UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLI
2003

BERNSTEIN QUASIANALYTIC FUNCTIONS ON ALGEBRAIC SETS

by Alicja Skiba

Abstract. We extend the notion of Bernstein quasianalytic functions to algebraic sets in $\mathbb{C}^n$. We prove a uniqueness principle for such functions.

1. Introduction. Let $E \subset \mathbb{R}$ be a compact interval. It is the well-known result of Bernstein that a function $f \in \mathcal{C}(E)$ can be extended to a holomorphic function in a neighbourhood $U \subset \mathbb{C}$ of the set E if and only if

$$\limsup_{k\to\infty} \sqrt[k]{\operatorname{dist}_E(f, \mathcal{P}_k)} < 1, \tag{1.1}$$

where $\mathcal{P}_k = \mathcal{P}_k(\mathbb{C})$ denotes the space of all polynomials of one complex variable of degree at most k and $\operatorname{dist}_E(f, \mathcal{P}_k) = \inf\{\| f - p \|_E \,;\, p \in \mathcal{P}_k\}$. If a function f satisfies (1.1) then obviously the following identity principle holds:

(IP) $\quad f = 0$ *on a subinterval of* E *implies that* f *vanishes on* E.

As was observed by Bernstein, to establish the above identity principle it is enough to assume that the function f satisfies the weaker condition

$$\liminf_{k\to\infty} \sqrt[k]{\operatorname{dist}_E(f, \mathcal{P}_k)} < 1. \tag{1.2}$$

If a function f has property (1.2) it is called *quasianalytic in the sense of Bernstein.* Condition (1.2) can be reformulated as follows. There exist a constant $\varrho \in (0, 1)$ and a strictly increasing sequence of positive integers $\{n_j\}$ such that

$$\limsup_{n_j\to\infty} \sqrt[n_j]{\operatorname{dist}_E(f, \mathcal{P}_{n_j})} = \varrho.$$

Szmuszkowiczówna [**17**] and, independently, Lelong [**5**] extended (IP) by proving that a quasianalytic function $f \neq 0$ defined on E can vanish only on a subset of E with a transfinite diameter equal to 0 (or, equivalently, on a polar subset of E).

An important property of quasianalytic functions was found by Mazurkiewicz [**8**]: *the set* $\mathcal{B}(E)$ *of all quasianalytic functions defined on* E *is residual in the space* $\mathcal{C}(E)$. Let us recall that a subset A of a topological space X is residual if the set $X \setminus A$ is a union of a countable number of nowhere dense subsets of X. Another interesting result was obtained by Markuszewicz [**7**]: for any function $f \in \mathcal{C}(E)$ there exist $f_1, f_2 \in \mathcal{B}(E)$ such that $f = f_1 + f_2$.

The notion of quasianalyticity has been extended to the n–dimensional case in [**9**]. The theory of quasianalytic functions of several variables in the sense of Bernstein has been developed by Pleśniak in [**11**].

The aim of this paper is to show that the notion of a quasianalytic function in the sense of Bernstein can be extended to algebraic subsets of $\mathbb{C}^m$. In particular, it has been proved (Theorem 3.7) that an identity principle for such functions also holds on some algebraic sets. We complete the paper by Section 4, in which we give some examples of compact sets $E \subset \mathbb{C}^m$ preserving the Szmuszkowiczówna–Lelong type identity principle (see Definition 3.6).

2. Preliminaries. A subset A of $\mathbb{C}^m$ is said to be *pluripolar* if there exists a plurisubharmonic function u on $\mathbb{C}^m$ such that $A \subset \{u = -\infty\}$. If for each point $a \in A$ there exist an open neighbourhood V of a and a plurisubharmonic function v on V such that $A \cap V \subset \{v = -\infty\}$, then the set A is said to be *locally pluripolar*. Josefson [**4**] proved that both notions are equivalent.

