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ON THE SET OF HOMOGENEOUS GEODESICS OF A LEFT-INVARIANT METRIC

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Abstract. We present results concerning the set of homogeneous geodesics of a left-invariant Riemannian metric on a compact semi-simple Lie group.

If $(M, \langle,\rangle)$ is a Riemannian manifold its geodesic $\gamma\colon \mathbb{R} \to M$ is said to be *homogeneous* if there is a 1-parameter group of isometries $\Phi\colon \mathbb{R} \times M \to M$ of the Riemannian manifold such that

$$\gamma(\tau) = \Phi(\tau, \gamma(0)),\ \ \tau \in \mathbb{R}$$

holds; in an alternative terminology γ is called a *stationary geodesic.* The existence of a homogeneous geodesic in the case of a given 1-parameter isometry group was established under various assumptions long ago (see e. g. [**5**], [**3**]). On the other hand, as the comprehensive papers by C. S. Gordon, O. Kowalski and L. Vanhecke show the condition that all the geodesics in a homogeneous Riemannian manifold are homogeneous is a useful starting point in the classification of the homogeneous Riemannian manifolds [**2**], [**8**]. Accordingly, the problem of the existence of homogeneous geodesics in homogeneous Riemannian manifolds seems to be an interesting one. Recently several results have been obtained concerning the above problem. First, it has been shown by V. V. Kajzer that if G is a Lie group and $\langle,\rangle_G$ a left-invariant Riemannian metric on G, then the Riemannian manifold $(G, \langle,\rangle_G)$ has at least 1 homogeneous geodesic [**4**]. Generalizing this result of Kajzer it has been shown that if $M = G/H$ is a homogeneous manifold and $\langle,\rangle$ an invariant metric on G/H, then the homogeneous Riemannian manifold $(G/H, \langle,\rangle)$ has at least 1 homogeneous geodesic [**7**]. Furthermore, it has been shown by Kowalski and Vlášek that the above result is the best one which holds in general, namely they have produced a construction which yields a homogeneous Riemannian manifold for each dimension ≥ 4 such that there is only 1 homogeneous geodesic issuing from a point [**9**]. But on the other hand, it has been shown that if G is a compact semi-simple Lie group of $rank \geq 2$ and $\langle , \rangle_G$ is a left-invariant Riemannian metric on G, then the Riemannian manifold $(G, \langle , \rangle_G)$ has infinitely many homogeneous geodesics issuing from the identity element [**10**]. In fact, homogeneous geodesics of a left-invariant Riemannian metric on a compact connected Lie group has been studied earlier also by V. I. Arnold generalizing Euler's theory of rigid body motion in basically mechanical settings where the term stationary geodesic was introduced [**1**]. An extension of results of [**10**] to left-invariant Lagrangians over compact connected Lie groups of $rank \ \geq 2$ has been also obtained [**11**].

Studying the set of homogeneous geodesics of a homogeneous Riemannian manifold $(G/H, \langle , \rangle)$ the concept of geodesic vector proved to be convenient [**8**]. Namely, let $\Phi \colon G \times (G/H) \to G/H$ be the canonical left-action, $\mathfrak{g}$ the Lie algebra of G and $Exp \colon \mathfrak{g} \to G$ its exponential map. Put $o = H \in G/H$, fix a tangent vector $v \in T_o(G/H) - \{0\}$ and consider the geodesic $\gamma \colon \mathbb{R} \to G/H$ defined by $v = \dot{\gamma}(0)$. It is said that v is a *geodesic vector* if γ is a homogeneous geodesic of $(G/H, \langle , \rangle)$; in other words if

$$\gamma(\tau) = \Phi(Exp(\tau X), o), \ \tau \in \mathbb{R}$$

holds with some $X \in \mathfrak{g}$. The study of the set of homogeneous geodesics of a homogenous Riemannian manifold is obviously reducible to the study of the set of its geodesic vectors. It seems that the set of the geodesic vectors of a homogeneous Riemannian manifold does not admit a simple description in general. Namely, O. Kowalski, S. Nikčević and Z. Vlášek have given several examples which show that the set of geodesic vectors may have various structures [**6**]. The results presented below concern the set of geodesic vectors of a homogeneous Riemannian manifold $(G, \langle , \rangle_G)$, where G is a compact semi-simple Lie group and $\langle , \rangle_G$ is a left-invariant Riemannian metric on G.

