(T_b T_a)^2 = T_{\partial_1} T_{\partial_2} T_y^{-4} T_{y'}^{-4}.$$

(See Figures 2, 4).

Proof By Theorem 3.7 $(a,b) = 2$, because otherwise w will be relatively pseudo-Anosov, with a support which is not a union of annuli, so it cannot be a multi-twist. Now apply Proposition 4.1 and Corollary 5.3. □

Remark 6.2 It is well known that $M_{1,0,1} = \mathrm{SL}(2,\mathbb{Z})$. Also,

$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

We have a short exact sequence [2]

$$0 \to \mathbb{Z} \to M_{1,1,0} \to M_{1,0,1} = \mathrm{SL}(2,\mathbb{Z}) \to 0.$$

The Dehn twists T_a and T_b in Figure 1 induce the matrices

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{Z})$, respectively. Clearly $sts = q$ and hence $\mathrm{SL}(2,\mathbb{Z}) = \langle t, s\rangle$. In this case, a word in $\langle T_a, T_b\rangle$ is pseudo-Anosov if and only if the trace of the corresponding matrix has absolute value of 2 or more. Such a word is a multi-twist if the corresponding matrix has trace 2.

Definition 6.3 A relation $w(T_a, T_b) = T_C$ is called lantern-like if T_a, T_b are Dehn twists, and T_C is a multi-twist with C having at least 3 components.

Theorem 6.4 *The only lantern-like relations in any mapping class group are described in Theorem 6.1.*

Proof We have to only show that, if $(a,b) = 1$, they cannot form a lantern-like relation. But in that case, a, b are supported in a once-punctured torus, hence T_C can be made of twists in the boundary and at most one simple closed curve in that torus. □

7 Groups generated by $n \geq 3$ powers of twists

In this section the phrase "$i \neq j \neq k$" means that i, j, k are distinct. Let $a_1, \ldots, a_n$ be $n \geq 3$ simple closed curves on a surface S such that $(a_i, a_j) > 0$ for $i \neq j$.

Let $\lambda_{ijk} > 1$ and $\mu_{ij} > 0$ (for $i \neq j \neq k$) be real numbers such that $\mu_{ji} = \mu_{ij}^{-1}$. Put $\lambda = (\lambda_{ijk})_{i\neq j\neq k}$ and $\mu = (\mu_{ij})_{i\neq j}$. Define the set of simple closed curves

$$N_{a_i} = N_{a_i;\lambda,\mu} = \left\{ x \mid (x, a_i) < \mu_{ij}(x, a_j), \ \frac{(x, a_k)}{(x, a_j)} < \lambda_{ijk}\frac{(a_i, a_k)}{(a_i, a_j)}, \ \forall j \neq k \neq i \right\},$$

for $i = 1, \ldots, n$. Note that $a_i \in N_{a_i}$.

Lemma 7.1 *Let $a_1, \dots, a_n$ be a set of $n \geq 3$ simple closed curves such that $\iota(a_i,a_j) \neq 0$ for $i \neq j$.*

(i) *The sets N_{a_i}, $i = 1, \dots, n$ are mutually disjoint.*

(ii) *For $1 \leq i \neq j \leq n$, we have $T_{a_i}^{m}(N_{a_j}) \subset N_{a_i}$ for*

$$
\begin{aligned}
m \geq \max \Big\{ & \frac{2}{\lambda_{ij}\,\iota(a_i,a_j)}, \\
& \frac{1}{\lambda_{ik}\,\iota(a_i,a_k)} + \mu_{jik}\frac{\iota(a_j,a_k)}{\iota(a_i,a_j)\iota(a_i,a_k)}, \\
& \frac{\mu_{jil}}{\lambda_{ikl}-1}\,\frac{\iota(a_j,a_l)}{\iota(a_i,a_l)\iota(a_j,a_i)} + \frac{\lambda_{ikl}\,\mu_{jik}}{\lambda_{ikl}-1}\,\frac{\iota(a_j,a_k)}{\iota(a_j,a_i)\iota(a_i,a_k)}, \\
& \frac{1}{(\lambda_{ikj}-1)\,\lambda_{ij}\,\iota(a_i,a_j)} + \frac{\lambda_{ikj}\,\mu_{jik}}{\lambda_{ikj}-1}\,\frac{\iota(a_j,a_k)}{\iota(a_j,a_i)\iota(a_i,a_k)}, \\
& \frac{\lambda_{ijl}}{(\lambda_{ijl}-1)\,\lambda_{ij}\,\iota(a_i,a_j)} + \frac{\mu_{jil}}{\lambda_{ijl}-1}\,\frac{\iota(a_j,a_l)}{\iota(a_j,a_i)\iota(a_i,a_l)} \Big\}_{k \neq l \neq i}.
\end{aligned}
$$

Proof (i) is clear. To prove (ii), consider $x \in N_{a_j}$. We have

$$\iota(T_{a_i}^m(x), a_j) \geq m\,\iota(a_i,a_j)\iota(x,a_i) - \iota(x,a_j) > \mu_{ji}\,\iota(x,a_i) = \mu_{ji}\,\iota(T_{a_i}^m(x), a_i)$$

for $m \geq \frac{2\mu_{ji}}{\iota(a_i,a_j)}$. Let $k \neq i, j$. Then

$$\iota(T_{a_i}^m(x), a_k) \geq m\,\iota(a_i,a_k)\iota(x,a_i) - \iota(x,a_k) > \mu_{ki}\,\iota(x,a_i)$$

if

$$m \geq \frac{1}{\lambda_{ik}\,\iota(a_i,a_k)} + \mu_{jik}\frac{\iota(a_j,a_k)}{\iota(a_i,a_j)\iota(a_i,a_k)}.$$