Let us recall that a set $A \subset \mathbb{C}^m$ is called *locally analytic* if for each point $a \in A$ there are an open neighbourhood U of a and functions $f_1, ..., f_s$ holomorphic in U such that $A \cap U = \{z \in U \ : \ f_1(z) = ... = f_s(z) = 0\}$. Let $\mathbb{M}$ be a locally analytic subset of $\mathbb{C}^m$ whose subset $\mathbb{M}_{\mathrm{reg}}$ of regular points is a complex submanifold of $\mathbb{C}^m$ of pure dimension k ($k \leqslant m$). A function defined on $\mathbb{M}$ is said to be *plurisubharmonic on* $\mathbb{M}$ if it is plurisubharmonic on $\mathbb{M}_{\mathrm{reg}}$ and locally bounded from above on $\mathbb{M}$. We say that a set $N \subset \mathbb{M}$ is *pluripolar in* $\mathbb{M}$ if there exists a plurisubharmonic function u on $\mathbb{M}$ such that $N \cap \mathbb{M}_{\mathrm{reg}} \subset \{u = -\infty\}$.

Let E be a subset of the space $\mathbb{C}^m$. The function

$$V_E(z) = \sup\{u(z) \ \ : \ \ u \in \mathcal{L}(\mathbb{C}^m), \ \ u_{|E} \leqslant 0 \ \},$$

where $\mathcal{L}(\mathbb{C}^m) = \{u \in \mathrm{PSH}(\mathbb{C}^m) \ \ ; \ \ sup_{z \in \mathbb{C}^m}[u(z) - \log(1 + |z|)] < \infty\}$ is the Lelong class of plurisubharmonic functions with minimal growth, is called the *extremal function* of the set E. Let $\mathcal{P} = \mathcal{P}(\mathbb{C}^m)$ be the space of all polynomials of m complex variables. For a compact set $E \subset \mathbb{C}^m$ Siciak [**15**] has introduced the function

$$\Phi_E(z) = \{|p(z)|^{1/deg\, p} \ \ : \ \ p \in \mathcal{P}(\mathbb{C}^m), \ \ deg\, p \geqslant 1, \ \ \|p\|_E = \sup_{z \in E} |p(z)| \leqslant 1\}$$

by now called *Siciak's extremal function.* It is known (Zakharyuta [**18**], Siciak [**16**]) that $V_E(z) = \log \Phi_E(z)$ for $z \in \mathbb{C}^m$. It follows directly from the definition of Φ_E that for any compact set E and any polynomial p the following *Bernstein–Walsh–Siciak inequality* holds:

$$|p(z)| \leqslant \| p \|_E \, [\Phi_E(z)]^{deg\, p} \quad , \quad z \in \mathbb{C}^m. \tag{2.1}$$

A set $D \subset \mathbb{C}^m$ is called negligible if there exists a family of functions $\{u_\iota\} \subset PSH(\mathbb{C}^m)$ locally bounded from above such that

$$D \subset \{z \in \mathbb{C}^m \;\; ; \;\; \sup u_\iota(z) < (\sup u_\iota(z))^*\},$$

where $h^*(z) = \limsup_{u \to z} h(u)$.

An essential role in this paper is played by the following Bedford–Taylor counterpart of the classical Kellogg lemma (see e.g. [**13**, Theorem 4.2.5]).

THEOREM 2.1. ([**2**, Theorem 7.1]) *Negligible sets in $\mathbb{C}^m$ are exactly pluripolar sets.*

Let now E be a compact subset of $\mathbb{C}^m$. Observe that

$$F := \{z \in E \;\; ; \;\; V_E \text{ is not continuous in } z\} = \{z \in E \;\; ; \;\; V_E^*(z) > V_E(z) = 0\}.$$

Therefore F is a negligible subset of $\mathbb{C}^m$ and, by the above Bedford–Taylor version of the Kellogg lemma, F must be pluripolar.

3. Bernstein quasianalytic functions on algebraic sets.

DEFINITION 3.1. Let $\mathbb{M} \subset \mathbb{C}^m$ be an algebraic set, and let K be a compact subset of $\mathbb{M}$. A function f defined on K, with values in $\mathbb{C}$, is said to be quasianalytic on K in the sense of Bernstein if there exist a strictly increasing sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ and a sequence of polynomials $p_{n_j} \in P_{n_j}(\mathbb{C}^m)$, $j = 1, 2, \ldots$, such that

$$\limsup_{j \to +\infty} \sqrt[n_j]{\| f - p_{n_j} \|_K} \; < \; 1. \tag{3.1}$$

The set of all quasianalytic functions on K is denoted by $\mathcal{B}(K)$.