1. The restricted quadratic form and the existence of geodesic vectors. Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra, $Ad \colon G \times \mathfrak{g} \to \mathfrak{g}$ the adjoint action, $G(X) = \{Ad(g)X \ |g \in G\} \subset \mathfrak{g}$ the orbit of an element $X \in \mathfrak{g}$ and $G_X < G$ the isotropy subgroup at X. The set G/G_X of left-cosets of G_X endowed with its canonical smooth manifold structure admits the canonical left-action

$$\Phi_X \colon G \times (G/G_X) \ni (g, aG_X) \mapsto (ga)G_X \in G/G_X$$

which is also smooth. Moreover, a smooth bijection $\alpha_X \colon G/G_X \to G(X)$ is defined by $\alpha_X \colon G/G_X \ni aG_X \mapsto Ad(a)X \in G(X) \subset \mathfrak{g}$ which thus yields an injective immersion into $\mathfrak{g}$ which is equivariant with respect to the actions Φ_X and Ad.

Now consider a symmetric positive definite bilinear form $A\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, then A defines a left-invariant Riemannian metric $\langle,\rangle_G$ on G by

$$\langle u,v\rangle_G = A(T_g\lambda_g^{-1}u, T_g\lambda_g^{-1}v),\ \ u,v\in T_gG,$$

where $\lambda_g\colon G\to G$ is the left translation by $g\in G$. There is also the quadratic form $Q\colon \mathfrak{g}\to\mathbb{R}$ given by $Q(X)=A(X,X),\ X\in\mathfrak{g}$ and a smooth function

$$q_X\colon Q\circ\alpha_X\colon G/G_X\to\mathbb{R},$$

which is called the *restricted quadratic form* on G/G_X. The following result has been obtained earlier [**10**]:

PROPOSITION 1.1. *Let G be a connected Lie group, $A\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ a positive definite symmetric bilinear form defined on its Lie algebra and $\langle,\rangle_G$ the left-invariant Riemannian metric defined by A on G. Then $U\in G(X)\subset\mathfrak{g}=T_eG$ is a geodesic vector if and only if $\alpha_X^{-1}(U)\in G/G_X$ is a critical point of the restricted quadratic form $q_X=Q\circ\alpha_X$.*

Essentially the same result was also obtained earlier by Arnold but in the framework of analytical mechanics [**1**].

If G is also compact and semi-simple then the manifold G/G_X becomes compact, and the restricted quadratic form $q\colon G/G_X\to\mathbb{R}$ has at least two critical points and consequently infinitely many geodesic vectors $X\in\mathfrak{g}=T_eG$ exist if *rank* $G\geq 2$ holds and these critical points yield infinitely many homogeneous geodesics emanating from the identity element $e\in G$ of the group [**10**]. The above result raises the problem of finding more information about the number and type of the critical points of the restricted quadratic form $q_X\colon G/G_X\to\mathbb{R}$ in order to obtain a detailed description of the set of geodesic vectors and consequently of the set of homogeneous geodesics.

2. The gradient of the restricted quadratic form. Let G be a compact semi-simple Lie group G and $K\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ its Cartan-Killing form which being negative definite yields a euclidean inner product by $\langle,\rangle=-K$ on $\mathfrak{g}$ which in turn canonically induces a Riemannian metric $\langle,\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ by the requirement that the canonical isomorphisms

$$\iota_Z\colon \mathfrak{g}\to T_Z\mathfrak{g},\ \ Z\in\mathfrak{g}$$

are isometries. There is also a unique Riemannian metric $\langle,\rangle_X$ on the homogeneous manifold G/G_X which renders the injective immersion α_X into $(\mathfrak{g},\langle,\rangle_{\mathfrak{g}})$ isometric.

