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HOLONOMY THEORY AND 4-DIMENSIONAL LORENTZ MANIFOLDS

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Abstract. This lecture describes the holonomy group for a 4-dimensional Hausdorff, connected and simply connected manifold admitting a Lorentz metric and shows, briefly, some applications to Einstein's space-time of general relativity.

1. Holonomy Theory on 4-Dimensional Lorentz Manifolds. Let M be a 4-dimensional, smooth, Hausdorff, connected, and simply connected manifold admitting a smooth Lorentz metric g (and thus is paracompact) and associated smooth Levi-Civita connection Γ and covariant derivative ∇ with curvature $\mathcal{R}$. The curvature components are denoted by $R^a{}_{bcd}$ and the corresponding Ricci tensor components by $R_{ab} \equiv R^c{}_{acb}$. Let Φ be the holonomy group of M derived in the usual way by parallel transport of the members of the tangent space T_mM to M at m around smooth closed curves at m. (In fact Φ is independent of the differentiability class C^k $(k \geq 1)$ of these curves – for this and more details about holonomy theory see [**6**]). It follows that Φ is a connected Lie group and hence a connected Lie subgroup of the identity component $\mathcal{L}_0$ of the Lorentz group $\mathcal{L}$. Thus Φ is determined by its Lie algebra ϕ which is then a subalgebra of the Lie algebra A of $\mathcal{L}_0$. The problem with the study of Φ arises from the Lorentz signature of g and from the resulting more diverse nature of the subgroup structure of $\mathcal{L}_0$.

The Lorentz group and its Lie algebra can be represented by

$$\mathcal{L} = \{B \in GL(4,\mathbb{R}) : B\eta B^T = \eta\}, \quad A = \{C \in M_n\mathbb{R} : \eta C + (\eta C)^T = 0\}$$

where $\eta = \mathrm{diag}(-1,1,1,1)$. The members of A are skew self-adjoint with respect to η and can be represented in Minkowski space (i.e. the manifold $\mathbb{R}^4$ with metric η) by skew symmetric matrices $F_{ab} = \eta_{ac}C^c{}_b$ for $C \in A$. If $C \neq 0$, F has (matrix) rank two or four and in the former case it can be written as $F_{ab} = p_a q_b - q_a p_b$ for $p, q \in \mathbb{R}^4$ and is then denoted by $p \wedge q$. Now let $\{\ell, n, x, y\}$ be a null tetrad in Minkowski space so that the only non-vanishing inner products between them are $\eta(\ell, n) = \eta(x, x) = \eta(y, y) = 1$ and let $\{u, x, y, z\}$ be a pseudo-orthonormal tetrad where the only non-vanishing inner products are $-\eta(u, u) = \eta(x, x) = \eta(y, y) = \eta(z, z) = 1$. Then the possible subalgebras of A can be collected together using this notation (and with $0 \neq \rho \in \mathbb{R}$) in Table 1 (see e.g. [**8**] from which the notational labeling is taken).

Label	Basis	Dim	Label	Basis	Dim
R_1	0	0	R_8	$l \wedge x,\ l \wedge y$	2
R_2	$l \wedge n$	1	R_9	$l \wedge x,\ l \wedge y,\ l \wedge n$	3
R_3	$l \wedge x$	1	R_{10}	$l \wedge x,\ n \wedge x,\ l \wedge n$	3
R_4	$x \wedge y$	1	R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	3
R_5	$l \wedge n + \rho x \wedge y$	1	R_{12}	$l \wedge x,\ l \wedge y,\ l \wedge n + \rho x \wedge y$	3
R_6	$l \wedge n,\ l \wedge x$	2	R_{13}	$x \wedge y,\ x \wedge z,\ y \wedge z$	3
R_7	$l \wedge n,\ x \wedge y$	2	R_{14}	$l \wedge x,\ l \wedge y,\ l \wedge n,\ x \wedge y$	4
			R_{15}	$(= A)$	6

TABLE 1.