This definition extends the notion of quasianalyticity of functions defined on a compact subset of the space $\mathbb{C}^k$ to those defined on pieces of algebraic sets.

LEMMA 3.2. *Let h be a holomorphic mapping defined on an open set $U \subset \mathbb{C}^k$, with values in a locally analytic set $\mathbb{M}$ in $\mathbb{C}^m$ of pure dimension k ($k \leqslant m$). Assume that h is non-degenerate, which means that $rank_U h = \max_{z \in U} rank_z h = k$. Let E be a compact subset of U and $F = h(E)$. If a set $N \subset F$ is non-pluripolar in $\mathbb{M}$, then the set $h^{-1}(N) \cap E$ is non-pluripolar in $\mathbb{C}^k$.*

Proof. Let $\mathbb{M}_{\rm sing}$ be the set of singular points of $\mathbb{M}$. Then $\mathbb{M}_{\rm sing}$ is an analytical subset of $\mathbb{M}$ and $\dim \mathbb{M}_{\rm sing} < k$ (see e.g. [**6**, Chapter IV.2.4]). Hence, in view of Josefson's theorem, the set $\mathbb{M}_{\rm sing}$ is pluripolar in $\mathbb{M}$. The set $A = \{z \in U \ : \mathrm{rank}_z h < k\}$ is analytic. By the well-known theorem (see e.g. [**3**, Chapter 1.3.8]) the set $h(A)$ is contained in at most countable family of locally analytic sets in $\mathbb{M}$ of dimension less than k. Consequently, by Josefson's theorem, $h(A)$ is a pluripolar subset of $\mathbb{M}$. Hence the set $\widetilde{N} := N \backslash (h(A) \cup \mathbb{M}_{\rm sing})$ is non-pluripolar in $\mathbb{M}$. So applying again Josefson's theorem we show that there exists a point $a \in \widetilde{N}$ such that

(3.2) for each $\varepsilon > 0$ the set $B(a,\varepsilon) \cap \widetilde{N}$ is non-pluripolar in $\mathbb{M}$.

Since a is a regular point of the set $\mathbb{M}$, we can find a constant $\varepsilon_0 > 0$ and a biholomorphism ϕ of the set $B(a,\varepsilon_0) \cap \mathbb{M}$ onto the unit ball B in $\mathbb{C}^k$. Let us take a point $b \in h^{-1}(a) \cap E$. Then $b \notin A$, so one can find an open neighbourhood $V \subset U$ of the point b such that $h_{|V}$ is a biholomorphism of V onto $h(V)$ and $a \in h(V) \subset B(a,\varepsilon_0)$. By (3.2) the set $\phi(h(V) \cap \widetilde{N})$ is non-pluripolar. Hence, since the mapping $\phi \circ h_{|V}$ is a biholomorphism of V onto $\phi(h(V))$, the set $(\phi \circ h_{|V})^{-1}\big(\phi(h(V) \cap \widetilde{N})\big) = h_{|V}^{-1}(h(V) \cap \widetilde{N})$ is non-pluripolar in $\mathbb{C}^k$. Since $\widetilde{N} \subset F = h(E)$ and $h_{|V}$ is a biholomorphism, we have $h_{|V}^{-1}(h(V) \cap \widetilde{N}) \subset E$. Consequently, the set $E \cap h^{-1}(N)$ must be non-pluripolar in $\mathbb{C}^k$. □

Lemma 3.3. *Let $E \subset \mathbb{C}^k$ be a non-pluripolar, polynomially convex compact set. Let h be a non-degenerate holomorphic mapping defined on an open neighbourhood U of E, with values in an algebraic set $\mathbb{M} \subset \mathbb{C}^m$ $(k \leqslant m)$. Let K be a compact subset of $\mathbb{M}$ such that $h(E) \subset K$. If a function f is quasianalytic on K, then the function $g = f \circ h$ is quasianalytic on E.*

To prove this lemma we shall need a characterization of algebraic sets in $\mathbb{C}^m$ given by Sadullaev [**14**].