For $U\in\mathfrak{g}$ consider the corresponding infinitesimal generator $\tilde{U}\in\mathcal{T}(G/G_X)$ of the canonical action Φ_X. Since α_X is equivariant the following holds

$$\alpha_X(\Phi_X(Exp(\tau U),gG_X)) = \alpha_X(Exp(\tau U)gG_X)$$
$$= Ad(Exp(\tau U),\alpha_X(gG_X)) = Ad(Exp(\tau U))(Ad(g)X),\ \ \tau\in\mathbb{R},\ \ g\in G.$$

Differentiating the above equality yields obviously the following one

$$T\alpha_X \tilde{U}(gG_X) = \iota_{Ad(g)X}[U, Ad(g)X].$$

Consider now the isotropy subalgebra $\mathfrak{g}_X < \mathfrak{g}$ of the adjoint action at X and also its orthogonal complement $\mathfrak{m}_X \subset \mathfrak{g}$ with respect to $\langle,\rangle$. Thus the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{m}_X \oplus \mathfrak{g}_X$$

yields the following orthogonal decomposition of the tangent space

$$T_Z\mathfrak{g} = \iota_Z(\mathfrak{m}_X) \oplus \iota_Z(\mathfrak{g}_X), \;\; Z \in \mathfrak{g}.$$

Accordingly, in what follows for a $W \in T_Z\mathfrak{g}$ the decomposition $W = W' + W''$ will be meant to be taken with respect to the above orthogonal direct sum decomposition. Moreover, since by

$$T_{gG_X}G/G_X \ni \tilde{U}(gG_X) \mapsto \iota^{-1}_{Ad(g)X} \circ T\alpha_X\tilde{U}(gG_X) \in \mathfrak{m}_{Ad(g)X}$$

a vector space isomorphism is defined, it has an inverse isomorphism

$$\omega_{Ad(g)X} \colon \mathfrak{m}_{Ad(g)X} \to T_{gG_X}G/G_X$$

which will be repeatedly applied in what follows.

Now consider a symmetric positive definite bilinear form $A \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, then a unique vector space automorphism $\kappa \colon \mathfrak{g} \to \mathfrak{g}$ is defined by

$$A(X, Y) = \langle \kappa X, Y\rangle, \;\; X, Y \in \mathfrak{g}$$

which is symmetric with respect to $\langle,\rangle$. The gradient of the restricted quadratic form q_X with respect to the Riemmanian metric $\langle,\rangle_X$ is given by the next proposition.

PROPOSITION 2.1. *Let G be a compact semi-simple Lie group, $A \colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ a positive definite symmetric bilinear form and $Q \circ \alpha_X = q_X : G/G_X \to \mathbb{R}$ for $X \in \mathfrak{g}$ the corresponding restricted quadratic form. Then its gradient field is given by*

$$G/G_X \ni gG_X \mapsto grad\; q_X(gG_X) = 2\omega_X(\kappa(Ad(g)X))' \circ \alpha_X$$

with respect to the Riemannian metric $\langle,\rangle_X$ where ω_X and κ are the corresponding vector space isomorphisms.

PROOF. In fact, the gradient of the quadratic form Q in the euclidean vector space $(\mathfrak{g}, \langle,\rangle)$ is obviously given by

$$grad\; Q(Z) = 2\kappa Z, \;\; Z \in \mathfrak{g}.$$

Thus the corresponding gradient field in the Riemannian manifold $(\mathfrak{g}, \langle,\rangle_{\mathfrak{g}})$ is given by

$$\mathfrak{g} \ni Z \mapsto 2\iota_Z \circ \kappa Z \in T_Z\mathfrak{g}.$$

For $U \in \mathfrak{m}_{Ad(g)X}$ let $\tilde{U} \in T_{gG_X} G/G_X$ be the corresponding infinitesimal generator of Φ_X, then the following holds