Let $k, l \neq i$. Then

$$\iota(T_{a_i}^m(x), a_l)/\iota(T_{a_i}^m(x), a_k) < \lambda_{ikl}\,\iota(a_i,a_l)/\iota(a_i,a_k)$$

if and only if

$$\iota(a_i,a_k)\,\iota(T_{a_i}^m(x), a_l) < \lambda_{ikl}\,\iota(a_i,a_l)\,\iota(T_{a_i}^m(x), a_k).$$

This will hold if

$$\iota(a_i,a_k)\big(m\,\iota(a_i,a_l)\iota(x,a_i) + \iota(x,a_l)\big) < \lambda_{ikl}\,\iota(a_i,a_l)\big(m\,\iota(a_i,a_k)\iota(x,a_i) - \iota(x,a_k)\big). \quad (1)$$

The inequality (1) is equivalent to

$$m\,\iota(a_i,a_l)(\lambda_{ikl}-1) > \frac{\iota(x,a_l)}{\iota(x,a_i)} + \lambda_{ikl}\frac{\iota(x,a_k)\iota(a_i,a_l)}{\iota(x,a_i)\iota(a_i,a_k)}. \quad (2)$$

(One has $(x;a_i) > 0$ since $x \in N_{a_j}$.) Therefore for $l \neq j$ and $k \neq j$, it is enough to have

$$(a_i;a_l)(\ _{ikl} - 1) \qquad _{jil}\frac{(a_j;a_l)}{(a_j;a_i)} + \ _{ikl}\ _{jik}\frac{(a_j;a_k)(a_i;a_l)}{(a_j;a_i)(a_i;a_k)};$$

i.e.,

$$\frac{_{jil}}{_{ikl} - 1}\frac{(a_j;a_l)}{(a_i;a_l)(a_j;a_i)} + \frac{_{ikl}\ _{jik}}{_{ikl} - 1}\frac{(a_j;a_k)}{(a_j;a_i)(a_i;a_k)}:$$

If $l = j$ (and so $k \neq j$) then one can replace the inequality (2) with

$$(a_i;a_l)(\ _{ikl} - 1) \qquad _{ji} + \ _{ikl}\frac{(x;a_k)(a_i;a_l)}{(x;a_i)(a_i;a_k)}$$

which gives

$$\frac{1}{(\ _{ikj} - 1)\ _{ij}(a_i;a_j)} + \frac{_{ikj}\ _{jik}}{_{ikj} - 1}\frac{(a_j;a_k)}{(a_j;a_i)(a_i;a_k)}:$$

If $k = j$ (and so $l \neq j$) one similarly needs

$$\frac{_{ijl}}{(\ _{ijl} - 1)\ _{ij}(a_i;a_j)} + \frac{_{jil}}{_{ijl} - 1}\frac{(a_j;a_l)}{(a_j;a_i)(a_i;a_l)}:$$

□

This lemma conveys the idea that if the set $f(a_i;a_j)g_{i\neq j}$ is not \too spread around" then the group $\Gamma = \langle T_{a_1}; \quad ;T_{a_n}\rangle$ is free on n generators, as follows:

Theorem 7.2 *Let $a_1; \quad ;a_n$ be $n \quad 3$ simple closed curves on a surface S such that $M \quad m^2{=}6$ where $M = \max f(a_i;a_j)g_{i\neq j}$ and $m = \min f(a_i;a_j)g_{i\neq j}$. Then*

$$\Gamma = \langle T_{a_1}; \quad ;T_{a_n}\rangle = \mathbb{F}_n:$$

More generally, suppose that for all $i \neq j \neq k$ we have

$$\frac{(a_i;a_k)}{(a_i;a_j)(a_j;a_k)} \qquad \frac{1}{6}:$$

Then the same conclusion holds.

Proof Put $_{ij} = 1$ and $_{ijk} = 2$ in Lemma 7.1. By assumption, for all $i \neq j \neq k$,

$$\frac{(a_i;a_k)}{(a_i;a_j)(a_j;a_k)} \qquad \frac{1}{6}:$$

This implies $i(a_i, a_j) \geq 6$ for all $i \neq j$, since otherwise it is impossible for both of

$$\frac{i(a_i, a_k)}{i(a_i, a_j)\, i(a_j, a_k)} \text{ and } \frac{i(a_j, a_k)}{i(a_i, a_j)\, i(a_i, a_k)}$$

to be $\leq 1/6$. Therefore, it is easily seen that $\lambda = 1$ satisfies the requirements of Lemma 7.1. □

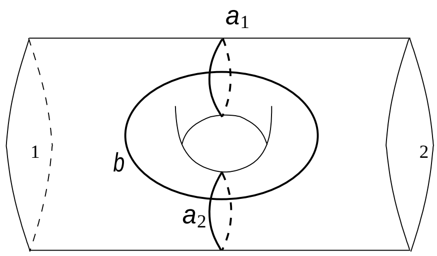

Figure 11: $(T_{a_1} T_{a_2} T_b)^4 = T_{\delta_1} T_{\delta_2}$

8 Questions

We end this paper by looking at some questions. Consider the group $\Gamma = \langle T_{a_1}^{m_1} T_{a_2}^{m_2}, T_b^n \rangle$, where the simple closed curves are defined in Figure 11, and they satisfy the torus relation $(T_{a_1} T_{a_2} T_b)^4 = T_{\delta_1} T_{\delta_2}$. It is interesting to find out if there are any torus-like relations. Theorems 3.2 and 3.7 will restrict the search. In particular:

Question 1 Is it true that $\langle T_{a_1}^2 T_{a_2}, T_b \rangle = \mathbb{F}_2$?

Question 2 Under what conditions is $\Gamma = \langle T_{a_1}, \cdots, T_{a_n} \rangle$ relatively pseudo-Anosov?

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