UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVI
2008

HOMOGENEOUS POLYNOMIAL INVARIANTS FOR CUBIC-HOMOGENEOUS FUNCTIONS

by Gaetano Zampieri

Abstract. This note introduces the concept of homogeneous polynomial invariant in connection with the cubic-homogeneous functions with constant Jacobian determinant. These last functions are sufficient to study the Jacobian conjecture. The new concept hopefully permits to deepen the research on the line of the linear dependence problem recently solved by de Bondt's counterexample.

1. Introduction. A famous problem included in Smale's list [**10**] is the *Jacobian conjecture*, originated by Keller [**8**]: is a polynomial mapping $f\colon \mathbb{C}^n \to \mathbb{C}^n$ with nonzero constant Jacobian determinant necessarily one-to-one? The problem is still open, although many interesting results have been obtained in connection with it. Most of them can be studied in the rich book [**6**] by van den Essen who seems to conjecture the possibility of a counterexample, see [**5**]. Among the known fact: a polynomial mapping $\mathbb{C}^n \to \mathbb{C}^n$ that is one-to-one must always be onto and the inverse is itself a polynomial mapping, briefly a polynomial automorphism of $\mathbb{C}^n$, see Rudin [**9**].

To deal with this problem it is sufficient to consider the special class of the 'cubic-homogeneous' functions

$$f(x) = x - g(x)\,, \qquad \det f'(x) = 1\,, \qquad x \in \mathbb{C}^n\,, \tag{1}$$

where $g\colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping *homogeneous of degree 3*. Indeed, by the *reduction of degree theorem*, Yagzhev [**11**] and Bass, Connel and Wright [**1**], the Jacobian conjecture is true if and only if it is verified for these

1991 *Mathematics Subject Classification.* 14R15.
Key words and phrases. Cubic-homogeneous maps, linear dependence problem.
Supported by the PRIN 06-07 directed by Fabio Zanolin.

cubic functions. Incidentally, a further important reduction, to the 'cubic-linear' polynomial mappings, is due to Drużkowski [**4**].

Many theorems and several conjectures deal with the cubic-homogeneous functions (1). One of them is the 'linear dependence conjecture' which says that the components $g_1, \dots, g_n$ of g should be linearly dependent over $\mathbb{C}$, namely there should exist $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, not all zero, such that

$$\lambda_1 g_1 + \cdots + \lambda_n g_n = 0 \,. \tag{2}$$

Partial positive answers to this problem started with [**1**] 25 years ago. The counterexample, in dimension 10 and higher, came with de Bondt [**2**], 2006 (and with [**3**], using a technique of [**7**], for the cubic-linear case). Such examples seem very important and, possibly, they are steps toward a counterexample (if any) to the Jacobian conjecture itself.

The aim of this paper is 'to raise the stakes' by introducing a new concept more general than the linear dependence above. Our starting point will be de Bondt's counterexample, more precisely, a variant of it that we are going to give in dimension 9.

2. Counterexamples to linear dependence. First let us remind de Bondt's 10-dimensional counterexample

$$f(x_1, x_2, \dots, x_{10}) = (x_1, x_2, \dots, x_{10}) - g(x_1, x_2, \dots, x_{10}), \tag{3}$$

$$g(x_1, x_2, \dots, x_{10}) = \begin{pmatrix} x_1x_9x_{10} - x_2x_{10}^2 \\ x_1x_9^2 - x_2x_9x_{10} \\ x_3x_9x_{10} - x_4x_{10}^2 \\ x_3x_9^2 - x_4x_9x_{10} \\ x_5x_9x_{10} - x_6x_{10}^2 \\ x_5x_9^2 - x_6x_9x_{10} \\ (x_1x_4 - x_2x_3)x_9 \\ (x_3x_6 - x_4x_5)x_9 \\ (x_1x_4 - x_2x_3)x_8 - (x_3x_6 - x_4x_5)x_7 \\ x_9^3 \end{pmatrix}, \tag{4}$$

where we wrote the last member as a column vector to have a nicer formula. Our 9-dimensional counterexample is obtained by means of slight modifications

$$f(x_1, x_2, \dots, x_9) = (x_1, x_2, \dots, x_9) - g(x_1, x_2, \dots, x_9), \tag{5}$$

$$(6)\qquad g(x_1,x_2,\ldots,x_9)=\begin{pmatrix} x_1x_7x_9+x_2x_9^2\\ -x_1x_7^2-x_2x_7x_9\\ x_3x_7x_9+x_4x_9^2\\ -x_3x_7^2-x_4x_7x_9\\ x_5x_7x_9+x_6x_9^2\\ -x_5x_7^2-x_6x_7x_9\\ (x_1x_4-x_2x_3)x_9\\ (x_3x_6-x_4x_5)x_9\\ (x_1x_4-x_2x_3)x_8-(x_3x_6-x_4x_5)x_7 \end{pmatrix}.$$

We may easily check that $\det f'(x)=1$ for all $x\in\mathbb{C}^9$. To show that the components of the function g in (6) are linearly independent we consider the polynomial curve $t\mapsto g(t,t^3,t^2,t^5,t^4,t^9,1,t^{20},t^{10})$ and delete all terms but the lower order ones in each component. In this way we get $(t^{11},-t,t^{12},-t^2,t^{14},-t^4,$ $-t^{15},-t^{19},t^9)$ whose powers are all different. This fact implies the linear independence.