$$\begin{aligned} \langle \tilde{U}, grad\ q_X \rangle_X = \tilde{U} q_X = \tilde{U}(Q \circ \alpha_X) = ((T\alpha_X \tilde{U})Q) \circ \alpha_X \\ = \langle T\alpha_X \tilde{U}, grad\ Q \rangle_{\mathfrak{g}} \circ \alpha_X = \langle T\alpha_X \tilde{U}, 2\iota_{Ad(g)X} \circ \kappa(Ad(g)X) \rangle_{\mathfrak{g}} \circ \alpha_X \\ = \langle T\alpha_X \tilde{U}, 2\iota_{Ad(g)X}(\kappa(Ad(g)X))' \rangle_{\mathfrak{g}} \circ \alpha_X \\ = \langle \tilde{U}, 2(\omega_{Ad(g)X}(\kappa(Ad(g)X))' \rangle_X. \end{aligned}$$

But then the gradient field of the smooth function $q_X = Q \circ \alpha_X$ in the Riemannian manifold $(G/G_X, \langle, \rangle_X)$ is given by

$$G/G_X \ni gG_X \mapsto 2\omega_{Ad(g)X}(\kappa(Ad(g)X))' \in T_{gG_X} G/G_X$$

since any element of $T_{gG_X} G/G_X$ is obtainable as $\tilde{U}(gG_X)$ with a suitable $U \in \mathfrak{m}_{Ad(g)X}$. □

Corollary 1. *Let a left-invariant Riemannian metric be given on a compact semi-simple Lie group by a positive definite symmetric bilinear form A. Then $X \in T_e G = \mathfrak{g}$ is a geodesic vector if and only if*

$$\kappa X \in \mathfrak{g}_X$$

where κ is the vectorspace automorphism associated with A and $\mathfrak{g}_X < \mathfrak{g}$ is the isotropy subalgebra.

Proof. The *corollary* is a direct consequence of *propositions 1.1* and *2.1*. □

Corollary 2. *Let a left-invariant Riemannian metric be given on a compact semi-simple Lie group G by a symmetric positive definite bilinear form A. If $E \in \mathfrak{g}$ is an eigenvector of the corresponding symmetric automorphism $\kappa\colon \mathfrak{g} \to \mathfrak{g}$, then E is a geodesic vector of $\langle, \rangle_G$.*

Proof. Let $\lambda \in \mathbb{R}$ be the eigenvalue corresponding to the eigenvector E. Then

$$\kappa E = \lambda E \in \mathfrak{g}_E$$

since $E \in \mathfrak{g}_E$. But then the preceding *corollary* applies. □

In fact, the observation that an eigenvector of κ yields a geodesic vector was the starting point for the results of Kajzer [**4**].

Corollary 3. *Let a left-invariant Riemannian metric be given on a compact semi-simple Lie group by a positive definite bilinear form A. Then $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle X, \kappa(\mathfrak{m}_X) \rangle = \{0\}$$

where κ is the automorphism associated with A.

Proof. In fact, $\kappa X \in \mathfrak{g}_X$ if and only if $\langle \kappa X, \mathfrak{m}_X\rangle = \{0\}$. Furthermore the equality

$$\langle \kappa X, \mathfrak{m}_X\rangle = \langle X, \kappa(\mathfrak{m}_X)\rangle$$

follows from the fact that the automorphism κ is symmetric. $\square$

Theorem 2.2. *Let G be a compact semi-simple Lie group endowed with a left-invariant Riemannian metric $\langle,\rangle_G$ and let $\mathfrak{h} < \mathfrak{g}$ be a Cartan subalgebra of its Lie algebra. Then the set of those elements $X \in \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ which are geodesic vectors of $\langle,\rangle_G$ is either empty or there is a subspace $\mathfrak{s} \subset \mathfrak{h}$ such that*

1. *dim $\mathfrak{s} \geq 1$.*
2. *Every element of $\mathfrak{s}$ is a geodesic vector.*
3. *Each regular element of $\mathfrak{h}$ which is a geodesic vector is contained by $\mathfrak{s}$.*

Proof. Consider the $\langle,\rangle$-orthogonal complement $\mathfrak{m} = \mathfrak{h}^{\perp}$ of the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Then by

$$\mathfrak{s} = \mathfrak{h} \cap (\kappa(\mathfrak{m}))^{\perp}$$

a subspace of the Cartan subalgebra $\mathfrak{h}$ is defined.

