UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLV
2007

REMARKS ON THE CAYLEY–VAN DER WAERDEN–CHOW FORM

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Abstract. It is known that a variety in projective space is uniquely determined by its Cayley–van der Waerden–Chow form. An algebraic formulation and a proof (for an arbitrary base field) of this classical result are given in view of applications to the Stückrad–Vogel intersection cycle.

1. Introduction. It is well-known that, given a k-dimensional projective variety $X \subset \mathbb{P}^n_K$, all $(n-k-1)$-dimensional projective subspaces meeting X form a hypersurface in the Grassmannian of $(n-k-1)$-dimensional projective subspaces in $\mathbb{P}^n$ from which X can be recovered. The homogeneous form in the Plücker coordinates defining this hypersurface is known as the *(Cayley–van der Waerden–) Chow form* of X. It was introduced by Cayley [**2, 3**] and later generalized by Chow and van der Waerden [**4**]. Since then there has appeared a vast literature on the subject (see e.g. [**5, 7**], and the references given there).

In this note we present some results on generic hyperplane sections of affine or projective varieties and an algebraic formulation and proof of the classical result that a variety $X \subset \mathbb{P}^n_K$, where K is an *arbitrary* field, is determined by its Cayley–van der Waerden–Chow form. These algebraic results, for which we could not find any suitable reference, are very useful in order to study movable components of the intersection cycle (and its intersection numbers) in the algebraic approach to intersection theory of Stückrad and Vogel [**8, 6**] (see our forthcoming paper [**1**]).

1991 *Mathematics Subject Classification.* P14C05, 13B99, 14N05.

Key words and phrases. Cayley–van der Waerden–Chow form, generic residual intersections.

The support by the "Zentrum für Höhere Studien" at the University of Leipzig and by the Italian Ministry of University and Research (MIUR) is gratefully acknowledged. The first author is a member of GNSAGA of INdAM.

In Section 2 we will establish a 1-1 correspondence between components of the closure of the difference of two varieties and the components of a generic residual intersection (see Proposition 2.5). This result is important in the study of the Stückrad–Vogel intersection algorithm in order to control components outside the intersection (see [**1**]).

In Section 3, using Lemma 2.3, we shall prove the result on the Cayley–van der Waerden–Chow form (Proposition 3.3) and give an example that in positive characteristic the degree of the Chow form can be smaller than the degree of the variety.

2. Some algebraic preliminaries. Let R be a commutative noetherian ring with identity element. Further let $x_1, \ldots, x_r \in R$ and let $u_1, \ldots, u_r$ be indeterminates over R, where $r \in \mathbb{N}^+$. If M is an R-module, N a submodule of M and J an ideal in R, let

$$N :_M \langle J \rangle := \{m \in M \mid J^t \cdot m \subseteq N \text{ for some } t \in \mathbb{N}\}\,.$$

We will assume that not all of the elements $x_1, \ldots, x_r$ are nilpotent and set

$$S := R[u_1, \ldots, u_r], \quad I := (x_1, \ldots, x_r)R,$$
$$F := \sum_{i=1}^{r} x_i u_i \in S \text{ and } R' := S/FS :_S \langle IS \rangle\,.$$

We note that R' is an S-algebra which is not the zero ring because of our assumption.

In this section we will establish a 1-1 correspondence between the associated prime ideals of R and R' and we will prove that the corresponding primary ideals have the same length. We note that the geometric background of these investigations is the study of generic hyperplane sections of affine or projective varieties.

For this we will use a more general approach. First of all it is easy to see that the definition of R' can be extended to R-modules: For an R-module M let

$$M' := (M \otimes_R S)/F(M \otimes_R S) :_{M \otimes_R S} \langle IS \rangle.$$

M' is an S-module which may be considered as an R'- and an R-module as well.

If $f\colon M \to N$ is an R-linear map (M, N R-modules), then $f \otimes_R S\colon M \otimes_R S \to N \otimes_R S$ induces a homomorphism $f'\colon M' \to N'$, which is S-, R'- and R-linear.

It is clear that this defines an additive covariant functor from the category of R-modules to the category of S-modules (R'-, R-modules, resp.). We will denote it by $\mathfrak{G}_R(x_1, \ldots, x_r)$ and write $\mathfrak{G}$ if no confusion is possible. (By definition $\mathfrak{G}(M) = M'$ and $\mathfrak{G}(f) = f'$ for all R-modules M and all R-linear maps *f*.) $\mathfrak{G}$ commutes with localizations, i.e., if $T \subset R$ is multiplicatively closed then $(M')_T = (M_T)'$ for all R-modules M if we consider M' as an R- or an S-module.

Furthermore, let $\varphi_M \colon M \to M'$ be the composition of the embedding

$$M \subseteq M[u_1, \ldots\ldots, u_r] = M \otimes_R S$$

and the canonical epimorphism $M \otimes_R S \to M'$. It is clear that $\ker \varphi_M = 0 :_M \langle I \rangle$. By $\varphi_R^* \colon \operatorname{Spec} R' \to \operatorname{Spec} R$, we denote the morphism induced by φ_R.

Finally, we note that the map

$$V(FS) \setminus V(IS) \to \operatorname{Spec} R'$$

given by $\mathfrak{P} \mapsto \mathfrak{P}/FS :_S \langle IS \rangle$ for all $\mathfrak{P} \in V(FS) \setminus V(IS) \subseteq \operatorname{Spec} S$, is injective.

Lemma 2.1. *With the preceding notation, it holds*

(a) $\mathfrak{G}_R(x_1, \ldots, x_r)$ *respects monomorphisms and epimorphisms.*

(b) For all R-modules M, N and all R-linear maps $f \colon M \to N$ there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi_M} & M' \\ \Big\downarrow f & & \Big\downarrow f' \\ N & \xrightarrow{\varphi_N} & N', \end{array}$$

Moreover, f' is a monomorphism (isomorphism) if $\operatorname{Supp}(\ker f) \subseteq V(I)$ *(and f is an epimorphism).*

Remark 2.2. The commutative diagram in (b) says that the φ_M, M an R-module, provide a natural transformation of the identity functor of the category of R-modules into $\mathfrak{G}(x_1, \ldots, x_r)$ considered as a functor from the category of R-modules to itself.