The Lie group $\mathcal{L}_0$ is exponential, cfg. [**9**], (i.e. given $h \in \mathcal{L}_0$, $\exists C \in A$ such that $h = \exp C$) but its connected subgroups are not necessarily exponential. However, if H is a connected Lie subgroup of $\mathcal{L}_0$ (whose Lie algebra A' is therefore one of those in Table 1), then given $h \in H$, $\exists C_1, \ldots, C_n \in A'$ such that $h = \exp C_1 \circ \cdots \circ \exp C_n$.

Now let $m \in M$ and let V be a non-trivial proper subspace of $T_m M$. Then V is called *holonomy invariant* if it is carried into itself upon parallel transport along any closed curve at m and, if so, its parallel transport to any other point m' of M along a curve c is independent of c and is also holonomy invariant. Each such V gives rise to a smooth m-dimensional Frobenius-type distribution on M (where $m = \dim V$) which is then in an obvious sense holonomy invariant. If such a V exists, the holonomy group is called *reducible* (and otherwise *irreducible*). Again, for such a V, if it contains no non-trivial proper holonomy invariant subspace it is called *irreducible* (and otherwise *reducible*).

2. Decomposition Theorems. For a Hausdorff, connected, simply connected manifold with a positive definite metric, the existence of irreducible holonomy invariant subspaces of $T_m M$ leads to a natural decomposition of $T_m M$ and to an associated metric product neighbourhood of each point of M (and also to M being a *global* metric product if M is geodesically complete).

This is the well-known theorem of de Rham (see e.g. [**6**]). For Lorentz metrics the situation is a little more complicated. So with M as in Section 1 let V be a non-trivial proper subspace of T_mM. If the metric $g(m)$ induces a (non-degenerate) metric on V, then V is called *non-null* (and, in particular, *spacelike* if this induced metric is positive definite and *timelike* if it is Lorentz). Otherwise V is called *null*. (For more details see [**2**].) Now a subspace of T_mM is called *weakly irreducible* if it is either $\{0\}$ or T_mM or non-null and if, in addition, it contains no non-trivial, proper, non-null holonomy invariant subspaces. Thus a weakly irreducible subspace may not be irreducible. For Lorentz metrics weak irreducibility plays the role that irreducibility plays for positive definite metrics. The holonomy group is now called *non-degenerately reducible* if for some $m \in M$, T_mM contains a (non-trivial, proper) non-null holonomy invariant subspace. The following theorem due to Wu [**11**] is the analogue of de Rham's theorem in the Lorentz case.

THEOREM 2.1. *Let M be a simply connected space-time with metric g and let $m \in M$. Suppose the subspace V_0 of T_mM on which the holonomy group Φ of M acts trivially is either $\{0\}$ or proper and non-null (the trivial case $V_0 = T_mM$ is excluded since M is then flat. Then the orthogonal complement of V_0 is holonomy invariant (and either equals T_mM or else is non-null) and may be written as a direct sum $V_1 \oplus \cdots \oplus V_k$ of subspaces of T_mM which are weakly irreducible, mutually orthogonal and holonomy invariant. Further, if M_i are the maximal integral manifolds through m of the holonomy invariant distributions on M corresponding to the subspaces V_i $(0 \leq i \leq k)$ each with their metric g_i induced from g, there exists an open neighbourhood U of m which, as an open submanifold of M with metric induced from g, is isometric to $U_0 \times U_1 \times \cdots \times U_k$ where each U_i is an open submanifold of M_i with metric induced from g_i. If V_0 is not trivial, M_0 is locally Euclidean.*

If, in addition, M is geodesically complete, then each M_i $(0 \leq i \leq k)$ is connected, simply connected, totally geodesic and geodesically complete and M is isometric to the metric product $M_0 \times M_1 \times \cdots \times M_k$. Also M_0 is Euclidean.