Assume that $\mathfrak{s} \neq \{0\}$ and consider a $Y \in \mathfrak{s} - \{0\}$ and also the corresponding isotropy subalgebra $\mathfrak{g}_Y < \mathfrak{g}$. Then

$$\mathfrak{h} < \mathfrak{g}_Y$$

since $\mathfrak{h}$ is commutative and $\mathfrak{g}_Y$ is the maximal algbera formed by elements commuting with Y. Let $\mathfrak{m}_Y$ be the $\langle,\rangle$-orthogonal complement of $\mathfrak{g}_Y$ in $\mathfrak{g}$, then

$$\mathfrak{m} \supset \mathfrak{m}_Y$$

by the preceding observations. But then $Y \perp \kappa(\mathfrak{m})$ and the following holds

$$\{0\} = \langle Y, \kappa(\mathfrak{m})\rangle = \langle \kappa Y, \mathfrak{m}\rangle \supset \langle \kappa Y, \mathfrak{m}_Y\rangle$$

which implies that $\kappa Y \in \mathfrak{g}_Y$. But then by *corollary 1* of *proposition 2.1* the origin $o_Y \in G/G_Y$ is a critical point of q_Y. Therefore Y is a geodesic vector according to *proposition 1.1*.

Additionally assume that $X \in \mathfrak{h}$ is a regular element of $\mathfrak{g}$ and also a geodesic vector. Then $\mathfrak{g}_X = \mathfrak{h}$, $\mathfrak{m} = \mathfrak{m}_X$ and by the above *corollary 1* $\kappa X \in \mathfrak{g}_X$. Therefore the following is obtained

$$\{0\} = \langle \kappa X, \mathfrak{m}_X\rangle = \langle \kappa X, \mathfrak{m}\rangle = \langle X, \kappa(\mathfrak{m})\rangle.$$

Consequently $X \in \mathfrak{h} \cap (\kappa(\mathfrak{m}))^{\perp} = \mathfrak{s}$ as well. $\square$

3. The Hessian of the restricted quadratic form. As the second step in studying the critical points of a restricted quadratic form q_X the Hessian of q_X is calculated at a critical point.

PROPOSITION 3.1. *Let G be a compact semi-simple Lie group, $A\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ a positive definite bilinear form, $X\in\mathfrak{g}$ such that o_X is a critical point of q_X. Then the Hessian $H_{o_X}q\colon T_{o_X}(G/G_X)\times T_{o_X}(G/G_X)\to\mathbb{R}$ is given by*

$$H_{o_X}q(\tilde{U},\tilde{V})=\langle U,[[V,X],\kappa X]+[X,\kappa[V,X]]\rangle\circ\alpha_X,\ \ U,V\in\mathfrak{m}_X$$

where $U,V\in\mathfrak{m}_X$ and $\tilde{U},\tilde{V}\in\mathcal{T}(G/G_X)$ are the associated infinitesimal generators of the canonical action Φ_X.