Theorem 3.4. (Sadullaev's criterion) *An analytic subset A of $\mathbb{C}^m$ is algebraic if and only if Siciak's extremal function Φ_E is locally bounded on A for some (and hence for each) non-pluripolar compact subset E of A.*

An important role in the following proof is played by a uniform version of the Bernstein–Walsh–Siciak theorem [**10**, Lemma 1].

Theorem 3.5. *Let $\mathcal{A}(U)$ be the space of bounded holomorphic functions defined in an open set $U \subset \mathbb{C}^k$, and let $\| f \|_U := \sup_{z \in U} |f(z)|$. For every polynomially convex compact subset E of U there exist constants $M > 0$ and $a \in (0,1)$ such that*

$$\operatorname{dist}_E(f,\mathcal{P}_l) := \inf\{\| f-p\|_E \,;\, p\in\mathcal{P}_l\} \leqslant M \,\| f\|_U \, a^l$$

for $f\in\mathcal{A}(U)$ and $l\in\mathbb{N}$.

Proof of Lemma 3.3. The definition of a quasianalytic function (3.1) implies that for such a function f one can choose a sequence of polynomials $p_{n_j}\in\mathcal{P}_{n_j}$ of m variables and a constant $\varrho\in(0,1)$ such that

$$\| f-p_{n_j}\|_K \leqslant \varrho^{n_j} \quad \text{for} \quad j\geqslant j_0. \tag{3.3}$$

Since $K\supset h(E)$, we have

$$\| f\circ h-p_{n_j}\circ h\|_E = \| f-p_{n_j}\|_{h(E)} \leqslant \| f-p_{n_j}\|_K \leqslant \varrho^{n_j} \quad \text{for} \quad j\geqslant j_0. \tag{3.4}$$

We may assume that h is bounded on U. Then taking a constant $R>0$ sufficiently large, we have $h(U)\subset B(0,R)\cap\mathbb{M}$. Hence, by the Bernstein–Walsh–Siciak inequality (2.1), we get

$$\sup_{z\in U}|p_{n_j}\circ h(z)| \leqslant \sup_{w\in h(U)}|p_{n_j}(w)| \leqslant \| p_{n_j}\|_K \big[\sup_{w\in h(U)}\Phi_K(w)\big]^{n_j}. \tag{3.5}$$

By [**1**, Lemma 0.1], $h(E)$ is a non-pluripolar subset of the algebraic set $\mathbb{M}$, whence by Sadullaev's criterion we get

$$C_1 := \sup_{w\in h(U)}\Phi_K(w) < \infty. \tag{3.6}$$

Now, since

$$\| p_{n_j}\|_K \leqslant \| f-p_{n_j}\|_K + \| f\|_K \leqslant 1+\| f\|_K$$

for all $j\geqslant j_0$, by (3.5) and (3.6) one can find a constant $C>0$ such that

$$\sup_{z\in U}|p_{n_j}\circ h(z)| \leqslant C^{n_j}\,, \quad j\geqslant j_0.$$

Owing to this we can apply Theorem 3.5 to the family of functions $\{p_{n_j}\circ h\}$ and we get

$$\operatorname{dist}_K(p_{n_j}\circ h, P_l(\mathbb{C}^k)) \leqslant M\,\| p_{n_j}\circ h\|_U \, a^l \leqslant MC^{n_j}a^l$$

for each $j,l\in\mathbb{N}$ with suitably chosen constants $M>0$ and $a\in(0,1)$. Consequently, for each $j\in\mathbb{N}$ and each $l\in\mathbb{N}$ there exists $r_l\in P_l(\mathbb{C}^k)$ such that

$$\| p_{n_j}\circ h-r_l\|_K \leqslant MC^{n_j}a^l.$$

Choosing an integer t such that $Ca^t\leqslant a$ and putting $l=tn_j$ gives

$$\| p_{n_j}\circ h-r_{tn_j}\|_K \leqslant Ma^{n_j}. \tag{3.7}$$

By the triangle inequality, (3.4) and (3.7), for $j\geqslant j_0$ we get

$$\| f\circ h-r_{tn_j}\|_E \leqslant \varrho^{n_j}+Ma^{n_j} \leqslant 2M\eta^{n_j} = 2M(\eta^{\frac{1}{t}})^{tn_j},$$

where $\eta=\max\{\varrho,a\}<1$.