Before embarking on the proof of Lemma 2.1 we introduce the following notion: A prime ideal $\mathfrak{P} \in \operatorname{Spec} S$ is called *R-rational* if there is a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{P} = \mathfrak{p} \cdot S$. In this situation we have $\mathfrak{p} = \mathfrak{P} \cap R$ and $\operatorname{ht}_S(\mathfrak{P}) = \operatorname{ht}_R(\mathfrak{p})$, where $\operatorname{ht}(I)$ denotes the height of an ideal I. If M is an R-module, then all prime ideals of $\operatorname{Ass}_S(M \otimes_R S)$ are R-rational, for $R \subset S$ is a special case of a flat extension of rings.

Proof. We set

$$\tilde{X} := X \otimes_R S / F(X \otimes_R S),$$

X an R-module. It is enough to show the following: For any exact sequence

$$(1) \qquad 0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

of R-modules f' is a monomorphism and g' an epimorphism. If, moreover, $\operatorname{Supp} M \subseteq V(I)$ then g' is an isomorphism. Tensoring (1) with S/FS we obtain an exact sequence

$$0 \to C \to \tilde{M} \to \tilde{N} \to \tilde{P} \to 0 \tag{2}$$

with a suitably defined S-module C. Since $C = \operatorname{coker}(0 :_{N \otimes_R S} F \to 0 :_{P \otimes_R S} F)$ we have $\operatorname{Supp}_S(C) \subseteq \operatorname{Supp}_S(0 :_{P \otimes_R S} F)$.

On the other hand, $\operatorname{Ass}_S(P \otimes_R S)$ consists only of R-rational primes. Therefore, $0 :_{P \otimes_R S} F \subseteq 0 :_{P \otimes_R S} \langle IS \rangle$, i.e.,

$$\operatorname{Supp}_S(C) \subseteq \operatorname{Supp}_S(0 :_{P \otimes_R S} F) \subseteq V(IS).$$

Hence $H^0_{IS}(C) = C$, $H^i_{IS}(C) = 0$ for all $i > 0$ and from (2) we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0_{IS}(C) \arrow[r] \arrow[d, "\Vert"'] & H^0_{IS}(\tilde{M}) \arrow[r] \arrow[d, "\rho", "\cap\vert"'] & H^0_{IS}(\tilde{N}) \arrow[r, "\tilde{g}"] \arrow[d, "\sigma", "\cap\vert"'] & H^0_{IS}(\tilde{P}) \arrow[d, "\tau", "\cap\vert"'] & \\
0 \arrow[r] & C \arrow[r] & \tilde{M} \arrow[r] & \tilde{N} \arrow[r] & \tilde{P} \arrow[r] & 0
\end{tikzcd}$$

Therefore, passing to cokernels, we get a monomorphism

$$M' = \operatorname{coker} \rho \xrightarrow{f'} \operatorname{coker} \sigma = N'$$

and an epimorphism

$$N' = \operatorname{coker} \sigma \xrightarrow{g'} \operatorname{coker} \tau = P'$$

as claimed. If $\operatorname{Supp} M \subseteq V(I)$ then $\tilde{g}$ is an epimorphism, $\rho = \mathrm{id}$ and therefore g' is an isomorphism. $\square$

We note that the following result can be obtained by analyzing the proofs in [**9**]. For the convenience of the reader we will give here an independent proof. We begin with two lemmata.

LEMMA 2.3. *Let M be an R-module. Then we have for $\mathfrak{P} \in \operatorname{Ass}_S M'$ and $\mathfrak{p} := \mathfrak{P} \cap R$:*

(a) $\mathfrak{p} \in \operatorname{Ass}_R(M) \setminus V(I)$,
(b) $\mathfrak{P} = (\mathfrak{p}S + FS) :_S \langle IS \rangle$.

PROOF. Since

$$\operatorname{Ass}_S M' = \operatorname{Ass}_S (M \otimes_R S)/F(M \otimes_R S) \setminus V(IS)$$

we have $\mathfrak{P} \in V(FS) \setminus V(IS)$ and therefore $I \not\subseteq \mathfrak{p}$, i. e., $\mathfrak{p} \notin V(I)$. Without loss of generality we assume $x_1 \notin \mathfrak{p}$.

Considering M' as an R-module, we have

$$\begin{aligned}(M')_{\mathfrak{p}} &= (M_{\mathfrak{p}})'\\ &= M_{\mathfrak{p}}[u_1,\dots,u_r]/FM_{\mathfrak{p}}[u_1,\dots,u_r] \quad (\text{since } IS_{\mathfrak{p}} = S_{\mathfrak{p}})\\ &\cong M_{\mathfrak{p}}[u_2,\dots,u_r],\end{aligned}$$

where the last isomorphism is induced by the inclusion

$$M_{\mathfrak{p}}[u_2,\dots,u_r] \subset M_{\mathfrak{p}}[u_1,\dots,u_r]$$

(note that the image of x_1 in $R_{\mathfrak{p}}$ is a unit). Because $R\backslash\mathfrak{p} \subseteq S\backslash\mathfrak{P}$ and $I \not\subseteq \mathfrak{p}$ this isomorphism gives rise to the following isomorphisms (note that $IS_{\mathfrak{P}} = S_{\mathfrak{P}}$)

$$(M')_{\mathfrak{P}} = ((M')_{\mathfrak{p}})_{\mathfrak{P}S_{\mathfrak{p}}} \cong M_{\mathfrak{p}}[u_2,\dots,u_r]_{\mathfrak{P}'}\,,$$

where $\mathfrak{P}'$ denotes the image of $\mathfrak{P}S_{\mathfrak{p}}$ in $R_{\mathfrak{p}}[u_2,\dots,u_r]$ under the map given by the composition of the canonical epimorphism $S_{\mathfrak{p}} \to S_{\mathfrak{p}}/FS_{\mathfrak{p}} = (R')_{\mathfrak{p}}$ and the isomorphism $S_{\mathfrak{p}}/FS_{\mathfrak{p}} \cong R_{\mathfrak{p}}[u_2,\dots,u_r]$. Therefore, we have

$$\mathfrak{P}S_{\mathfrak{p}}/FS_{\mathfrak{p}} = (\mathfrak{P}'S_{\mathfrak{p}} + FS_{\mathfrak{p}})/FS_{\mathfrak{p}},$$

i.e., $\mathfrak{P}S_{\mathfrak{p}} = \mathfrak{P}'S_{\mathfrak{p}} + FS_{\mathfrak{p}}$. Since $\mathfrak{P} \in \operatorname{Ass}_S M'$ we have $\mathfrak{P}S_{\mathfrak{P}} \in \operatorname{Ass}_{S_{\mathfrak{P}}}(M')_{\mathfrak{P}}$ and consequently

$$\mathfrak{P}'R_{\mathfrak{p}}[u_2,\dots,u_r]_{\mathfrak{P}'} \in \operatorname{Ass}_{R_{\mathfrak{p}}[u_2,\dots,u_r]_{\mathfrak{P}'}} M_{\mathfrak{p}}[u_2,\dots,u_r]_{\mathfrak{P}'}.$$