PROOF. In order to calculate the Hessian of q_X at the critical point o_X consider $U,V\in\mathfrak{m}_X$ and the corresponding infinitesimal generators $\tilde{U}$, $\tilde{V}\in\mathcal{T}(G/G_X)$ of the canonical action Φ_X. Then

$$\begin{aligned}
H_{o_X}q_X(\tilde{U}(o_X),\tilde{V}(o_X)) &= \tilde{V}(o_X)(\tilde{U}q_X)\\
&=\lim_{\tau\to0}\frac{1}{\tau}\{(\tilde{U}q_X)(Exp(\tau V)G_X)-(\tilde{U}q_X)(G_X)\}=\\
&=\lim_{\tau\to0}\frac{1}{\tau}\{((T\alpha_X\tilde{U}(Exp(\tau V)G_X)Q)\circ\alpha_X(Exp(\tau V)G_X)\\
&\quad-(T\alpha_X\tilde{U}(G_X)Q)\circ\alpha_X(G_X)\}
\end{aligned}$$

by the definition of the tangent linear map since $q_X=Q\circ\alpha_X$. But then, by the already calculated expression of the image of an infinitesimal generator and its interior product with the field *grad* Q presented in the proof of *proposition 2.1*, the above expression is equal to the following one:

$$\begin{aligned}
&\lim_{\tau\to0}\frac{1}{\tau}\{\langle\iota_{Ad(Exp(\tau V))X}[U,Ad(Exp(\tau V)X],2\iota_{Ad(Exp(\tau V))X}\circ\kappa(Ad(Exp(\tau V)X)\rangle_{\mathfrak{g}}\\
&\qquad-\langle\iota_X[U,X],2\iota_X\circ\kappa X\rangle_{\mathfrak{g}}\}\\
&=2\lim_{\tau\to0}\frac{1}{\tau}\{\langle[U,Ad(Exp(\tau V)X],\kappa(Ad(Exp\tau V)X)\rangle-\langle[U,X],\kappa X\rangle\}\\
&=2\lim_{\tau\to0}\{\langle\frac{[Ad(Exp(\tau V))U-U}{\tau},X],\kappa(Ad(Exp(\tau V)X)\rangle\\
&\qquad+\langle[U,X],\frac{\kappa(Ad(Exp(\tau V)X-\kappa X}{\tau}\rangle\}\\
&=2\{\langle[[V,U],X],\kappa X\rangle+\langle[U,X],\kappa[V,X]\rangle\}.
\end{aligned}$$

But this was to be proved. Since any element of $T_{o_X}(G/G_X)$ is obtainable through infinitesimal generators of Φ, the Hessian is given as above. $\square$

Remark 1. *Let $o_X \in G/G_X$ be a critical point of q_X then the vector space isomorphism*

$$\varpi_X = \omega_X \circ ad(X) \colon \mathfrak{m}_X \to T_{o_X} G/G_X$$

pulls back the Hessian $H_{o_X} q_X$ to a symmetric bilinear form

$$\varpi_X^* H_{o_X} \colon (U, V) \mapsto 2\langle U, [[V, X], \kappa X] + [X, \kappa[V, X]]\rangle$$

which is defined on $\mathfrak{m}_X$.

Remark 2. *The Hessian of the restricted quadratic form as given above is symmetric at a critical point.*

Proof. Let $X \in \mathfrak{g}$ be such that $o_X \in G/G_X$ is a critical point of the restricted quadratic form q_X. For $U, V \in \mathfrak{m}_X$ consider the corresponding infinitesimal generators $\tilde{U},\ \tilde{V} \in \mathcal{T}(G/G_X)$ of the canonical left action. Then the following holds:

$$\begin{aligned}
\frac{1}{2} H_{o_X} q(\tilde{U}, \tilde{V}) &= \{\langle U, [[V, X], \kappa X] + [X, \kappa[V, X]]\rangle\} \circ \alpha_X \\
&= \{\langle [\kappa X, U], [V, X]\rangle + \langle [U, X], \kappa[V, X]\rangle\} \circ \alpha_X \\
&= \{\langle \kappa X, [U, [V, X]]\rangle + \langle \kappa[U, X], [V, X]\rangle\} \circ \alpha_X \\
&= \{\langle \kappa X, -[X, [U, V]] - [V, [X, U]]\rangle + \langle [V, X], \kappa[U, X]\rangle\} \circ \alpha_X \\
&= \{\langle [\kappa X, X], [U, V]\rangle + \langle [V, \kappa X], [U, X]\rangle + \langle V, [X, \kappa[U, X]\rangle\} \circ \alpha_X \\
&= \{\langle V, [[U, X], \kappa X] + [X, [\kappa[U, X]\rangle\} \circ \alpha_X = \frac{1}{2} H_{o_X} q(\tilde{V}, \tilde{U}).
\end{aligned}$$