The local metric decomposability in the non-degenerately reducible cases guaranteed by Wu's theorem is rather useful in performing local calculations. In constructing the holonomy group the well-known theorem of Ambrose and Singer (see e.g. [**6**]) is useful. This theorem can be stated in the following very simple form; a representation of the holonomy algebra ϕ can be obtained by first fixing a point $m \in M$ and then for any other point $m' \in M$ choosing a smooth curve c from m to m'. Then construct at m' the set of all skew-symmetric tensors in the "range" of the curvature tensor at m' i.e. all such tensors of the form $R_{abcd}e^c f^d$ for $e, f \in T_{m'}M$. Then parallel transport this latter set along c to m. If this is repeated for all points m' and all curves c from m to m' the collection obtained at m is then a representation of ϕ.

As an example, from Table 1, if the holonomy type of M is R_{13}, T_mM splits irreducibly into the product of holonomy invariant subspaces of dimension one (u) and three (x, y, z), the former leading to a global smooth covariantly constant vector field on M satisfying $\nabla u = 0$ and $g(u, u) = -1$ on M. The holonomy group is then $SO(3)$. The holonomy types R_2, R_3, R_4, R_6, R_7, R_{10} and R_{13} are non-degenerately reducible whilst the types R_8, R_9, R_{11}, R_{12} and R_{14} are reducible but not non-degenerately reducible. The type R_5 cannot exist as a space-time holonomy group (see, e.g. [**4**]). Of these latter types R_8 and R_{11} admit a non-trivial global covariantly constant null vector field ℓ (so that $\nabla\ell = 0$ and $g(\ell, \ell) = 0$). However, the types R_9, R_{12} and R_{14} admit at each $m \in M$ a 1-dimensional null holonomy invariant subspace which gives rise to a smooth global (non-trivial) null vector field ℓ on M which is recurrent, ($\nabla\ell = \ell \otimes r$, for some smooth global 1-form r) but which cannot be globally scaled to be (non-trivially) covariantly constant. It should be noted here that any global nowhere-zero recurrent vector field X is such that the sign $(+, -, 0)$ of $g(X, X)$ is constant on M and if this sign is not 0, i.e. if X is not null on M then X may always be globally scaled so that it is non-trivially covariantly constant. The holonomy type R_{15} is irreducible. For details of all this see [**4**].

3. A Partial Holonomy Classification. General Relativity is the most successful theory of the gravitational field to date and describes space-time as a 4-dimensional Lorentz manifold of the type given in Section 1 (not necessarily with the simply connected condition, but this will be retained here for convenience). The distribution of the gravitational sources is described by a tensor called the energy-momentum tensor and the prescription of this tensor on M is equivalent to prescribing the Ricci tensor on M. It is perhaps more appropriate for these lectures to work with the Ricci tensor rather than to get involved with the physics of the energy-momentum tensor. Full details of this latter tensor can be found in [**10**].

In the problem of classifying gravitational fields, the Petrov classification of the Weyl tensor of M [**7**] has proved most useful. But this classification is a pointwise algebraic classification and the "type" of the Weyl tensor may vary from point to point. A holonomy classification has the advantage that it deals with the whole space-time and is not pointwise. However, there is a sense in which it is too "fine" in some categories and too coarse in others. (The Petrov classification also suffers to some extent from this.) To proceed with this partial holonomy classification the method employed is to consider first the more commonly studied energy-momentum tensors (which here means

Ricci tensors) and to see which holonomy groups are possible for such space-times. The choice of Ricci tensor is made on physical grounds and is described using an algebraic classification of symmetric second order tensors according to their Segre type (Jordan canonical form). It is assumed here (just as it is in the physical approaches to these space-times) that the algebraic type of the Ricci tensor (which may also vary from point to point) *is the same everywhere on M.*

At $m \in M$ the Ricci tensor can be regarded as a linear map $T_mM \to T_mM$ according to $k^a \to R^a{}_b k^b$ and as such can be classified according to its Segre (Jordan) type. Because of the Lorentz signature of $g(m)$ the possible Segre types are restricted to $\{1,111\}$, $\{211\}$, $\{31\}$ and $\{z\bar{z}11\}$ together with their degeneracies (which are denoted by including the appropriate digits in the Segre type symbol inside round brackets). In the first type the Ricci tensor is diagonalisable over $\mathbb{R}$ and this is the only type to admit timelike eigenvectors, the "timelike eigenvalue" being separated from the others by a comma. In the last type the Ricci tensor is diagonalisable over $\mathbb{C}$ admitting two real and two non-real (conjugate) eigenvalues and is the only type to admit non-real eigenvalues. The third and fourth types are less interesting physically since they fail the so-called "energy conditions". Full details of this classification can be found in [**7, 3, 5**].