Therefore,

$$\mathfrak{P}' \in \operatorname{Ass}_{R_{\mathfrak{p}}[u_2,\dots,u_r]} M_{\mathfrak{p}}[u_2,\dots,u_r] = \{\mathfrak{q}R_{\mathfrak{p}}[u_2,\dots,u_r] \mid \mathfrak{q} \in \operatorname{Ass} M_{\mathfrak{p}}\},$$

i.e., there is some $\mathfrak{q} \in \operatorname{Ass} M_{\mathfrak{p}}$ with $\mathfrak{P}' = \mathfrak{q}R_{\mathfrak{p}}[u_2,\dots,u_r]$. For this prime ideal $\mathfrak{q}$ we have

$$\mathfrak{q} = \mathfrak{P}' \cap R_{\mathfrak{p}} = (\mathfrak{P}'S_{\mathfrak{p}} + FS_{\mathfrak{p}}) \cap R_{\mathfrak{p}} = \mathfrak{P}S_{\mathfrak{p}} \cap R_{\mathfrak{p}} = (\mathfrak{P} \cap R)_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}\,.$$

Therefore, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ and thus $\mathfrak{p} \in \operatorname{Ass}_R M$, which shows (a).

For proving (b) we note that $\mathfrak{q} = \mathfrak{p}R_{\mathfrak{p}}$ implies $\mathfrak{P}' = \mathfrak{p}R_{\mathfrak{p}}[u_2,\dots,u_r]$ and therefore we get with $\mathfrak{p}^* := (\mathfrak{p}S + FS) :_S \langle IS \rangle$ (note that $I \not\subseteq \mathfrak{p}$):

$$\mathfrak{P}S_{\mathfrak{p}} = \mathfrak{P}'S_{\mathfrak{p}} + FS_{\mathfrak{p}} = \mathfrak{p}S_{\mathfrak{p}} + FS_{\mathfrak{p}} = \mathfrak{p}^*S_{\mathfrak{p}}\,.$$

Let $\tilde{\mathfrak{P}} \in \operatorname{Ass}_S S/\mathfrak{p}^*$. Since $S/\mathfrak{p}^* \cong (R/\mathfrak{p})'$, (a) implies $(\tilde{\mathfrak{P}}/\mathfrak{p}S) \cap (R/\mathfrak{p}) = 0$, i.e., $\tilde{\mathfrak{P}} \cap R = \mathfrak{p}$ and therefore we have $R \setminus \mathfrak{p} \subseteq S \setminus \tilde{\mathfrak{P}}$. Now $\mathfrak{P}S_{\mathfrak{p}} = \mathfrak{p}^*S_{\mathfrak{p}}$ implies $\mathfrak{P} = \mathfrak{p}^*$. □

Lemma 2.4. *For $\mathfrak{p} \in \operatorname{Spec} R$ we set $\mathfrak{p}^* := (\mathfrak{p}S + FS) :_S \langle IS \rangle$. Then the following conditions are equivalent:*

(i) $\mathfrak{p}^* \in \operatorname{Spec} S$,

(ii) $\mathfrak{p} \notin V(I)$,

(iii) $\mathfrak{p}^* \cap R = \mathfrak{p}$.

In this case $\mathfrak{p}^* \in V(FS) \setminus V(IS)$ *and* $\operatorname{ht}_S(\mathfrak{p}^*) = 1 + \operatorname{ht}_R(\mathfrak{p})$.

Proof. (ii) $\Rightarrow$ (i), (iii): Passing from R to $R/\mathfrak{p}$, we can assume without loss of generality that R is an integral domain and that $\mathfrak{p} = 0$. Then $S/\mathfrak{p}^* = R'$. Since

$$(FS :_S \langle I \rangle) \cap R = \ker \varphi_R = 0 :_R \langle I \rangle = 0,$$

it is sufficient to show that $FS :_S \langle I \rangle$ is a prime ideal in S.

Let $\mathfrak{P} \in \operatorname{Ass}_S R'$. Then $\mathfrak{P} \cap R \in \operatorname{Ass} R = \{0\}$ by Lemma 2.3(a), i.e. $\mathfrak{P} \cap R = 0$. By Lemma 2.3(b), we therefore obtain $\mathfrak{P} = FS :_S \langle I \rangle$, i.e. $FS :_S \langle I \rangle$ is a prime ideal.

(i), (iii) $\Rightarrow$ (ii): Assume that $\mathfrak{p} \in V(I)$. Then $IS \subseteq \mathfrak{p}S$ and therefore $\mathfrak{p}^* = S \notin \operatorname{Spec} S$ and $\mathfrak{p}^* \cap R = S \cap R = R \neq \mathfrak{p}$.

It is clear that $\mathfrak{p}^* \in V(FS) \setminus V(IS)$ in this case. Then

$$\mathfrak{p}^* S_{\mathfrak{p}^*} = (\mathfrak{p}S_{\mathfrak{p}^*} + FS_{\mathfrak{p}^*}) :_{S_{\mathfrak{p}^*}} \langle IS_{\mathfrak{p}^*} \rangle = \mathfrak{p}S_{\mathfrak{p}^*} + FS_{\mathfrak{p}^*}.$$

Since $F \notin \mathfrak{p}S_{\mathfrak{p}^*}$,

$$\operatorname{ht}_S(\mathfrak{p}^*) = \operatorname{ht}_{S_{\mathfrak{p}^*}}(\mathfrak{p}^* S_{\mathfrak{p}^*}) = 1 + \operatorname{ht}_{S_{\mathfrak{p}^*}}(\mathfrak{p}S_{\mathfrak{p}^*}) = 1 + \operatorname{ht}_R(\mathfrak{p})$$

by Krull's Hauptidealsatz. $\square$

Now we can prove:

Proposition 2.5. *For any R-module M the map $\varphi_R^*\colon \operatorname{Spec} R' \to \operatorname{Spec} R$ induces a bijection $\operatorname{Ass}_{R'} M' \to \operatorname{Ass}_R(M) \setminus V(I)$.*

Proof. By our above remarks, it is sufficient to show that the restriction map $\operatorname{Spec} S \to \operatorname{Spec} R$ induces a bijection $\operatorname{Ass}_S(M') \to \operatorname{Ass}_R(M) \setminus V(I)$. For $\mathfrak{p} \in \operatorname{Spec} R \setminus V(I)$, we set again $\mathfrak{p}^* := (\mathfrak{p}S + FS) :_S \langle IS \rangle$. Then $\mathfrak{p}^* \in V(FS) \setminus V(IS) \subseteq \operatorname{Spec} S$ and $\mathfrak{p}^* \cap R = \mathfrak{p}$ by Lemma 2.4.