Finally, setting $u_j := tn_j$ gives

$$\limsup_{j\to\infty} \sqrt[u_j]{\| f \circ h - r_{u_j} \|_E} \leqslant \eta^{\frac{1}{t}} < 1,$$

whence $f \circ h$ is a quasianalytic function on E. $\square$

DEFINITION 3.6. A compact set $E \subset \mathbb{C}^k$ is said to satisfy condition (NB) if for every $f \in \mathcal{B}(E)$ and every non-pluripolar set $F \subset E$, $f = 0$ on F implies $f = 0$ on E.

THEOREM 3.7. *Let E be a polynomially convex compact set in $\mathbb{C}^k$ satisfying condition (NB). Let*

$$h\ : U \supset E \ \mapsto \ \mathbb{M} \subset \mathbb{C}^m$$

be a non-degenerate holomorphic mapping in an open neighbourhood U of E, with values in an algebraic set $\mathbb{M}$ of pure dimension k $(k \leqslant m)$, and $K = h(E)$. If f is quasianalytic on K and $f(z) = 0$ for $z \in N \subset K$, where N is a non-pluripolar subset of $\mathbb{M}$, then $f \equiv 0$ on K.

PROOF. In view of Lemma 3.2 the set $F = h^{-1}(N) \cap E$ is a non-pluripolar subset of E on which the function $f \circ h$ vanishes. By Lemma 3.3 the function $f \circ h$ is quasianalytic on E. Consequently, by condition (NB), $f \circ h \equiv 0$ on E, and therefore $f \equiv 0$ on $K = h(E)$. $\square$

4. Sets satisfying condition (NB). By the Szmuszkowiczówna–Lelong theorem every closed subinterval of $\mathbb{C}$ satisfies condition (NB). In [**9**] it has been proved that this condition is satisfied by subsets of $\mathbb{C}^n$ of type $E = E_1 \times \ldots \times E_n$ where each set E_i is a continuum in $\mathbb{C}$. By Theorem 2.1 and (IP) we derive that if E is a convex compact subsets of $\mathbb{C}^n$ then it satisfies (NB). Now we shall prove essentially more, viz. that the sets whose two arbitrary points can be connected by an analytic curve belong to the class of NB–sets. We shall need a counterpart of Lemma 3.3 in the case where h is a holomorphic mapping with values in $\mathbb{C}^m$.

LEMMA 4.1. *Let $E \subset \mathbb{C}^k$ be a non-pluripolar, polynomially convex compact set, and let $K \subset \mathbb{C}^m$ be a non-pluripolar compact set. Let h be a holomorphic map defined in an open neighbourhood U of E, with values in $\mathbb{C}^m$, such that $h(E) \subset K$. Then for every quasianalytic function f on K the function $f \circ h$ is quasianalytic on E.*

REMARK 4.2. The dimensions k and m can be arbitrary.

The proof of Lemma 4.1 is similar to that of Lemma 3.3. Now, the constant C_1 in (3.6) is finite, since Siciak's extremal function associated with a non-pluripolar compact set is locally bounded (see [**16**, Lemma 3.4, Corollary 3.9 and Theorem 3.10]).

THEOREM 4.3. *Let E be a non-pluripolar compact subset of $\mathbb{C}^n$. Assume that for any two different points of E there exists an analytic mapping $l_{a,b}$ defined in a neighbourhood of $[0,1]$ such that $l_{a,b}([0,1]) \subset E$, $l_{a,b}(0) = a$ and $l_{a,b}(1) = b$. Then the set E satisfies (NB).*

PROOF. Let us take a function $f \in \mathcal{B}(E)$ and a non-pluripolar set $F \subset E$ such that $f(z) = 0$ for $z \in F$. By the continuity of the function f it can be assumed that the set F is compact. The definition of a quasianalytic function implies that for some $\varrho \in (0,1)$ and a sequence of polynomials $\{p_{n_j}\}$ with $deg\, p_{n_j} \leqslant n_j$ we have

$$\| f - p_{n_j} \|_F \leqslant \| f - p_{n_j} \|_E \leqslant \varrho^{n_j} \quad \text{for } j \geqslant j_0.$$