Since any tangent vector in $T_{o_X} G/G_X$ is obtainable this way, the assertion holds. $\square$

Proposition 3.2. *Let G be a compact connected semi-simple Lie group, A a symmetric positive definite bilinear form on $\mathfrak{g}$ and $X \in \mathfrak{g}$ such that o_X is a critical point of q_X. Then o_X is a degenerate critical point if and only if there is a $V \in \mathfrak{m}_X - \{0\}$ such that*

$$[[V, X], \kappa X] + [X, \kappa[V, X]] \in \mathfrak{g}_X$$

where κ is the symmetric automorphism associated with A.

Proof. In fact, $H_{o_X} q$ and $\varpi_X^* H_{o_X}$ are degenerate simultaneously. But $\varpi_X^* H_{o_X}$ is degenerate if and only if there is a $V \in \mathfrak{m}_X$ such that for all $U \in \mathfrak{m}_X$ the following holds:

$$\frac{1}{2} H_{o_X} q(\tilde{U}, \tilde{V}) = \langle U, [[V, X], \kappa X] + [X, \kappa[V, X]]\rangle \circ \alpha_X = 0.$$

But the above equality is valid for each $U \in \mathfrak{m}_X$ if and only if

$$[[V, X], \kappa X] + [X, \kappa[V, X]] \in \mathfrak{g}_X,$$

but this is the assertion of the *proposition*. $\square$

Corollary 1. *Let Let $X \in \mathfrak{g}$ be an element of the Lie algebra such that $o_X \in G/G_X$ is a critical point of qX. Then o_X is a degenerate critical point if and only if there is a $V \in \mathfrak{g}_X - \{0\}$ such that*

$$[[V, X], \kappa X] + ([X, \kappa[V, X]])' = 0$$

where the component is defined by the orthogonal decomposition $\mathfrak{g} = \mathfrak{m}_X \oplus \mathfrak{g}_X$.

Proof. In fact, the condition of the preceding *proposition* is satisfied if and only if there is a $V \in \mathfrak{m}_X - \{0\}$ such that the following holds

$$0 = ([[V, X], \kappa X] + [X, \kappa[V, X]])' = [[V, X], \kappa X] + ([X, \kappa[V, X]])'$$

since $\kappa X \in \mathfrak{g}_X$ by *corollary 1* of *proposition 2.1* and then $[[V, X], \kappa X] \in \mathfrak{m}_X$. $\square$

Corollary 2. *Let $X \in \mathfrak{g}$ be such that o_X a critical point of q_X. Then o_X is degenerate if and only if there is a $V \in \mathfrak{m}_X - \{0\}$ such that*

$$[\kappa X, V] - (\kappa[X, V])' = 0,$$

where $\mathfrak{m}_X \subset \mathfrak{g}$ is the $\langle , \rangle$-orthogonal complement of $\mathfrak{g}_X$.

Proof. By the preceding *proposition* o_X is a degenerate critical point of q_X if and only if there is a $V \in \mathfrak{m}_X - \{0\}$ such that

$$[[V, X], \kappa X] + [X, \kappa[V, X]] \in \mathfrak{g}_X.$$

By the *Jacobi identity* the preceding relation is equivalent to the following one:

$$[X, [\kappa X, V]] + [V, [X, \kappa X]] + [X, \kappa[V, X]] \in \mathfrak{g}_X.$$