The following types of space-times can now be described in terms of the algebraic nature of the Ricci tensor. The proof is omitted, full details being available in [**4**]. In some cases a brief physical description of the gravitational field described will be given.

Theorem 3.1. *Let M be a (simply connected) space-time as described in Section 1.*

(i) *If M represents a non-flat vacuum space-time (i.e. the Ricci tensor is the zero tensor on M), then its holonomy type is either R_8, R_{14} or R_{15}.*
(ii) *If M represents an Einstein space (i.e.the Ricci tensor is everywhere a multiple of the metric tensor) but not a vacuum space-time, then the holonomy type of M is R_7, R_{14} or R_{15}.*
(iii) *If M is conformally flat, but not flat, then the holonomy type of M is R_7, R_8, R_{10}, R_{13}, R_{14} or R_{15} and the possibilities for the algebraic type of the Ricci tensor are easily calculated.*
(iv) *If the Ricci tensor for M is everywhere of Segre type $\{(211)\}$ with eigenvalue zero (this represents a null Einstein–Maxwell field, i.e. the gravitational field arising from an idealised form of a pure electromagnetic radiation field), then the holonomy type of M is R_3, R_8, R_{10}, R_{14} or R_{15}.*
(v) *If the Ricci tensor is everywhere of Segre type $\{(1,1)(11)\}$ with equal and opposite eigenvalues (this represents a non-null Einstein-Maxwell field,*

i.e. the gravitational field arising from a general pure electromagnetic field), then the holonomy type of M is R_7, R_{14} or R_{15}.

(vi) *If the Ricci tensor of M is everywhere of Segre type* $\{1,(111)\}$ *(this represents the gravitational field of a perfect fluid whose density and isotropic pressure are certain combinations of these two Ricci eigenvalues and which do not simultaneous vanish over any non-empty open subset of M), then the holonomy type of M is R_{10}, R_{13} or R_{15}.*

It is remarked that each holonomy type (except R_5) can occur as the holonomy type of a space-time [**1, 4**]. Further details can be found in the author's forthcoming book "Curvature Structure and Symmetries in General Relativity" (World Scientific, Singapore).

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References

1. Debever R., Cahen M., *Sur les Espaces–Temps qui admettent un champ de vecteurs isotropes parallèles,* Bull. Acad. Belg. Cl. Sci., **47** (1962), 491.
2. Hall G.S., *Symmetries in 4–dimensional Lorentz Manifolds,* Univ. Iagel. Acta Math., **40** (2002), 207–214.
3. Hall G.S., *The Classification of Second Order Symmetric Tensors in General Relativity Theory,* in: Differential Geometry, Banach Centre Publications, Vol. 12, PWN, Warsaw, 1984.
4. Hall G.S., Lonie D.P., *Holonomy groups and Space–Times,* Class. Quant. Grav., **17** (2000), 1369.
5. Hawking S.W., Ellis G.F.R., *The Large Scale Structure of Space–Time C.U.P.,* Cambridge, 1973.
6. Kobayashi S., Nomizu K., *Foundations of Differential Geometry,* Vol. 1, Interscience, New York, 1963.
7. Petrov A.Z., *Einstein Spaces,* Pergamon, 1969.
8. Schell J.F., *Classification of 4–dimensional Riemannian Spaces,* J. Math. Phys., **2** (1961), 291.
9. Shaw R., *The subgroup structure of the homogeneous Lorentz group,* Q. J. Math., **21** (1970), 101.
10. Stephani H., Kramer D., MacCallum M.A.H., Hoenselears C., Herlt E., *Exact Solutions of Einstein's Field Equations,* Cambridge University Press, 2003.
11. Wu H., *On the de Rham Decomposition Theorem,* Illinois J. Math., **8** (1964), 291.

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