Since $f_{|F} \equiv 0$, it follows that $\| p_{n_j} \|_F \leqslant \varrho^{n_j}$. Due to the above estimate and inequality (2.1), we get

$$|p_{n_j}(z)| \leqslant \varrho^{n_j} [\Phi_F(z)]^{n_j}. \tag{4.1}$$

By Theorem 2.1 the set of all points of F at which Siciak's extremal function Φ_F is not continuous must be pluripolar. So there exists a point $a \in F$ at which Φ_F is continuous. If we take $\eta \in (0, \frac{1}{\varrho})$ then we can choose a constant $\varepsilon > 0$ such that $|\Phi_F(z)| \leqslant \eta$ for $z \in B(a,\varepsilon)$. Applying this estimate to inequality (4.1) gives $\| p_{n_j} \|_{B(a,\varepsilon)} \leqslant (\varrho\,\eta)^{n_j}$ for $j \geqslant j_0$. Since $(\varrho\,\eta) < 1$, $f(z) = 0$ for $z \in B(a,\varepsilon) \cap E$. Now let us choose an arbitrary point $b \in E \setminus \{a\}$ and an analytic map $l_{a,b}$ satisfying the assumptions of the theorem. Let us note that the function $f \circ l_{a,b}$ defined in a neighbourhood of the interval $[0,1]$ is quasianalytic on $[0,1]$ (Lemma 4.1) and that there exists $\alpha > 0$ such that $l_{a,b}([0,\alpha]) \subset B(a,\varepsilon) \cap E$. Hence $f \circ l_{a,b} \equiv 0$ on $[0,\alpha]$. By the classical Bernstein theorem it follows that $h \circ l_{a,b} \equiv 0$ on $[0,1]$. Hence, in particular, $f(b) = 0$. By the arbitrariness of the choice of $b \in E$ we derive that $f \equiv 0$ on E. □

By Lemma 3.2 and Lemma 3.3, property (NB) is invariant under biholomorphic mappings. Under certain conditions, we can prove more, namely

THEOREM 4.4. *Let $W \subset \mathbb{C}^k$ be a compact set satisfying (NB). Assume moreover that for every point $a \in W$ and for every constant $\varepsilon > 0$*

$$\text{the set } B(a,\varepsilon) \cap W \text{ is non-pluripolar.} \tag{4.2}$$

Let h be a non-degenerate holomorphic mapping defined in an open neighbourhood U of the set W, with values in $\mathbb{C}^m$ $(k \geqslant m)$. Then $h(W)$ satisfies (NB).

PROOF. Condition (4.2) implies that the set W is non-pluripolar. Since h is a non-degenerate holomorphic mapping, the set $h(W)$ is non-pluripolar (see [**12**, Lemma 2.5]). Let us take a quasianalytic function q on $h(W)$ and a non-pluripolar set $N \subset h(W)$ such that $q = 0$ on N. We can proceed like in the first part of the proof of Theorem 4.3. Namely we may assume that N

is compact and we can choose a sequence of polynomials $p_{n_j} \in \mathcal{P}_{n_j}$ such that $\| q - p_{n_j} \|_{h(W)} \leqslant \varrho^{n_j}$ for a certain $\varrho \in (0,1)$ and a sequence of positive integers $n_j \nearrow \infty$.

Then, due to the non-pluripolarity of N, by Theorem 2.1 and the Bernstein–Walsh–Siciak inequality (2.1), there exist a point $b \in N$ and constants $r > 0$, $\gamma \in (\varrho, 1)$ such that $\| p_{n_j} \|_{B(b,r)} \leqslant \gamma^{n_j}$ for $j \geqslant j_0$. Since h is a continuous mapping, the set $h^{-1}(B(b,r))$ is an open subset of U. Let $a \in h^{-1}(b) \cap W$. Obviously, there is a constant $\delta > 0$ such that $B(a,\delta) \subset h^{-1}(B(b,r))$. By assumption (4.2) the set $F := B(a,\delta) \cap W$ is not pluripolar. Let us note that

$$\begin{aligned} h(F) &= h(B(a,\delta) \cap W) \subset h(h^{-1}(B(b,r)) \cap W) \\ &\subset h(h^{-1}(B(b,r))) \cap h(W) = B(b,r) \cap h(W). \end{aligned}$$