3. Homogeneous polynomial invariants. Let us see how the function $f:\mathbb{C}^9\to\mathbb{C}^9$ in (5) and (6) gives a polynomial inverse. First, we consider the last 3 components in the change of variables $y=f(x)$

$$(7)\qquad \begin{cases} y_7=x_7-(x_1x_4-x_2x_3)x_9\\ y_8=x_8-(x_3x_6-x_4x_5)x_9\\ y_9=x_9-(x_1x_4-x_2x_3)x_8+(x_3x_6-x_4x_5)x_7, \end{cases}$$

$$(8)\qquad \begin{pmatrix}y_7\\y_8\\y_9\end{pmatrix}=\begin{pmatrix}x_7\\x_8\\x_9\end{pmatrix}-\underbrace{\begin{pmatrix}0&0&(x_1x_4-x_2x_3)\\0&0&(x_3x_6-x_4x_5)\\-(x_3x_6-x_4x_5)&(x_1x_4-x_2x_3)&0\end{pmatrix}}_{=:M(x)}\begin{pmatrix}x_7\\x_8\\x_9\end{pmatrix}.$$

The matrix $M(x)$ we just defined is nilpotent, namely $M(x)^3=0$ for all $x\in\mathbb{C}^9$. Moreover, we can check that it satisfies the *invariance* property $M(f(x))=M(x)$ for all $x\in\mathbb{C}^9$. Thus the inverse change of variables is

$$(9)\qquad \begin{pmatrix}x_7\\x_8\\x_9\end{pmatrix}=\begin{pmatrix}y_7\\y_8\\y_9\end{pmatrix}+M(y)\begin{pmatrix}y_7\\y_8\\y_9\end{pmatrix}+M(y)^2\begin{pmatrix}y_7\\y_8\\y_9\end{pmatrix}.$$

Now, we consider the first two components

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \underbrace{\begin{pmatrix} x_7x_9 & x_9^2 \\ -x_7^2 & -x_7x_9 \end{pmatrix}}_{=:N(x_7,x_9)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} . \tag{10}$$

Also $N(x_7, x_9)$ is nilpotent for all $x_7, x_9 \in \mathbb{C}$, so

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + N(x_7, x_9) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{11}$$

and we get the inversion of the first two variables by replacing x_7, x_9 in (11) with the right hand sides of the corresponding components in formula (9). Exactly the same arguments can be used to get the inversion formulas for the second pair (y_3, y_4) and the third (y_5, y_6).

We have just seen that the invertibility of the cubic-homogeneous function f in (5) and (6) is related to the invariance of M, whose non zero components are homogeneous quadratic polynomial functions. This crucial fact suggests the following general definition

Definition 1. Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a cubic-homogeneous polynomial function (1). We say that the homogeneous polynomial function $k\colon \mathbb{C}^n \to \mathbb{C}$ is an invariant for f if $k \circ f = k$.

When the degree of the polynomial function k is 1 we have a linear function, and the existence of a nonzero linear invariant for a cubic-homogeneous function $f(x) = x - g(x)$ is equivalent to the linear dependence of the components of g.

Proposition 1. *There do not exist nonzero homogeneous polynomial invariants of degree 1 for $f(x) = x - g(x)$ in* (6). *The functions*

$$x \mapsto x_1x_4 - x_2x_3 \, , \qquad x \mapsto x_3x_6 - x_4x_5 \, , \qquad x \mapsto x_1x_6 - x_2x_5 \tag{12}$$

are homogeneous polynomial invariants of degree 2 and

$$x \mapsto -(x_1x_4 - x_2x_3)x_8 + (x_3x_6 - x_4x_5)x_7 \tag{13}$$

is an invariant of degree 3.

The first part of Proposition 1 was proved in Section 2. The other statements are easily checked. Finally

Problems 1. Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a cubic-homogeneous polynomial function (1). Can f have homogeneous polynomial invariants of degree 3 without lower degree homogeneous polynomial invariants? Can f fail to have any nonzero homogeneous polynomial invariant of degree $d \leq 3$?

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Received January 14, 2008

Dipartimento di Informatica
Università di Verona
Strada Le Grazie, 15
I-37134 Verona, Italy
e-mail: gaetano.zampieri@univr.it