By Lemma 2.3(a), the restriction map $\operatorname{Spec} S \to \operatorname{Spec} R$ induces a map

$$\psi_M\colon \operatorname{Ass}_S(M') \to \operatorname{Ass}_R(M) \setminus V(I).$$

By Lemma 2.3(b), we have $\mathfrak{P} = \psi_M(\mathfrak{P})^*$ for all $\mathfrak{P} \in \operatorname{Ass}_S M'$ and therefore ψ_M is injective.

Let now $\mathfrak{p} \in \operatorname{Ass}_R(M) \setminus V(I)$, i.e., there is a monomorphism $R/\mathfrak{p} \to M$. By Lemma 2.1(a), this monomorphism induces a monomorphism $S/\mathfrak{p}^* \cong (R/\mathfrak{p})' \to M'$, i.e., $\mathfrak{p}^* \in \operatorname{Ass}_S M'$. Since $\psi_M(\mathfrak{p}^*) = \mathfrak{p}$, ψ_M is surjective and hence bijective. $\square$

Corollary 2.6. *Let M be a finitely generated R-module and let $\mathfrak{p} \in \min \operatorname{Ass}_R M$ with $I \not\subseteq \mathfrak{p}$. If $\mathfrak{p}' \in \operatorname{Ass}_{R'} M'$ is the uniquely determined prime ideal such that $\mathfrak{p} = \varphi_R^*(\mathfrak{p}')$ then $\mathfrak{p}' \in \min \operatorname{Ass}_{R'} M'$ and* $\operatorname{length}(R_{\mathfrak{p}}) = \operatorname{length}(R'_{\mathfrak{p}'})$.

Proof. Passing from R to $R/\operatorname{Ann}_R M$, we may assume without loss of generality that $\mathfrak{p}$ is minimal in $\operatorname{Spec} R$. Since $I \not\subseteq \mathfrak{p}$, we get $IS \not\subseteq \mathfrak{p}^*$ and $F \notin \mathfrak{p}S$. Using Lemma 2.4, we obtain

$$\operatorname{Ass} S_{\mathfrak{p}^*} = \{\mathfrak{q}S_{\mathfrak{p}^*} | \mathfrak{q} \in \operatorname{Ass} R, \mathfrak{q} \subseteq \mathfrak{p}^*\} = \{\mathfrak{q}S_{\mathfrak{p}^*} | \mathfrak{q} \in \operatorname{Ass} R, \mathfrak{q} \subseteq \mathfrak{p}^* \cap R\} = \{\mathfrak{p}S_{\mathfrak{p}^*}\}$$

by the minimality of $\mathfrak{p}$ in $\operatorname{Spec} R$. Since $\dim S_{\mathfrak{p}^*} = \operatorname{ht}_S(\mathfrak{p}^*) = 1 + \operatorname{ht}_R(\mathfrak{p}) = 1$ by Lemma 2.4, F is a parameter element in $S_{\mathfrak{p}^*}$ which is a nonzerodivisor. Therefore,

$$\operatorname{length} S_{\mathfrak{p}^*}/FS_{\mathfrak{p}^*} = e(FS_{\mathfrak{p}^*}, S_{\mathfrak{p}^*}) < \infty\,.$$

Now

$$R'_{\mathfrak{p}'} \cong R'_{\mathfrak{p}^*} = S_{\mathfrak{p}^*}/(FS_{\mathfrak{p}^*} :_{S_{\mathfrak{p}^*}} \langle IS_{\mathfrak{p}^*}\rangle) = S_{\mathfrak{p}^*}/FS_{\mathfrak{p}^*}\,,$$

i.e., we obtain $\operatorname{length} R'_{\mathfrak{p}'} = e(FS_{\mathfrak{p}^*}, S_{\mathfrak{p}^*}) < \infty$, in particular, $\mathfrak{p}'$ is minimal in $\operatorname{Ass} R'$. Further, by the addition and reduction theorem for multiplicities, we get

$$e(FS_{\mathfrak{p}*}, S_{\mathfrak{p}^*}) = \operatorname{length}(S_{\mathfrak{p}S}) \cdot e(FS_{\mathfrak{p}*}, S_{\mathfrak{p}^*}/\mathfrak{p}S_{\mathfrak{p}^*})\,,$$

for $\mathfrak{p}S_{\mathfrak{p}^*}$ is the unique minimal prime ideal in $S_{\mathfrak{p}^*}$. Since $R \subset S$ is flat, $R_{\mathfrak{p}} \subset S_{\mathfrak{p}S}$ is flat too with fibre $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$. Therefore, $\operatorname{length}(S_{\mathfrak{p}S}) = \operatorname{length}(R_{\mathfrak{p}})$. Furthermore, since $F \notin \mathfrak{p}$, we get

$$e(FS_{\mathfrak{p}*}, S_{\mathfrak{p}^*}/\mathfrak{p}S_{\mathfrak{p}^*}) = \operatorname{length}(S_{\mathfrak{p}^*}/\mathfrak{p}S_{\mathfrak{p}^*} + FS_{\mathfrak{p}^*}) = \operatorname{length}(S_{\mathfrak{p}^*}/\mathfrak{p}^*S_{\mathfrak{p}^*}) = 1,$$

as required. □

Corollary 2.7. *Assume R is reduced (an integral domain). Then R' is again reduced (an integral domain) with the same number of associated primes as R.*

3. The Cayley–van der Waerden–Chow form. Let $u_0, \dots, u_n$, $n \geq 1$, be indeterminates of degree one. We will use the following notation: Let R be a ring. For a polynomial $f \in R[u_0, \dots, u_n]$ we denote by $\operatorname{supp} f$ the (finite) set of all monomials in $u_0, \dots, u_n$ occurring in f with non-zero coefficient.

Lemma 3.1. *Let $L|K$ be an algebraic field extension. For $\xi := (\xi_1, \dots, \xi_n) \in L^n$, set*

$$G_\xi := u_1\xi_1 + \dots + u_n\xi_n \in L[u_1, \dots, u_n] \quad \text{and}$$
$$F_\xi := u_0 + G_\xi \in L[u_0, u_1, \dots, u_n].$$

Then:

(a) There is an irreducible homogeneous polynomial $f_\xi \in K[u_0, u_1, \dots, u_n]$ of positive degree with the following properties:

(i) $f_\xi \cdot K[u_0, u_1, \dots, u_n] = F_\xi L[u_0, u_1, \dots, u_n] \cap K[u_0, u_1, \dots, u_n]$.