So we have

$$\| q \circ h \|_F = \| q \|_{h(F)} \leqslant \| q - p_{n_j} \|_{h(F)} + \| p_{n_j} \|_{h(F)} \leqslant \varrho^{n_j} + \gamma^{n_j} \leqslant 2\gamma^{n_j}.$$

Hence the function $q \circ h$ vanishes on a non-pluripolar subset F of W. Since W satisfies (NB) and, by Lemma 3.3, the function $q \circ h$ is quasianalytic, we have $q \circ h \equiv 0$ on W. Consequently, $q \equiv 0$ on $h(W)$. □

Other examples of NB–sets are yielded by the following

Proposition 4.5. *Let E_1 and E_2 be compact subsets of $\mathbb{C}^k$ satisfying (NB). If the set $E_1 \cap E_2$ is non-pluripolar then the set $E := E_1 \cup E_2$ also satisfies (NB).*

Proof. Let $q \in \mathcal{B}(E)$, and let $F \subset E$ be a non-pluripolar set such that q vanishes on F. We may assume that $F \cap E_1$ is a non-pluripolar set. Since $E_1 \in (NB)$, we get $q \equiv 0$ on E_1. Hence $q = 0$ on the non-pluripolar subset $E_1 \cap E_2$ of E_2. Since $E_2 \in (NB)$, $q = 0$ on E_2.

□

Acknowledgments. The author is indebted to Professor Wiesław Pleśniak for his valuable suggestions and remarks.

References

1. Baran M., Pleśniak W., *Polynomial inequalities on algebraic sets*, Studia Math., **141(3)** (2000), 209–219.
2. Bedford E., Taylor B.A., *A new capacity for plurisubharmonic functions*, Acta Mathematica, **149** (1982), 1–40.
3. Chirka E.M., *Complex Analytic Sets*, Mathematics and Its Applications, Vol. **46**, 1985.
4. Josefson B., *On the equivalence between locally and globally polar sets for plurisubharmonic functions in $\mathbb{C}^n$*, Arkiv för Matematik, **16** (1978), 109–115.
5. Lelong P., *Sur une propriété simple des polynômes*, R. Acad. Sci. Paris, **224** (1947), 883–885.

6. Łojasiewicz S., *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991.
7. Markuszewicz A.I., *On the best approximation*, Doklady AN USSR, **44** (1944), 290–292 (in Russian).
8. Mazurkiewicz S., *Les fonctions quasi-analytiques dans l'espace fonctionnel*, Mathematica (Cluj), **13** (1937), 16–21.
9. Pleśniak W., *Quasianalytic functions of several complex variables*, Zeszyty Nauk. Uniw. Jagiell., **15** (1971), 135–145.
10. ______, *On superposition of quasianalytic functions*, Ann. Polon. Math., **26** (1972), 73–84.
11. ______, *Quasianalytic functions in the sense of Bernstein*, Dissertationes Mathematicae, **147** (1977).
12. ______, *Invariance of the L–regularity of compact sets in $\mathbb{C}^N$ under holomorphic mappings*, Trans. Amer. Math. Soc., **246** (1978), 373–383.
13. Ransford T., *Potential theory in the complex plane*, Cambridge Univ. Press, 1995.
14. Sadullaev A., *An estimate for polynomials on analytic sets*, Math. USSR Izv., **20** (1983), 493–502.
15. Siciak J., *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc., **105** (1962), 322–357.
16. ______, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Ann. Polon. Math., **39** (1981), 175–211.
17. Szmuszkowiczówna H., *Un théorème sur les polynômes et son application à la théorie des fonctions quasi-analytiques*, C.R. Acad. Sci. Paris, **198** (1934), 1119–1120.
18. Zakharyuta V.P., *Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein–Walsh theorem for analytic functions of several complex variables*, Ann. Polon. Math., **33** (1976), 137–148 (in Russian).

Received June 17, 2003

Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków, Poland
e-mail: skiba@im.uj.edu.pl