But the fact that o_X is a critical point of q_X implies by the *corollary 1* of *Proposition 2.1* that $\kappa X \in \mathfrak{g}_X$ and therefore $[X, \kappa X] = 0$. Therefore the preceding condition is equivalent to the following one

$$[X, [\kappa X, V] - \kappa[X, V]] \in \mathfrak{g}_X.$$

But then the following holds:

$$[X, [\kappa X, V] - (\kappa[X, V])'] = [X, [\kappa X, V] - (\kappa[X, V] - (\kappa[X, V])'')]$$

$$= [X, [\kappa X, V] - \kappa[X, V]] \in \mathfrak{g}_X.$$

Yet $[\kappa X, V] \in \mathfrak{m}_X$, $(\kappa[X, V])' \in \mathfrak{m}_X$ imply $[X, [\kappa X, V] - (\kappa[X, V])'] \in \mathfrak{m}_X$. But then

$$[X, [\kappa X, V] - (\kappa[X, V])'] = 0$$

is valid. Therefore $[\kappa X, V] - (\kappa[X, V])' \in \mathfrak{g}_X$ follows by a basic property of $\mathfrak{g}_X$. Thus $[\kappa X, V] - (\kappa[X, V])' \in \mathfrak{m}_X \cap \mathfrak{g}_X = \{0\}$ is obtained. $\square$

THEOREM 3.3. *Let a left-invariant Riemannian metric be given on a compact semi-simple Lie group G by a positive definite symmetric bilinear form $A\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ such that the eigenspaces of the corresponding symmetric automorphism κ are 1-dimensional. If $E\in\mathfrak{g}$ is an eigenvector of κ with maximal eigenvalue, then the corresponding $o_E\in G/G_E$ is a non-degenerate critical point of q_E.*

PROOF. The point $o_E\in G/G_E$ is a critical point of q_E in consequence of *corollary 2* of *proposition 2.1*. Moreover, o_E is degenerate critical point by the preceding *corollary* if and only if there is a $V\in\mathfrak{m}_E-\{0\}$ such that

$$[\kappa E,V]+(\kappa[E,V])'=0.$$

Let now $\lambda\in\mathbb{R}$ be the maximal eigenvalue of κ corresponding to E. Then the preceding equality is equivalent to the following one:

$$\lambda[E,V]+(\kappa[E,V])'=0.$$

Here $[E,V]\neq 0$, since $V\in\mathfrak{m}_E-\{0\}$.

First consider the case when $(\kappa[E,V])'=\kappa[E,V]$ holds. Then the above equality is valid if and only if $[E,V]$ is an eigenvector of κ with eigenvalue λ. But $E\perp[E,V]$. Therefore the above equality holds if and only if the eigenspace of A corresponding to λ has dimension ≥ 2; but this contradicts the assumption that all eigenspaces are 1-dimensional.

Secondly consider the case when $(\kappa[E,V])'\neq\kappa[E,V]$ is valid. Then the equality

$$(\kappa[E,V])'=[\kappa E,V]=\lambda[E,V]$$

is equivalent to the orthogonal decomposition

$$\kappa[E,V]=\lambda[E,V]+(\kappa[E,V])''$$

with non-zero terms on the left side. But this decomposition is equivalent to the inequality

$$\|\kappa[E,V]\|>\lambda\|[E,V]\|$$

which contradicts the assumption that A is positive definite and λ is the maximal eigenvalue of κ. □

COROLLARY. *Let G be a compact connected semi-simple Lie group with a left-invariant Riemannian metric $\langle,\rangle_G$ defined by a positive definite symmetric bilinear form $A\colon\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ such that the corresponding symmetric automorphism $\kappa\colon\mathfrak{g}\to\mathfrak{g}$ has only 1-dimensional eigenspaces. If E is an eigenvector of the maximal eigenvalue, then it is a geodesic vector which is isolated in the set of geodesic vectors on the orbit $G(E)\subset\mathfrak{g}$ under the adjoint action.*

PROOF. A direct consequence of *corollary 1* to *proposition 2.1* and of the preceding *theorem.* □

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