(ii) f_ξ is integral with respect to u_0, that is, $u_0^{\deg f_\xi} \in \operatorname{supp} f_\xi$.

(iii) $g_\xi := f_\xi(X, u_1, \ldots u_n) \in K(u_1, \ldots, u_n)[X]$ *(X an indeterminate) is the minimal polynomial of* $-G_\xi \in L(u_1, \ldots, u_n)$ *over* $K(u_1, \ldots, u_n)$.

(iv) If all but possibly one of the elements $\xi_1, \ldots, \xi_n$ *are separable over* K*, then* $\deg f_\xi = [K[\xi_1, \ldots, \xi_n] : K]$*. Otherwise, there exists a non-negative integer* ε *with* $p^\varepsilon \deg f_\xi = [K[\xi_1, \ldots, \xi_n] : K]$*, where* $p := \operatorname{char} K > 0$.

(b) If $\zeta := (\zeta_1, \ldots, \zeta_n) \in L^n$*, then* $f_\xi = f_\zeta$ *(up to a non zero constant factor) if and only if there exists a* K*-isomorphism* $\varphi\colon K[\xi_1, \ldots, \xi_n] \to K[\zeta_1, \ldots, \zeta_n]$ *with* $\varphi(\xi_i) = \zeta_i$*,* $i = 1, \ldots, n$.

Proof. For any field Z with $K \subseteq Z \subseteq L$ we set

$$S_Z := Z[u_0, \ldots, u_n], \quad Q_Z := Z(u_0, \ldots, u_n) = Q(S_Z) \text{ and}$$
$$S'_Z := Z[u_1, \ldots, u_n], \quad Q'_Z := Z(u_1, \ldots, u_n) = Q(S'_Z).$$

Since F_ξ is homogeneous of degree one, it is obviously irreducible in S_L and in $Q'_L[u_0]$. Therefore, $F_\xi S_L$ is a homogeneous prime ideal in S_L (note that S_L is factorial) and $P_\xi := F_\xi S_L \cap S_K$ is a homogeneous prime ideal in S_K.

(a) Since

$$P_\xi = F_\xi S_L \cap S_K = F_\xi S_L \cap S_{K[\xi_1,\ldots,\xi_n]} \cap S_K = F_\xi S_{K[\xi_1,\ldots,\xi_n]} \cap S_K\,,$$

we may assume without loss of generality that $L = K[\xi_1, \ldots, \xi_n]$ and hence that $L|K$ is finite. Furthermore, $P_\xi \cap S'_K \subseteq F_\xi S_L \cap S'_L = 0$ so that we have a chain of monomorphisms and isomorphisms, respectively,

$$S'_K \to S_K/P_\xi \to S_L/F_\xi S_L \cong S'_L\,,$$

where the composition map $S'_K \to S'_L$ is given by the inclusion $K \subseteq L$ and by $u_i \mapsto u_i$, $i = 1, \ldots, n$. Since $L|K$ is finite, $S'_L (\cong S'_K \otimes_K L)$ is a finitely generated graded S'_K-module and hence S_K/P_ξ is a finitely generated graded S'_K-module as well. Therefore, there exists an irreducible homogeneous polynomial f_ξ in P_ξ which is integral with respect to u_0. It is clear that $\deg f_\xi = \deg_{u_0} f_\xi > 0$. Moreover,

$$\operatorname{ht}(P_\xi) = \dim S_K - \dim S_K/P_\xi = n + 1 - \dim S'_K = n + 1 - n = 1,$$

so that actually $P_\xi = f_\xi \cdot S_K$, and (i) and (ii) have been proved.

Since f_ξ is irreducible in $S_K = S'_K[u_0]$, $f_\xi(X, u_1, \ldots, u_n)$ is irreducible in $Q'_K[X]$. Let $g := f_\xi(-G_\xi, u_1, \ldots, u_n) \in S'_L$. Since $f_\xi = f_\xi(F_\xi - G_\xi, u_1, \ldots, u_n)$, we have $g - f_\xi \in F_\xi S_L$ and therefore

$$g = g - f_\xi + f_\xi \in F_\xi S_L \cap S'_L = 0.$$

Thus (iii) has been proved.

If all but possibly one of the elements $\xi_1, \dots, \xi_n$ are separable over K, then all but possibly one of them, viewed as elements of $Q_L = Q_K[\xi_1, \dots, \xi_n]$, are separable over Q_K with the same minimal polynomials as over K. Therefore, G_ξ is a primitive element of the finite field extension $Q_K[\xi_1, \dots, \xi_n]|Q_K$ and, taking into account (a)(ii) and (iii) and using $Q_K[\xi_1, \dots, \xi_n] \cong Q_K \otimes_K K[\xi_1, \dots, \xi_n]$, one has

$$\begin{aligned}\deg f_\xi &= \deg_X f_\xi(X, u_1, \dots, u_n) = [Q_K[-G_\xi] : Q_K] = \\ &= [Q_K[\xi_1, \dots, \xi_n] : Q_K] = [K[\xi_1, \dots, \xi_n] : K]\,.\end{aligned}$$

Assume that at least two of the elements $\xi_1, \dots, \xi_n$ are not separable over K. Then, in particular, K is not perfect. Since $\xi_1^{p^e}, \dots, \xi_n^{p^e}$ are separable over K for $e \gg 1$, $G_\xi^{p^e}$ is a primitive element of $Q_K[\xi_1^{p^e}, \dots, \xi_n^{p^e}]|Q_K$. Therefore,

$$\xi_i^{p^e} \in Q_K[G_\xi^{p^e}] \subseteq Q_K[G_\xi] \text{ for all } i = 1, \dots, n.$$

As above,

$$\begin{aligned}\deg f_\xi &= [Q_K[-G_\xi] : Q_K] = [Q_K[G_\xi] : Q_K] = \\ &= \frac{[Q_K[\xi_1, \dots, \xi_n] : Q_K]}{[Q_K[\xi_1, \dots, \xi_n] : Q_K[G_\xi]]} = \frac{[K[\xi_1, \dots, \xi_n] : K]}{[Q_K[\xi_1, \dots, \xi_n] : Q_K[G_\xi]]}\,.\end{aligned}$$

Since $Q_K[\xi_1^{p^e}, \dots, \xi_n^{p^e}] \subseteq Q_K[G_\xi]$, the degree $[Q_K[\xi_1, \dots, \xi_n] : Q_K[G_\xi]]$ divides

$$[Q_K[\xi_1, \dots, \xi_n] : Q_K[\xi_1^{p^e}, \dots, \xi_n^{p^e}]],$$

but the latter is a power of p. Therefore, $[Q_K[\xi_1, \dots, \xi_n] : Q_K[G_\xi]] = p^\varepsilon$ for some $\varepsilon \in \mathbb{N}$, which shows (iv).

(b) If $f_\xi = f_\zeta =: f$, then $-G_\xi$ and $-G_\zeta$ are roots of $f(X, u_1, \dots, u_n)$. Therefore, there is a Q_K-isomorphism

$$\psi\colon Q_K[G_\xi] \to Q_K[G_\zeta] \text{ with } \psi(G_\xi) = G_\zeta.$$

We choose a positive integer q as follows: If K is perfect, we set $q := 1$. If K is not perfect and $p := \operatorname{char} K$, then $q := p^e$, where $e \in \mathbb{N}$ has been chosen as in the proof of (a)(iv), that is, such that $\xi_1^{p^e}, \dots, \xi_n^{p^e} \in Q_K[G_\xi]$ and $\zeta_1^{p^e}, \dots, \zeta_n^{p^e} \in Q_K[G_\zeta]$. Then

$$\sum_{i=1}^n u_i^q \zeta_i^q = G_\zeta^q = \psi(G_\xi)^q = \psi(G_\xi^q) = \psi \sum_{i=1}^n u_i^q \xi_i^q) = \sum_{i=1}^n u_i^q \psi(\xi_i^q)$$

and we obtain $\zeta_i^q = \psi(\xi_i^q)$, $i = 1, \dots, n$, since $\psi(\xi_i^q) \in Q_K[G_\zeta]$ is even algebraic over K (with the same minimal polynomial as ξ_i^q). Thus the restriction of ψ to $K^q[\xi_1^q, \dots, \xi_n^q]$ is a K^q-isomorphism

$$\psi_q\colon K^q[\xi_1^q, \dots, \xi_n^q] \to K^q[\zeta_1^q, \dots, \zeta_n^q].$$

The Frobenius homomorphism $\varphi_q\colon L \to L$ induces an isomorphism $\tilde{\varphi}_q\colon L \to L^q$. Now we set

$$\varphi := (\tilde{\varphi}_q^{-1}|_{K^q[\zeta_1^q,\ldots,\zeta_n^q]}) \circ \psi_q \circ (\tilde{\varphi}_q|_{K[\xi_1,\ldots,\xi_n]})\,.$$

The homomorphism φ has the desired property.

Conversely, let $\varphi\colon K[\xi_1,\ldots,\xi_n] \to K[\zeta_1,\ldots,\zeta_n]$ be a K-isomorphism with $\varphi(\xi_i) = \zeta_i$ for $i = 1,\ldots,n$. Then the restriction of $\varphi \otimes_K Q_K$ to $Q_K[G_\xi]$ yields a Q_K-isomorphism $\psi\colon Q_K[G_\xi] \to Q_K[G_\zeta]$ such that $\psi(G_\xi) = G_\zeta$. Hence the minimal polynomials of $-G_\xi$ and $-G_\zeta$ over Q_K coincide, that is, $f_\xi = f_\zeta$ by (a)(iii). This finishes the proof of Lemma 3.1. $\square$

LEMMA 3.2. *Let K be a field, and let $X_0,\ldots,X_n$ be further indeterminates of degree one. Set $R := K[X_0,\ldots,X_n]$, $T := K[u_0,\ldots,u_n]$, $S := T[X_0,\ldots,X_n] = R[u_0,\ldots,u_n]$ and $F := u_0X_0 + \cdots + u_nX_n \in S$. Furthermore, let $P \subset R$ be a homogeneous prime ideal with $r := \mathrm{ht}(P) \le n$ and set*

$$P^* := (PS + FS) :_S \langle X_0,\ldots,X_n\rangle.$$

Then:

(a) P^* is a prime ideal in S of height $r+1$ such that $P^* \cap R = P$, and P^* is homogeneous both with respect to $X_0,\ldots,X_n$ and $u_0,\ldots,u_n$.

(b) If $\mathrm{ht}(P) = n$, then there exists an irreducible and (with respect to $u_0,\ldots,u_n$) homogeneous polynomial $f_P \in T$ with the following properties:

(i) $f_PT = P^* \cap T$.

(ii) Let $I := \{i \in \{0,\ldots,n\} \mid X_i \notin P\}$. Then $f_P \in K[\{u_i \mid i \in I\}]$ and, for any $i \in I$, f_P is integral with respect to u_i, that is,

$$u_i^{\deg f_P} \in \mathrm{supp}\, f_P.$$

(iii) $\deg f_P = h_0(R/P)$ if $\mathrm{char}\, K = 0$. If $p := \mathrm{char}\, K > 0$,

$$\deg f_P = p^{-\varepsilon} h_0(R/P) \le h_0(R/P)$$

for a suitable $\varepsilon \in \mathbb{N}$.

(c) If $P' \subset R$ is another homogeneous prime ideal with $\mathrm{ht}(P)' = \mathrm{ht}(P) = n$, then $f_P = f_{P'}$ (up to multiplication by a nonzero constant) if and only if $P = P'$.

PROOF. (a) follows from Lemma 2.4 (it is clear, that P^* is homogeneous with respect to $X_0,\ldots,X_n$ and $u_0,\ldots,u_n$).

(b) Suppose that $X_0 \notin P$. Then $X_0 \notin P^*$ by (a). We set

$$R' := K[X_1,\ldots,X_n],\quad S' := R'[u_0,\ldots,u_n],\quad \text{and}$$
$$F' := u_0 + u_1X_1 + \cdots + u_nX_n \in S'.$$

Furthermore, let $\lambda_R\colon R' \to R_{X_0}$ be the K-homomorphism defined by

$$\lambda_R(X_i) = \frac{X_i}{X_0},\ i = 1, \dots, n.$$

Moreover, we set

$$\lambda_S := \lambda_R \otimes_K T\colon S' \to S_{X_0}.$$

By definition, λ_S is a T-homomorphism. Then $\tilde{P} := \lambda_R^{-1}(PR_{X_0})$ and $\widetilde{P^*} := \lambda_S^{-1}(P^* S_{X_0})$ are prime ideals in R' and S', respectively. More precisely, $\tilde{P} = \{f|_{X_0=1} \mid f \in P\}$ and $\widetilde{P^*} = \{f^*|_{X_0=1} \mid f^* \in P^*\}$ are the dehomogenizations of P and P^*, respectively, with respect to X_0. Therefore, $\mathrm{ht}(\tilde{P}) = \mathrm{ht}(P) = n$ (i.e. $\tilde{P}$ is a maximal ideal of R'), $\mathrm{ht}(\widetilde{P^*}) = \mathrm{ht}(P^*) = n+1$ and $\widetilde{P^*} = \tilde{P}S' + F'S'$. Moreover, $\tilde{P}S' \cap T = PS \cap T = 0$ and $\widetilde{P^*} \cap T = P^* \cap T$. Hence T may be considered as a subring of $S'/\tilde{P}S'$ and we have $P^* \cap T = \left(\widetilde{P^*}/\tilde{P}S'\right) \cap T$. Since $\tilde{P}$ is a maximal ideal in R', we have $R'/\tilde{P} = K[\xi_1, \dots, \xi_n] =: L$, where $\xi_i := X_i + \tilde{P} \in \overline{K}$ for $i = 1, \dots, n$. With the notation of the proof of Lemma 3.1 we therefore have $S'/\tilde{P}S' = S_L$, where $F' + \tilde{P}S' = F_\xi$. By Lemma 3.1 (a)(i), we obtain (note that $T = S_K$)

$$\begin{aligned} P^* \cap T &= \left(\widetilde{P^*}/\tilde{P}S'\right) \cap T = \left((\tilde{P}S' + F'S')/\tilde{P}S'\right) \cap T \\ &= F_\xi S_L \cap S_K = f_\xi T = f_P T \end{aligned}$$

with $f_P := f_\xi$ and (i) is shown. Moreover, f_P depends (up to constant factors) only on P and not on the choice of the indeterminate X_i such that $X_i \notin P$. Therefore, by Lemma 3.1 (a)(ii), f_P is integral with respect to all u_i, $i \in I$. On the other hand, if $X_j \in P$ for $j \in \{0, \dots, n\}$ then with

$$P' := P \cap R_j,\ \text{where } R_j := K[X_0, \dots, X_{j-1}, X_{j+1}, \dots, X_n]$$

it is immediately clear that $P^* = P'^*S + X_j S$, where

$$P'^* \subseteq R_j[u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_n]$$

is suitably defined. Thus $f_P = f_{P'}$ does not depend upon u_j, and (ii) is proved. (iii) follows from Lemma 3.1 (a)(iv) since

$$h_0(R/P) = \mathrm{rank}_K\, R'/\tilde{P} = [K[\xi_1, \dots, \xi_n] : K].$$

(c) Let $f_P = f_Q =: f$ and $I := \{i \in \{0, \dots, n\} \mid X_i \notin P\}$. Assume that $X_i \in Q$ for all $i \in I$. Then by (b)(ii) we had

$$\begin{aligned} f &= f_P \in K[\{u_i \mid i \in I\}] \ \text{ and} \\ f &= f_Q \in K[\{u_i \mid i \in \{0, \dots, n\} \setminus I], \end{aligned}$$

so that f would be constant, which is a contradiction. Hence there exists an $i \in \{0, \dots, n\}$ such that $X_i \notin P$ and $X_i \notin Q$. Without loss of generality we may assume that this happens for $i = 0$. Further let $R'/\tilde{P} \cong K[\xi_1, \dots, \xi_n]$ and $R'/\tilde{Q} \cong K[\zeta_1, \dots, \zeta_n]$ (cf. the proof of (b)). Then $f_\xi = f_P = f_Q = f_\zeta$ and, by Lemma 3.1 (b), there is a K-isomorphism

$$\varphi\colon K[\xi_1, \dots, \xi_n] \to K[\zeta_1, \dots, \zeta_n] \ \text{ with } \ \varphi(\xi_i) = \zeta_i, \ i = 1, \dots, n.$$

But this implies for any $g \in \tilde{Q}$ that

$$\varphi(g(\xi_1, \dots, \xi_n)) = g(\varphi(\xi_1), \dots, \varphi(\xi_n)) = g(\zeta_1, \dots, \zeta_n) = 0\,,$$

that is, $g(\xi_1, \dots, \xi_n) = 0$. Thus $g \in \tilde{P}$ and $\tilde{Q} \subseteq \tilde{P}$. Since $\tilde{Q}$, $\tilde{P}$ are maximal ideals in R', this implies $\tilde{P} = \tilde{Q}$ and hence $P = Q$. The converse is trivial. $\square$

Proposition 3.3 (Cayley–van der Waerden–Chow form). *Let K be a field, and let $X_0, \dots, X_n$ be indeterminates. Further let $P \subset R := K[X_0, \dots, X_n]$ be a homogeneous prime ideal and $d := \dim R/P - 1 \geq 0$, that is, d is the degree of the Hilbert polynomial of R/P. For $i = 0, \dots, d$ and $j = 0, \dots, n$ let u_{ij} be further indeterminates and abbreviate $u_{i0}, \dots, u_{in}$ by $\mathbf{u}_i$. Moreover, set*

$$S := R[\mathbf{u}_0, \dots, \mathbf{u}_d], \quad T := k[\mathbf{u}_0, \dots, \mathbf{u}_d], \quad F_i := \sum_{j=0}^{n} u_{ij} X_j, \quad i = 0, \dots, d,$$

$$\text{and} \quad P^* := (PS + (F_0, \dots, F_d)S) : \langle X_0, \dots, X_n \rangle.$$

Then:

(a) P^* is a prime ideal in S of height $n+1$ with $P^* \cap R = P$, and P^* is homogeneous in each row of indeterminates $\mathbf{u}_0, \dots, \mathbf{u}_d$ as well as in $\mathbf{X} := X_0, \dots, X_n$.

(b) There exists an irreducible and with respect to each row of indeterminates $\mathbf{u}_0, \dots, \mathbf{u}_d$ homogeneous polynomial $f_P \in T$ with the following properties:

(i) $f_P T = P^* \cap T$.

(ii) $0 < \deg_{\mathbf{u}_0} f_P = \dots = \deg_{\mathbf{u}_d} f_P \leq h_0(R/P)$ with equality if $\operatorname{char} K = 0$.

(c) If $Q \subset R$ is another homogeneous prime ideal with $\operatorname{ht}(Q) = \operatorname{ht}(P)$, then $f_P = f_Q$ (up to multiplication by a nonzero constant) if and only if $P = Q$.

f_P is called Cayley–van der Waerden–Chow form of P.

Proof. (a) For $0 \leq i \leq d$ we set $S^{(i)} := R[\mathbf{u}_0, \dots, \mathbf{u}_i]$ and

$$P_i^* := (PS^{(i)} + (F_0, \dots, F_i)S^{(i)}) : \langle X_0, \dots, X_n \rangle\,.$$

Since $P_i^* = (P_{i-1}^* S^{(i)} + F_i S^{(i)}) : \langle X_0, \dots, X_n \rangle$ ($P_{-1}^* := P$), it follows from Lemma 3.2 by induction on i that P_i^* is a prime ideal in $S^{(i)}$ of height

$$\operatorname{ht}(P) + i = n - d + i + 1.$$

Now (a) follows since $P^* = P_d^*$.

(b) Let $I_P := P^* \cap T$. The ideal I_P is a prime ideal in T, which is homogeneous in each row of indeterminates $\mathbf{u}_0, \ldots, \mathbf{u}_d$. Now let

$$\begin{aligned} T_0 &:= K[\mathbf{u}_1, \ldots, \mathbf{u}_d], \\ S_0 &:= R[\mathbf{u}_1, \ldots, \mathbf{u}_d], \quad \text{and} \\ k &:= Q(T_0) = K(\mathbf{u}_1, \ldots, \mathbf{u}_d)\,. \end{aligned}$$

Furthermore, we set

$$P_0 := (PS_0 + (F_1, \ldots, F_d)S_0) : \langle X_0, \ldots, X_n \rangle\,.$$

By the proof of (a), P_0 is a prime ideal in S_0 of height $\mathrm{ht}(P) + d = n$. Since also

$$\begin{aligned} \mathrm{ht}(PS_0 + (F_1, \ldots, F_d)S_0) &= \mathrm{ht}(P) + d = n \\ &< n + 1 = \mathrm{ht}(X_0, \ldots, X_n)S_0\,, \end{aligned}$$

it follows that $P_0 \subseteq (X_0, \ldots, X_n)S_0$, and therefore

$$P_0 \cap T_0 \subseteq (X_0, \ldots, X_n)S_0 \cap T_0 = 0.$$

Furthermore, by Lemma 3.2 (a), $P^* \cap S_0 = P_0$ and hence

$$P^* \cap T_0 = P^* \cap S_0 \cap T_0 = P_0 \cap T_0 = 0,$$

that is, $I_P \cap T_0 = 0$. Now let

$$\begin{aligned} R_k &:= k[X_0, \ldots, X_n], \\ S_k &:= R_k[\mathbf{u}_0] = k[X_0, \ldots, X_n, u_{01}, \ldots, u_{0n}], \text{ and} \\ T_k &:= k[\mathbf{u}_0]. \end{aligned}$$

Then $R_k = S_0 \otimes_{T_0} k$ is the localization of S_0 at the multiplicatively closed subset $T_0 \setminus \{0\}$, and similar statements hold for S_k and T_k. Because of $I_P \cap T_0 = 0$ we have $\mathrm{ht}(I_P) = \mathrm{ht}(I_P T_k)$ and

$$I_P T_k = P^* S_k \cap T_k = (P_0 R_k)^* \cap T_k.$$

Since $P_0 R_k$ is a homogeneous prime ideal in R_k of height n, it follows by Lemma 3.2 (b) that $I_P T_k = f_{P_0 R_k} T_k$. Thus $\mathrm{ht}(I_P) = 1$, that is, $I_P = f_P T$ for some irreducible and with respect to each row of indeterminates $\mathbf{u}_0, \ldots, \mathbf{u}_d$ homogeneous polynomial $f_P \in T$ which shows (i). Any permutation of the rows of indeterminates $\mathbf{u}_0, \ldots, \mathbf{u}_d$ induces an automorphism of S and T, respectively, which maps the ideal P^* to itself and therefore leaves f_P unchanged. Hence $\deg_{\mathbf{u}_0} f_P = \cdots = \deg_{\mathbf{u}_d} f_P > 0$. If we set $P' := P_0 R_k$ then, by Lemma 3.2

(b)(iii) and the theorem of Bézout,

$$\begin{aligned}\deg_{\mathbf{u}_0} f_P &= \deg f_{P'} \leq h_0(R_k/P') \\ &= \deg_{\mathbf{X}} F_1 \cdot \ldots \cdot \deg_{\mathbf{X}} F_d \cdot h_0(R_k/PR_k) \\ &= h_0(R_k/PR_k) = h_0(R/P)\end{aligned}$$

with equality if $\operatorname{char} k = \operatorname{char} K = 0$, which finishes the proof of (ii).

(c) Let $f_P = f_Q$. Then, with the notation introduced in (b), we have

$$f_{P_0} = f_{PR_k} = f_{QR_k} = f_{Q_0}$$

and, by Lemma 3.2 (c), we conclude that $P_0 = Q_0$. Now from (a) and Lemma 3.2 (a) it follows that

$$\begin{aligned}P &= P^* \cap R = P^* \cap R_k \cap R = P_0 \cap R \\ &= Q_0 \cap R = Q^* \cap R_k \cap R = Q^* \cap R = Q\,,\end{aligned}$$

which finishes the proof. $\square$

The following example shows that the assumption on the characteristic in Proposition 3.3 (b)(ii) cannot be omitted.

Example 3.4. Let $K := (\mathbb{Z}/2\mathbb{Z})(s,t)$ with indeterminates s, t and

$$R := K[X_0, X_1, X_2].$$

Furthermore, let $f_1 = X_1^2 + sX_0^2$, $f_2 = X_2^2 + tX_0^2$ and $P := (f_1, f_2)R$. Then P is a homogeneous prime ideal in R. If we set

$$\begin{aligned}S &:= R[u_0, u_1, u_2], \\ F &:= u_0X_0 + u_1X_1 + u_2X_2, \\ \tilde{F} &:= u_0X_1X_2 + u_1X_0X_2 + u_2tX_0X_1 \quad \text{and} \\ f &:= u_0^2 + su_1^2 + tu_2^2,\end{aligned}$$

then we have

$$PS + FS = (f_1, f_2, F, \tilde{F}, f)S \cap (X_0^2, X_1^2, X_2^2, X_0X_1X_2, F)S\,,$$

where $(X_0^2, X_1^2, X_2^2, X_0X_1X_2, F)S$ is $(X_0, X_1, X_2)S$-primary.

Therefore, $P^* = (f_1, f_2, F, \tilde{F}, f)S$ and $f_P = f$, i.e., $\deg f_P = 2 < 4 = h_0(R/P)$.

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Received January 30, 2007

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